Résumé of Master's Thesis

Schanuel's Conjecture and Exponential Fields

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In recent years, Schanuel's Conjecture has played an important role in Transcendental Number Theory as well as decidability problems in Model Theory. The connection between these two areas was made by Boris Zilber in [5].

This project firstly gives an introduction to exponential fields, in particular to the complex exponential field $\mathbb{C}_{\exp}$, and secondly exhibits various applications of Schanuel's Conjecture in this context. Although Schanuel's Conjecture has been influencial in a wide range of mathematical areas, this survey mainly focuses on conclusions in exponential fields. Amongst others, papers by D'Aquino, Macintyre and Terzo [1], Marker [3] and Mantova [2] deal with such applications. The main objective is to highlight and compare the role Schanuel's Conjecture plays in the proofs of their results. Lastly, we summarise the idea behind Zilber's construction of fields imitating the complex exponential field and state a few recent results as well as open questions.

In the first chapter, we briefly summarise basic results of Field Theory to clarify the notion of the transcendence degree of a field extension. This allows us to state Schanuel's Conjecture, of which we present the two equivalent versions we refer to. These statements read as:

Version 1. Let $a_1, \ldots, a_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then

$$\mathrm{td}_{\mathbb{Q}}(\overline{a}, \exp(\overline{a})) \geq n,$$

where $\mathrm{td}_{\mathbb{Q}}(\overline{a}, \exp(\overline{a}))$ denotes the transcendence degree of the tuple $(\overline{a}, \exp(\overline{a}))$ over $\mathbb{Q}$.

Version 2. Let $a_1, \ldots, a_n$ be complex numbers. Then

$$\mathrm{td}_{\mathbb{Q}}(\overline{a}, \exp(\overline{a})) \geq \mathrm{ldim}_{\mathbb{Q}}(\overline{a}),$$

where $\mathrm{ldim}_{\mathbb{Q}}(\overline{a})$ denotes the linear dimension of the vector space over $\mathbb{Q}$ spanned by $a_1, \ldots, a_n$.

We introduce exponential fields axiomatically as follows: An exponential field is a field $(K, +, \cdot, 0, 1)$ equipped with a unary function $\exp : K \to K$, satisfying the following two axioms:

(E1)

$$\forall x, y \in K : \quad \exp(x + y) = \exp(x) \cdot \exp(y),$$

(E2)

$$\exp(0) = 1.$$

Next we give an overview of the two applications of Schanuel's Conjecture in the complex exponential field which we present in more detail in the second chapter. The first application is the proof of Shapiro's Conjecture, whose statement requires the notion of exponential polynomials: A function f over $\mathbb{C}$ of the form

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z},$$

where $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are complex coefficients, is called an exponential polynomial. The set of such functions forms a ring under the usual addition and multiplication denoted by $\mathcal{E}$.

Shapiro's Conjecture (see [1] p. 597). *If f and g are two exponential polynomials in $\mathcal{E}$ with infinitely many common zeros, then there exists an exponential polynomial h in $\mathcal{E}$ such that h is a common divisor of f and g in the ring $\mathcal{E}$, and h has infinitely many zeros in $\mathbb{C}$.*

The proof that Schanuel's Conjecture implies Shapiro's Conjecture is presented in a more general setting than the complex exponential field. One of the main steps of the proof is based on Ritt's Factorisation Theorem (see [4]), which allows us to reduce Shapiro's Conjecture to two distinct cases: The first case, namely that one of f and g is a so-called simples exponential polynomial, can be shown independently of Schanuel's Conjecture. The second case, namely that both f and g are irreducible in $\mathcal{E}$, requires Schanuel's Conjecture as the setting is changed to varieties over algebraically closed fields. Some arguments are slightly different from the ones used in [1] in order to make the proof more efficient.

The second result we focus on is the following statement proved by Marker (see [3] Theorem 1.6) under the assumption of Schanuel's Conjecture:

Theorem. Suppose that $p(X, Y) \in \overline{\mathbb{Q}}[X, Y]$ is irreducible and depends on both X and Y. Then there are infinitely many algebraically independent zeros of $f(z) = p(z, \mathrm{e}^z)$.

Marker actually states the theorem only for $p(X, Y) \in \mathbb{Q}[X, Y]$ but proves it more generally for polynomials in the algebraic closure of $\mathbb{Q}$. We present Marker's proof in more detail.

In [2] Mantova modifies this result and proves a more general statement, again assuming Schanuel's Conjecture:

Let $k \subset \mathbb{C}$ be a finitely generated field and $p(X, Y) \in k[X, Y]$. A solution v of $p(z, \mathrm{e}^z) = 0$ such that

$$\mathrm{td}_k(v, \mathrm{e}^v) = 1$$

is called generic over k:

Theorem. For any finitely generated field $k \subset \mathbb{C}$, and for any irreducible polynomial $p(X, Y) \in k[X, Y]$ depending on both X and Y, the equation

$$p(z, \mathrm{e}^z) = 0$$

has a solution generic over k.

We outline the main steps of the arguments leading to this result. Moreover, we explain how Marker's result can be deduced from Mantova's work.

Throughout the paper we carefully indicate when a result is based on the assumption of Schanuel's Conjecture and when it can be proved unconditionally.

Both Marker and Mantova refer to the simplest case of the strong exponential-algebraic closure property of $\mathbb{C}_{\exp}$ conjectured by Zilber in [5]. In this paper, the author constructs an axiomatisation of algebraically closed exponential fields of characteristic 0 satisfying Schanuel's Conjecture. Moreover he proves that in each uncountable cardinality there exists a model unique up to isomophism. It is an open question whether Zilber's Field of cardinality continuum is infact the complex exponential field. Since Zilber's paper is of great importance in the context of recent results on Schanuel's Conjecture, we describe and focus on his work in the third chapter. We firstly explain the basic notions of Infinitary Logic and secondly summarise the properties of Zilber's fields and how these are related to the applications of Schanuel's Conjecture presented in the second chapter.

The final chapter gives a few more examples of recent results related to Schanuel's Conjecture and states some open questions.

References

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[2] V. Mantova, *Generic Solutions of Polynomial Exponential Equations*, preprint, 2014. `arXiv:1402.0685`.

[3] David Marker, A Remark on Zilber's Pseudoexponentiation, *The Journal of Symbolic Logic* **71** (2006), no. 3, 791 – 798.

[4] J. F. Ritt, A factorization theory for functions $\sum_{i=1}^{n} a_i e^{\alpha_i z}$, *Transactions of the American Mathematical Society* **29** (1927), no. 3, 584 – 596.

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