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Schanuel's Conjecture and Exponential Fields

by Lothar Sebastian Krapp

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1 Introduction

In recent years, Schanuel's Conjecture has played an important role in Transcendental Number Theory (e. g. Waldschmidt [31]) as well as decidability problems in Model Theory (e. g. Macintyre and Wilkie [18]). The connection between these two areas was made by Boris Zilber in [36].

This dissertation will firstly give an introduction to exponential fields, in particular to the complex exponential field $\mathbb{C}_{\exp}$, and secondly exhibit various applications of Schanuel's Conjecture in this context. Although Schanuel's Conjecture has been influencial in a wide range of mathematical areas, this survey will mainly focus on conclusions in exponential fields. Amongst others, papers by D'Aquino, Macintyre and Terzo [6], Marker [21] and Mantova [19] deal with such applications. The main objective will be to highlight and compare the role Schanuel's Conjecture plays in the proofs of their results. Lastly, we will summarise the idea behind Zilber's construction of fields imitating the complex exponential field and state a few recent results as well as open questions.

In the first chapter, we will briefly summarise basic results of Field Theory to clarify the notion of the transcendence degree of a field extension. This allows us to state Schanuel's Conjecture, of which we will present the two equivalent versions we will refer to in this paper. We will also introduce exponential fields axiomatically and give an overview of the two applications of Schanuel's Conjecture in the complex exponential field which we will present in more detail in the second chapter. The first application is the proof of Shapiro's Conjecture on exponential polynomials with infinitely many common zeros. This will be done in a more general setting than the complex exponential field. The second application is the proof of the existence of infinitely many algebraically independent zeros of polynomial exponential equations. Throughout the paper we will carefully indicate when a result is based on the assumption of Schanuel's Conjecture and when it can be proved unconditionally.

Zilber's model theoretic approach by constructing fields satisfying Schanuel's Conjecture will be explained in the third chapter. We will firstly explain the basic notions of Infinitary Logic and secondly summarise the properties of Zilber's fields and how these are related to the applications of Schanuel's Conjecture.

The final chapter will give a few more examples of recent results related to Schanuel's Conjecture and state some open questions.

2 Schanuel's Conjecture

This chapter shall present an introduction to the matter of this dissertation, namely Schanuel's Conjecture itself. Although the mathematical language necessary to state this conjecture only requires basic results of field theory, it is important to make a few notions precise.

2.1 Background

The transcendence degree of a field extension L/K, which we will introduce in this section, is a measure of the "size" of the extension. In the case of finitely generated extensions, it indicates the minimal number of elements in L transcendental over K which must be adjoint to K in order to produce L. We will state two theorems, which can be found in basic field theory textbooks such as Howie [10] Theorem 10.6 and 10.7, to establish that the transcendence degree is well-defined.

Definition 2.1. Let K be a field of characteristic 0, and let L/K be a field extension. A subset $\{\alpha_1, \ldots, \alpha_n\}$ of L is called *algebraically independent* (over K) if for all polynomials $p(X_1, \ldots, X_n) \in K[X_1, \ldots, X_n]$,

$$p(\alpha_1, \ldots, \alpha_n) = 0 \text{ implies } p(X_1, \ldots, X_n) = 0.$$

Theorem 2.2. *Let L/K be a finitely generated field extension such that $L = K(\alpha_1, \ldots, \alpha_n)$. Then there exists an intermediate field extension E, that is, $L/E/K$, such that for some m such that $0 \le m \le n$ we have*

(i) $E = K(\beta_1, \ldots, \beta_m)$ for some set $\{\beta_1, \ldots, \beta_m\} \subseteq L$ which is algebraically independent over K, and

(ii) the degree $[L : E]$ of the field extension L/E is finite.

Theorem 2.3. *Using the same notation and conditions as in Theorem 2.2, suppose that there is another intermediate field extension $L/F/K$, such that*

(i) $F = K(\gamma_1, \ldots, \gamma_p)$ for some set $\{\gamma_1, \ldots, \gamma_p\} \subseteq L$ which is algebraically independent over K, and

(ii) $[L : F]$ is finite.

Then $p = m$.

The number m is called the *transcendence degree* of L over K and denoted by $\mathrm{td}(L/K)$ or $\mathrm{td}_K(L)$. For ease of notation, we also write $\mathrm{td}_K(\alpha_1, \ldots, \alpha_n)$ for the transcendence degree of $K(\alpha_1, \ldots, \alpha_n)$ over K and call it the transcendence degree of $\alpha_1, \ldots, \alpha_n$ over K.

Remark 2.4. Theorem 2.3 can be proved similarly to the Steinitz exchange lemma for bases of finite-dimensional vector spaces, showing that a maximal linearly independent set of a finite-dimensional vector space forms a basis. By analogy, we can call B a *transcendence basis* of L/K if it is algebraically independent and maximal with that property. The transcendence degree of L/K is then equal to the cardinality of such a transcendence basis. Note that in this way we can also define the transcendence degree of a general (not necessarily finitely generated) field extension L/K as the largest cardinality of an algebraically independent subset of L over K.

2.2 Equivalent statements

The language we have established in the previous section is sufficient to state the equivalent versions of Schanuel's Conjecture, whose applications we will consider in the following chapters.

Schanuel's Conjecture was first mentioned in the literature by Stephen Schanuel's doctoral supervisor Serge Lang in [15], p. 30 f. It is stated as follows:

"[...] if $\alpha_1, \ldots, \alpha_m$ are complex numbers, linearly independent over $\mathbb{Q}$, then the transcendence degree of

$$\alpha_1, \ldots, \alpha_m, \mathrm{e}^{\alpha_1}, \ldots, \mathrm{e}^{\alpha_m}$$

is at least m."

Lang also remarks the significance of this conjecture, as the algebraic independence of e and π could be proved by considering the complex numbers

$$1, 2\pi\mathrm{i}, \mathrm{e} \text{ and } \mathrm{e}^{2\pi\mathrm{i}},$$

having transcendence degree of at least 2.

In our terminology, this first statement reads as:

Version 1. *Let $a_1, \ldots, a_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then*

$$\mathrm{td}_{\mathbb{Q}}(a_1, \ldots, a_n, \exp(a_1), \ldots, \exp(a_n)) \geq n.$$

For ease of notation, we will from now on also write tuples $(a_1, \ldots, a_n)$ as $\overline{a}$ and $(\exp(a_1), \ldots, \exp(a_n))$ as $\exp(\overline{a})$.

Consider a $\mathbb{Q}$-linear combination $a_{n+1} = \lambda_1 a_1 + \ldots + \lambda_n a_n$. Let M be a non-zero integer such that $M\lambda_k$ is an integer for all k, and assume, without loss of generality, that $M\lambda_1, \ldots, M\lambda_s$ are non-negative and $M\lambda_{s+1}, \ldots, M\lambda_n$ are negative for some $0 \leq s \leq n$. Then for the polynomial

$$p(Y_1, \ldots, Y_{n+1}) = \prod_{k=1}^{s} Y_k^{M\lambda_k} - Y_{n+1}^{M} \prod_{k=s+1}^{n} Y_k^{-M\lambda_k}$$

we obtain

$$p(\exp(a_1), \ldots, \exp(a_{n+1})) = 0.$$

Therefore, both $(a_1, \ldots, a_{n+1})$ and $(\exp(a_1), \ldots, \exp(a_{n+1}))$ are algebraically dependent. Hence, adding a linear combination of $\overline{a}$ to the tuple does not change the transcendence degree of $(\overline{a}, \exp(\overline{a}))$.

This gives rise to the following equivalent version of Schanuel's Conjecture, which we will also use and refer to:

Version 2. *Let $a_1, \ldots, a_n$ be complex numbers. Then*

$$\mathrm{td}_{\mathbb{Q}}(\overline{a}, \exp(\overline{a})) \geq \mathrm{ldim}_{\mathbb{Q}}(\overline{a}),$$

where $\mathrm{ldim}_{\mathbb{Q}}(\overline{a})$ *denotes the* linear dimension *of the vector space over* $\mathbb{Q}$ *spanned by* $a_1, \ldots, a_n$.

By defining the *predimension* δ of a tuple of complex numbers $\overline{a}$ as

$$\delta(\overline{a}) := \mathrm{td}_{\mathbb{Q}}(\overline{a}, \exp(\overline{a})) - \mathrm{ldim}_{\mathbb{Q}}(\overline{a}),$$

this version can be restated as simply

$$\delta(\overline{a}) \geq 0.$$

For an overview of further variants of Schanuel's Conjecture and the (known) dependencies between them, see Kirby [12].

2.3 Exponential fields

Although we will mainly concentrate on applications of Schanuel's Conjecture in the complex exponential field $\mathbb{C}_{\exp}$, we also introduce a more general notion of exponential fields. This can be done in a model theoretic way (see e. g. Wolter [33]), which might be of interest for later parts of this work, but initially an axiomatic introduction will be sufficient for our purposes.

Definition 2.5. An *exponential field* is a field $(K, +, \cdot, 0, 1)$ equipped with a unary function

$$\exp : K \to K,$$

satisfying the following two axioms:

(E1)

$$\forall x, y \in K: \quad \exp(x + y) = \exp(x) \cdot \exp(y),$$

(E2)

$$\exp(0) = 1.$$

exp is said to be an exponential function on K.[1]

When we take $\mathbb{C}$ as our field and equip it with the usual exponential function $\exp : z \mapsto \mathrm{e}^z$, we obtain the *complex exponential field*

$$\mathbb{C}_{\exp} = (\mathbb{C}, +, \cdot, 0, 1, \exp).$$

Similarly, the *real exponential field* is defined as

$$\mathbb{R}_{\exp} = (\mathbb{R}, +, \cdot, 0, 1, \exp),$$

with standard exponentiation $x \mapsto \mathrm{e}^x$ on the real numbers.

Later we will also write e^x instead of $\exp(x)$ in general exponential fields, and e then stands for the element $\exp(1)$ in K.

2.4 Overview of applications

This section will give a brief overview of the applications of Schanuel's Conjecture which we will consider in detail in the next chapter.

Harold S. Shapiro stated in [24], p. 18, the following conjecture on exponential polynomials:

If two exponential polynomials have infinitely many zeroes in common, they are both multiples of some third (entire transcendental) exponential polynomial.

D'Aquino, Macintyre and Terzo prove Shapiro's Conjecture assuming Schanuel's Conjecture in [6]. We will restate Shapiro's Conjecture in a more symbolic language.

Definition 2.6. A function f over $\mathbb{C}$ of the form

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z},$$

where $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ are complex coefficients, is called an *exponential polynomial.* The set of such functions forms a ring under the usual addition and multiplication, which we will denote by $\mathcal{E}$.

[1] Some authors include $\exp(1) \neq 1$ as an axiom to disallow the trivial constant exponential function.

This definition of exponential polynomials differs from Shapiro's (see [24] p. 1), which allows complex polynomials in place of constant coefficients $\lambda_1, \ldots, \lambda_n$. We will therefore only consider a special case of Shapiro's Conjecture.

The notions of multiples and divisibility of exponential polynomials will be interpreted in the setting of the ring $\mathcal{E}$.

Remark 2.7. Every exponential polynomial is an *entire function*, that is, a complex-valued function which is holomorphic over the whole complex plane. Moreover, every non-constant exponential polynomial is *transcendental*, i. e. not a complex polynomial. This explains why Shapiro mentions these two properties in his original statement.

Shapiro's Conjecture (see [6] p. 597). *If f and g are two exponential polynomials in $\mathcal{E}$ with infinitely many common zeros, then there exists an exponential polynomial h in $\mathcal{E}$ such that h is a common divisor of f and g in the ring $\mathcal{E}$, and h has infinitely many zeros in $\mathbb{C}$.*

In fact, we will even consider Shapiro's Conjecture for a more general class of exponential fields which also contains $\mathbb{C}_{\exp}$.

The second result we focus on is the following statement proved by David Marker (see [21] Theorem 1.6):

Theorem 2.8 (SC[2])**.** *Suppose that $p(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ is irreducible and depends on both X and Y. Then there are infinitely many algebraically independent zeros of $f(z) = p(z, \mathrm{e}^z)$.*

Marker actually states the theorem only for $p(X,Y) \in \mathbb{Q}[X,Y]$ but proves it more generally for polynomials in the algebraic closure of $\mathbb{Q}$.

In [19] Vincenzo Mantova modifies this result and proves a more general statement.

Definition 2.9. Let $k \subset \mathbb{C}$ be a finitely generated field and $p(X,Y) \in k[X,Y]$. A solution v of $p(z, \mathrm{e}^z) = 0$ such that

$$\mathrm{td}_k(v, \mathrm{e}^v) = 1$$

is called *generic over k*.

Theorem 2.10 (SC)**.** (See [19] Theorem 1.2).
For any finitely generated field $k \subset \mathbb{C}$, and for any irreducible polynomial $p(X,Y) \in$

[2]This means that for the proof of this statement we assume Schanuel's Conjecture.

$k[X,Y]$ depending on both X and Y, the equation

$$p(z, \mathrm{e}^z) = 0$$

has a solution generic over k.

Both Marker and Mantova refer to the simplest case of the strong exponential-algebraic closure property of $\mathbb{C}_{\exp}$ conjectured by Zilber in [36]. Since Zilber's paper is of great importance in the context of recent results on Schanuel's Conjecture, we will describe and focus on his work in Chapter 4.

3 Survey of applications

In the following two sections, we will describe the proofs of the theorems stated in Section 2.4. Since this is a survey on the role Schanuel's Conjecture plays in those proofs, we will mainly focus on the arguments based on the assumption that Schanuel's Conjecture holds. The papers [21] and [19] which we consider work in the setting of the complex exponential field $\mathbb{C}_{\exp}$, whereas [6] proves Shapiro's Conjecture for slightly more general exponential fields. We will state all results in as general form as possible, particularly indicating where additional assumptions must be made, e. g. some results only hold in $\mathbb{C}_{\exp}$.

3.1 Shapiro's Conjecture

In this section we will establish how Schanuel's Conjecture implies Schapiro's Conjecture, following the work of D'Aquino, Macintyre and Terzo [6].

Recall the statement of Shapiro's Conjecture:
If f and g are two exponential polynomials in $\mathcal{E}$ with infinitely many common zeros, then there exists an exponential polynomial h in $\mathcal{E}$ such that h is a common divisor of f and g in the ring $\mathcal{E}$, and h has infinitely many zeros in $\mathbb{C}$.

We will consider two cases with distinct conditions on the reducibility of f and g. In both cases we can prove Shapiro's Conjecture in a more general setting with connections to Zilber's exponential field, which will be explained in Section 4.2. No arguments in the first case require Schanuel's Conjecture. In the second case the assumption of Schanuel's Conjecture, whose statement can easily be extended for this more general setting of exponential fields, is used. We will introduce some terminology and state the two different cases of Shapiro's Conjecture as theorems.

In this section, K will denote any exponential field with exponentiation $\exp(z) = \mathrm{e}^z$ and the following properties:

1. K has characteristic 0;
2. K is algebraically closed;
3. exp is surjective onto $K^\times$, the multiplicative group of K;
4. $\ker(\exp) = \{z \in K \mid \exp(z) = 1\}$ is an infinite cyclic group.

In Section 4.2, an exponential field satisfying the first three properties is called an ELA-field.

For the following definitions of constants we do not need the property that exp is surjective onto $K^\times$.

Kirby, Macintyre and Onshuus show in [14] that the set of the two possible generators of ker(exp) is first-order definable. Denote this set by $\{\alpha_0, -\alpha_0\}$. Fix a root of the polynomial $X^2 - 1$ in $K[X]$ and denote it by i. The two possible choices of i only differ by their sign and do not affect the properties of the following notions. We further define the sine function as

$$\sin(z) := \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}.$$

Since $\mathrm{e}^{\alpha_0} = \mathrm{e}^{-\alpha_0} = 1$, we can choose

$$\pi \in \left\{\frac{\alpha_0}{2\mathrm{i}}, \frac{-\alpha_0}{2\mathrm{i}}\right\}$$

such that

$$\sin\left(\frac{\pi}{2}\right) = 1.$$

These definitions result in constants π and i and a sine function sin in analogy to $\mathbb{C}_{\exp}$. Hence, many of their properties are similar to the ones of the corresponding notions in $\mathbb{C}_{\exp}$ and will not be pointed out in particular.

When we talk about an exponential polynomial

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z},$$

in this context, we take the coefficients $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ in K. The ring of such exponential polynomials is also denoted by $\mathcal{E}$. Note that its units are exactly the elements of the form $\lambda \mathrm{e}^{\mu z}$ for $\lambda \neq 0$.

Definition 3.1. Let $f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z}$ be an exponential polynomial. We call the vector space over $\mathbb{Q}$ generated by $\mu_1, \ldots, \mu_n$ the *support* of f and denote it by $\mathrm{supp}(f)$.[3]

[3] Since K is of characteristic 0, there is a natural copy of $\mathbb{Q}$ inside K.

Definition 3.2. An exponential polynomial f is called *simple* if $\dim_{\mathbb{Q}} \operatorname{supp}(f) = 1$.

For instance, $\sin(z)$ is simple.

Remark 3.3. Note that $\dim_{\mathbb{Q}} \operatorname{supp}(f) = \operatorname{ldim}_{\mathbb{Q}}(\overline{\mu})$. So if f is simple, then there exists a non-zero $\kappa \in K$ and $s_1, \ldots, s_n \in \mathbb{Z}$ such that

$$\mu_j = s_j \kappa$$

for all j.[4] Hence,

$$f(z) = \lambda_1 \mathrm{e}^{s_1 \kappa z} + \ldots + \lambda_n \mathrm{e}^{s_n \kappa z} = p(\mathrm{e}^{\kappa z})$$

for some polynomial $p(X) \in K[X]$. Since K is algebraically closed, $p(X)$ factorises into finitely many linear factors of the form $(X - \nu_j)$ for some $\nu_j \in K$. Thus $f(z)$ factorises into finitely many factors

$$(\mathrm{e}^{\kappa z} - \nu_j)\,.$$

Up to multiplication with a unit in $\mathcal{E}$, we can now express $f(z)$ as a finite product of factors of the form

$$(1 - \nu_j \mathrm{e}^{-\kappa z})\,.$$

We will quote Ritt's factorisation theorem to obtain the case distinction we are aiming for (see [23] p. 585):

Every function

$$1 + a_1 \mathrm{e}^{\alpha_1 x} + \ldots + a_n \mathrm{e}^{\alpha_n x},$$

distinct from unity, can be expressed in one and only one way as a product

$$(S_1 S_2 \ldots S_s)(I_1 I_2 \ldots I_i),$$

in which $S_1, \ldots, S_s$ are simple functions such that the coefficients of x in any one of them have irrational ratios to the coefficients of x in any other, and in which $I_1, \ldots, I_i$ are irreducible functions.

Ritt makes this statement over the exponential field $\mathbb{C}_{\exp}$, but since he only uses its properties of being algebraically closed and of characteristic 0, as [6] notes, we can use this result for our field K (even without the standard period property of exp). Using the terminology we established earlier and the fact that any function in $\mathcal{E}$ can be transformed into one of the form given in Ritt's factorisation theorem by multiplication with a unit in $\mathcal{E}$, this emerges as the following theorem:

[4] Ritt [23] uses this property to define simple exponential polynomials.

Theorem 3.4. *Let* $f(z) = \lambda_1 e^{\mu_1 z} + \ldots + \lambda_n e^{\mu_n z} \in \mathcal{E}$. *Then* f *can be written uniquely up to order and multiplication with a unit in* $\mathcal{E}$ *as the product in* $\mathcal{E}$

$$S_1 \ldots S_c I_1 \ldots I_d$$

where all S_j *are simple with* $\operatorname{supp}(S_j) \cap \operatorname{supp}(S_{j'}) = \{0\}$ *for* $j \neq j'$ *and all* I_k *are irreducible.*

Suppose that $f, g \in \mathcal{E}$ have infinitely many common zeros and are both not simple. Let

$$f = S_1 \ldots S_c I_1 \ldots I_d$$

and

$$g = T_1 \ldots T_u J_1 \ldots J_v$$

be the unique factorisations of f and g into simple S_j and T_i and irreducible I_k and J_ℓ, given in Theorem 3.4. A common zero of f and g must be a zero of a factor of each function. Hence, two factors $\tilde{f}$ and $\tilde{g}$, say, of f and g respectively, have infinitely many common zeros. If $\tilde{f}$ and $\tilde{g}$ have a common factor h in $\mathcal{E}$ with infinitely many zeros, then h is the common factor of f and g in Shapiro's Conjecture.

We deduce that in order to prove Shapiro's Conjecture, it suffices to show the existence of a common factor h of f and g with infinitely many zeros for the following two cases:

1. At least one of f and g is simple.

2. Both f and g are irreducible.

Case 1: For this case Shapiro's Conjecture has been proved for $\mathbb{C}_{\exp}$ by van der Poorten and Tijdeman [30] without the assumption of Schanuel's Conjecture. We will present an unconditional proof for general exponential fields as specified before.

Our first lemma is a special case of the Skolem–Mahler–Lech Theorem for arbitrary fields of characteristic 0 (Lech [16] p. 417). It is stated by [30], p. 62, for the complex exponential field. However, the proof also works for general exponential fields of characteristic 0, as is pointed out in [6].

Lemma 3.5. *Let* $f \in \mathcal{E}$ *and let* $A \subseteq \mathbb{Z}$ *be the set of integers on which* f *vanishes. Then* A *is the finite union of arithmetic progressions, that is, sets of the form* $\{m + kd \mid k \in \mathbb{Z}\}$ *for some* $m, d \in \mathbb{Z}$. *Moreover, if* A *is infinite, then at least one of these arithmetic progressions has a non-zero difference* d.

We need a second lemma using the notions of sin and π we defined earlier.

Lemma 3.6. *Let* $f \in \mathcal{E}$. *If* f *vanishes at all integers, then* $\sin(\pi z)$ *divides* f *in the ring* $\mathcal{E}$.

Proof. Let $f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z} \in \mathcal{E}$, with $\lambda_1, \ldots, \lambda_n \neq 0$, and suppose that f vanishes at all integers. If f is identically 0, then every exponential polynomial divides f. Otherwise we must have $n \geq 2$, as expressions of the form $\lambda_1 \mathrm{e}^{\mu_1 z}$ only become 0 if $\lambda_1 = 0$.

We will proceed by induction on the length n of f.

For $n = 2$,

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \lambda_2 \mathrm{e}^{\mu_2 z},$$

with $\lambda_1, \lambda_2 \neq 0$. By setting $z = 0$, we obtain

$$\lambda_1 + \lambda_2 = 0$$

and hence

$$f(z) = \lambda_1 \left(\mathrm{e}^{\mu_1 z} - \mathrm{e}^{\mu_2 z} \right).$$

Since $\lambda_1 \neq 0$, setting $z = 1$ gives us

$$\mathrm{e}^{\mu_1} - \mathrm{e}^{\mu_2} = 0.$$

Hence, μ_1 and μ_2 only differ by an integer multiple of the period of exp, i. e. $\mu_2 = \mu_1 + 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$, and

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} \left(1 - \mathrm{e}^{2k\pi\mathrm{i}z} \right).$$

Without loss of generality, we can assume that k is positive, as otherwise we can just switch the roles of μ_1 and μ_2. Recall that

$$\sin(z) := \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}.$$

Hence,

$$-2\mathrm{i}\mathrm{e}^{\pi\mathrm{i}z} \sin(\pi z) = 1 - \mathrm{e}^{2\pi\mathrm{i}z}.$$

Multiplying this by

$$1 + \mathrm{e}^{2\pi\mathrm{i}z} + \mathrm{e}^{4\pi\mathrm{i}z} + \ldots + \mathrm{e}^{2(k-1)\pi\mathrm{i}z}$$

gives us

$$1 - \mathrm{e}^{2k\pi\mathrm{i}z},$$

whence $\sin(\pi z)$ divides $f(z)$.

Now suppose that for all exponential polynomials $h(z)$ of length $n-1$ which vanish at all integers, $\sin(\pi z)$ divides $h(z)$. By setting z equal to $1, \ldots, n$ in $f(z)$, we obtain the system of identities

$$\lambda_1 \mathrm{e}^{\mu_1} + \ldots + \lambda_n \mathrm{e}^{\mu_n} = 0,$$
$$\lambda_1 (\mathrm{e}^{\mu_1})^2 + \ldots + \lambda_n (\mathrm{e}^{\mu_n})^2 = 0,$$

$$\vdots$$
$$\lambda_1(\mathrm{e}^{\mu_1})^n + \ldots + \lambda_n(\mathrm{e}^{\mu_n})^n = 0.$$

Let $\delta_j = \mathrm{e}^{\mu_j}$ for $j = 1, \ldots, n$. In matrix notation this system becomes

$$\begin{pmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \\ \delta_1^2 & \delta_2^2 & \cdots & \delta_n^2 \\ \vdots & \vdots & & \vdots \\ \delta_1^n & \delta_2^n & \cdots & \delta_n^n \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\lambda_1, \ldots, \lambda_n$ are all non-zero, they form a non-trivial solution of the corresponding system of linear equations. Hence, the determinant of the matrix must be zero. By taking out the factor δ_j from each column, we obtain

$$\delta_1\delta_2\ldots\delta_n \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_1 & \delta_2 & \cdots & \delta_n \\ \vdots & \vdots & & \vdots \\ \delta_1^{n-1} & \delta_2^{n-1} & \cdots & \delta_n^{n-1} \end{vmatrix} = 0.$$

This is the Vandermonde determinant, and so

$$\delta_1 \ldots \delta_n \cdot \prod_{1 \leq i < j \leq n} (\delta_i - \delta_j) = 0.$$

Since all δ_j are non-zero, $\delta_i = \delta_j$ for some $i < j$. Without loss of generality, we can assume that $\delta_1 = \delta_2$, i. e. $\mathrm{e}^{\mu_1} = \mathrm{e}^{\mu_2}$. Hence, for each $k \in \mathbb{Z}$,

$$(\lambda_1 + \lambda_2)\mathrm{e}^{\mu_1 k} + \sum_{j=3}^{n} \lambda_j \mathrm{e}^{\mu_j k} = f(k) + \lambda_2 \left(\mathrm{e}^{\mu_1 k} - \mathrm{e}^{\mu_2 k}\right) = 0.$$

The polynomial

$$h(z) = (\lambda_1 + \lambda_2)\mathrm{e}^{\mu_1 z} + \sum_{j=3}^{n} \lambda_j \mathrm{e}^{\mu_j z}$$

has length $n - 1$ and vanishes at all integers. By our inductive hypothesis, $\sin(\pi z)$ divides $h(z)$. Arguing as in the case $n = 2$, we also have that $\sin(\pi z)$ divides $\lambda_2 (\mathrm{e}^{\mu_1 z} - \mathrm{e}^{\mu_2 z})$. Since

$$f(z) = h(z) - \lambda_2 (\mathrm{e}^{\mu_1 z} - \mathrm{e}^{\mu_2 z}),$$

this gives us that $\sin(\pi z)$ divides $f(z)$. **Q. E. D.**

We will finally prove Shapiros Conjecture in the case of at least one simple exponential polynomial. This is where the property that exp maps surjectively onto the multiplicative group of K comes into play. We will fix a *logarithm* function from $K^\times$ to K, denoted by log, which satisfies for all $z \in K$,

$$\exp(\log z) = z$$

and

$$\log(\exp z) = z + 2k\pi \mathrm{i}$$

for some $k \in \mathbb{Z}$. In $\mathbb{C}$ the logarithm function is locally single-valued. Since we work in a general field, we cannot assume a similar property. However, for our purpose an arbitrary choice of values for log from point to point is sufficient.

Theorem 3.7. *Let f and g be two exponential polynomials in $\mathcal{E}$ with infinitely many common zeros such that at least one of f and g is simple. Then there exists an exponential polynomial h in $\mathcal{E}$ such that h is a common divisor of f and g in the ring $\mathcal{E}$, and h has infinitely many zeros in K.*

Proof. Let $f, g \in \mathcal{E}$ with infinitely many common roots, and assume, without loss of generality, that f is simple. By Remark 3.3 we can take $\alpha_1, \ldots, \alpha_L, \rho \in K$, where $\rho \neq 0$, and a unit $u(z) \in \mathcal{E}$ such that

$$f(z) = u(z) \cdot \prod_{j=1}^{L} (1 - \alpha_j \mathrm{e}^{\rho z}) .$$

Since this is a finite product of factors and f and g have infinitely many common zeros, g must have infinitely many common zeros with one of the factors, say $(1 - \alpha_1 \mathrm{e}^{\rho z})$. These zeros must be of the form

$$z = \frac{2k\pi \mathrm{i} + \log\left(\alpha_1^{-1}\right)}{\rho}$$

for $k \in \mathbb{Z}$. Thus, the exponential polynomial

$$g^*(z) = g\left(\frac{2z\pi \mathrm{i} + \log\left(\alpha_1^{-1}\right)}{\rho}\right)$$

vanishes at infinitely many integers. By Lemma 3.5, the set of integers on which g^* vanishes is the finite union of sets of the form $\{m + kd \mid k \in \mathbb{Z}\}$ for some $m, d \in \mathbb{Z}$, at least one of which has $d \neq 0$. Let $\{m + kd \mid k \in \mathbb{Z}\}$ be such an arithmetic progression with $d \neq 0$. Now $g^*(m + zd)$ is an exponential polynomial which vanishes at all integers. By Lemma 3.6, $\sin(\pi z)$ divides $g^*(m + zd)$.

Considering the divisor $(1 - \alpha_1 \mathrm{e}^{\rho z})$ of $f(z)$ again, we note that any z of the form

$$z = \frac{2k\pi \mathrm{i} + \log\left(\alpha_1^{-1}\right)}{\rho},$$

for some $k \in \mathbb{Z}$, is a zero of f. Hence, any integer is a zero of the exponential polynomial

$$f^*(z) = f\left(\frac{2z\pi\mathrm{i} + \log\left(\alpha_1^{-1}\right)}{\rho}\right).$$

In particular, all integers of the form $m+kd$, for some $k \in \mathbb{Z}$, are zeros of f^*. Hence, also $f^*(m+zd)$ is an exponential polynomial which vanishes at all integers and is thus divisible by $\sin(\pi z)$.

Tracing back the change of variable from f and g to f^* and g^*, we obtain that the simple exponential polynomial

$$h(z) = \sin\left(\frac{\pi}{d}\left(\frac{\rho z - \log\left(\alpha_1^{-1}\right)}{2\pi\mathrm{i}} - m\right)\right)$$

is a common divisor of $f(z)$ and $g(z)$ with infinitely many zeros.

Q. E. D.

Case 2: Now we consider the case that both f and g are irreducible. If $f = u \cdot g$ for some unit $u \in \mathcal{E}$, then g is the required common divisor of f and g in Shapiro's Conjecture. In the case that f and g are distinct irreducibles, that is, they are not unit multiples of each other, they cannot have a common divisor. It therefore suffices to prove the following theorem:

Theorem 3.8 (SC)**.** *Let f and g be distinct irreducible exponential polynomials in $\mathcal{E}$. Then f and g have at most finitely many common zeros.*

For the rest of this section, we will assume that f and g are distinct irreducibles with infinitely many common zeros and eventually lead this assumption to a contradiction. We will mostly follow the arguments of D'Aquino, Macintyre and Terzo [6], section 5, which are based on results on group varieties associated to exponential polynomials and in this context the work of Bombieri, Masser and Zannier [3], as well as a result on linear functions on groups of finite rank from Evertse, Schlickewei and Schmidt [7]. We only consider those steps in detail which assume Schanuel's Conjecture. D'Aquino, Macintyre and Terzo prove an equivalent version of Theorem 3.8 assuming Schanuel's Conjecture ([6] p. 606):

Let f and g be exponential polynomials, and assume f is irreducible. If f and g have infinitely many common zeros then f divides g.

We will therefore use a slightly different argument in the last step of our proof.

For $f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z}$ and $g(z) = \rho_1 \mathrm{e}^{\sigma_1 z} + \ldots + \rho_m \mathrm{e}^{\sigma_m z}$, let L denote the linear dimension of the union of the supports of f and g, i. e.

$$L = \mathrm{ldim}_{\mathbb{Q}}\left(\mathrm{supp}(f) \cup \mathrm{supp}(g)\right).$$

Fix a $\mathbb{Z}$-basis $B = \{b_1, \ldots, b_L\}$ of the group generated by $\overline{\mu}, \overline{\sigma}$, that is, the additive subgroup of $(K, +)$ generated by $\mu_1, \ldots, \mu_n, \sigma_1, \ldots, \sigma_m$.

Since $\{b_1, \ldots, b_L\}$ forms a $\mathbb{Z}$-basis of the group generated by $\overline{\mu}, \overline{\sigma}$, each $\mathrm{e}^{\mu_i z}$ and $\mathrm{e}^{\sigma_j z}$ can be expressed as a finite product of integer powers of elements in $\left\{\mathrm{e}^{b_1}, \ldots, \mathrm{e}^{b_L}\right\}$. By introdrucing the variables

$$Y_1 = \mathrm{e}^{b_1 z}, \ldots, Y_L = \mathrm{e}^{b_L z},$$

we can express $f(z)$ and $g(z)$ as the corresponding Laurent polynomials

$$\tilde{F}(Y_1, \ldots, Y_L), \tilde{G}(Y_1, \ldots, Y_L) \in \mathbb{Q}\left(\overline{\lambda}, \overline{\rho}\right)\left[Y_1^{\pm 1}, \ldots, Y_L^{\pm 1}\right],$$

that is, $\tilde{F}(Y_1, \ldots, Y_L)$ and $\tilde{G}(Y_1, \ldots, Y_L)$ are polynomials in L variables over $\mathbb{Q}\left(\overline{\lambda}, \overline{\rho}\right)$ allowing both non-negative and negative powers of the variables. By multiplying $\tilde{F}$ and $\tilde{G}$ by suitable monomials, we obtain regular polynomials

$$F(Y_1, \ldots, Y_L), G(Y_1, \ldots, Y_L) \in \mathbb{Q}\left(\overline{\lambda}, \overline{\rho}\right)[Y_1, \ldots, Y_L]$$

respectively.

Remark 3.9. If s is a common zero of f and g, then $\left(\mathrm{e}^{b_1 s}, \ldots, \mathrm{e}^{b_L s}\right)$ is a common zero of F and G. Since we only consider zeros of F and G whose components are all non-zero, it does not make a difference whether we consider common zeros of F and G or of $\tilde{F}$ and $\tilde{G}$ for the study of common zeros of f and g.

Let

$$V(F) = \left\{(v_1, \ldots, v_L) \in (K^\times)^L \mid F(v_1, \ldots, v_L) = 0\right\}$$

and

$$V(G) = \left\{(v_1, \ldots, v_L) \in (K^\times)^L \mid G(v_1, \ldots, v_L) = 0\right\}.$$

These are algebraic varieties over the L^{th} power of the multiplicative group of K. In the next section, we will explain more details of algebraic varieties, in particular over $\mathbb{C}$.

Remark 3.10. The dimension of an algebraic variety $V \subseteq (K^\times)^L$ over K indicates the maximal number of components of a point in V we can choose arbitrarily, or equivalently the maximal number of algebraically independent components of which a point in V can consist. Let H be the algebraic closure of $\mathbb{Q}\left(\overline{\lambda}, \overline{\rho}\right)$. A factorisation of F in $H[Y_1, \ldots, Y_L]$ would determine a factorisation of f in $\mathcal{E}$. Since f and g are distinct irreducibles in $\mathcal{E}$, the corresponding polynomials F and G are irreducibles in $H[Y_1, \ldots, Y_L]$ which are distinct, in the sense that neither is a monomial times the other. Hence, we obtain

$$\dim V(F) = \dim V(G) = L - 1,$$

and for $V(F,G) = \left\{(v_1,\dots,v_L) \in (K^\times)^L \mid F(v_1,\dots,v_L) = G(v_1,\dots,v_L) = 0\right\}$,

$$\dim V(F,G) \le L-2.$$

Let S be a set of infinitely many common non-zero zeros of f and g. For any $T \subseteq S$ and $s \in S$ we introduce some shorthand notations:

$$\bar{b}s = (b_1 s, \dots, b_L s),$$

$$\mathrm{e}^{\bar{b}s} = (\mathrm{e}^{b_1 s}, \dots, \mathrm{e}^{b_L s}),$$

$$Bs = \{b_1 s, \dots, b_L s\},$$

$$BT = \bigcup_{t \in T} Bt,$$

and

$$\mathrm{e}^{BT} = \{\mathrm{e}^a \mid a \in BT\}.$$

By Remark 3.9,

$$\left\{\mathrm{e}^{\bar{b}s} \mid s \in S\right\}$$

is an infinite set of common zeros of F and G. For finite $T \subset S$, we denote by $D(T)$ the linear dimension of $\mathrm{span}_{\mathbb{Q}}(BT)$, the vector space over $\mathbb{Q}$ spanned by BT,

$$D(T) = \mathrm{ldim}_{\mathbb{Q}}(BT).$$

Note that $D(T) \le |B||T| = L|T|$, and $D(\{s\}) = L$ for any $s \in S$, as $0 \notin S$. Moreover, let $\tau_1 = \mathrm{td}_{\mathbb{Q}}(\overline{\lambda}, \overline{\rho})$ and $\tau_2 = \mathrm{td}_{\mathbb{Q}}(\overline{\mu}, \overline{\sigma})$.

The next two lemmas give us upper bounds on $D(T)$ and rely on the assumption of Schanuel's Conjecture.

Lemma 3.11 (SC)**.** *Let T be a finite subset of S with $D(T) = L|T|$. Then $|T| \le \tau_1 + \tau_2$.*

Proof. Suppose that $T \subset S$ is finite with $D(T) = L|T|$. For any $t \in T$, we obtain a corresponding common zero of F and G,

$$\mathrm{e}^{\bar{b}t} \in V(F,G).$$

By Remark 3.10, $\dim V(F,G) \le L-2$. Since F and G are polynomials with coefficients in $\mathbb{Q}\left(\overline{\lambda}, \overline{\rho}\right)$, we have that

$$\mathrm{td}_{\mathbb{Q}(\overline{\lambda},\overline{\rho})}\left(\mathrm{e}^{\bar{b}t}\right) \le L-2.$$

Hence,

$$\mathrm{td}_{\mathbb{Q}}\left(\mathrm{e}^{BT}\right) \le (L-2)|T| + \tau_1.$$

Moreover, since $BT \subset \mathbb{Q}(\overline{\mu}, \overline{\sigma}, T)$, we also have that

$$\operatorname{td}_{\mathbb{Q}}(BT) \leq |T| + \tau_2.$$

Hence,

$$\operatorname{td}_{\mathbb{Q}}\left(BT, \mathrm{e}^{BT}\right) \leq (L-1)|T| + \tau_1 + \tau_2.$$

Schanuel's Conjecture, Version 2, implies

$$\operatorname{td}_{\mathbb{Q}}\left(BT, \mathrm{e}^{BT}\right) \geq D(T).$$

Using $D(T) = L|T|$, this gives us the estimate

$$L|T| \leq (L-1)|T| + \tau_1 + \tau_2,$$

whence

$$|T| \leq \tau_1 + \tau_2.$$

Q. E. D.

Let k_0 be the maximal cardinality of a finite subset T of S such that $D(T) = L|T|$ holds, and fix $T_0 \subset S$ of cardinality k_0 with $D(T) = Lk_0$. The following lemma gives us an upper bound on $D(T)$ for finite extensions of T_0.

Lemma 3.12 (SC)**.** *Let $T \subset S$ be a finite extension of T_0 by k elements. Then*

$$D(T_0) \leq D(T) \leq \tau_1 + \tau_2 + k(L-1).$$

Proof. Suppose that $T_0 \subseteq T \subset S$ such that $|T| - |T_0| = k$. Clearly $D(T_0) \leq D(T)$. Assume that there existed $t \in T \setminus T_0$ such that for all $1 \leq j \leq L$,

$$b_j t \notin \operatorname{span}_{\mathbb{Q}}(BT_0 \cup \{b_1 t, \ldots, b_{j-1} t\}).$$

Then

$$\begin{aligned} D(T_0 \cup \{t\}) &= \operatorname{ldim}_{\mathbb{Q}}(BT_0 \cup Bt) \\ &= \operatorname{ldim}_{\mathbb{Q}}(BT_0) + \operatorname{ldim}_{\mathbb{Q}}(Bt) \\ &= D(T_0) + D(\{t\}) \\ &= L(k_0 + 1). \end{aligned}$$

This contradicts the maximality of k_0 with the property that $D(T_0) = |T_0|$. Hence, for any of the k possible elements $t \in T \setminus T_0$, there exists at least one element $b_j \in B$ such that $b_j t \in \operatorname{span}_{\mathbb{Q}}(BT_0 \cup \{b_1 t, \ldots, b_{j-1} t\})$. Using the result $|T_0| \leq \tau_1 + \tau_2$ from the previous lemma, this gives us the upper bound on $D(T)$

$$D(T) \leq \tau_1 + \tau_2 + k(L-1).$$

Q. E. D.

Using arguments similar to the ones used in the proofs of the previous two lemmas, we can find an upper bound on the transcendence degree of BS over $\mathbb{Q}$.

Lemma 3.13. *The transcendence degree of BS over $\mathbb{Q}$ is at most $\tau_1 + 2\tau_2$.*

Proof. Fix any $s \in S \setminus T_0$. Arguing as in the proof of Lemma 3.12, we obtain that there exists at least one element $b_j \in B$ such that $b_j s \in \mathrm{span}_{\mathbb{Q}}(BT_0 \cup \{b_1 s, \dots, b_{j-1} s\})$. Hence, there exist $\alpha_1, \dots, \alpha_{j-1} \in \mathbb{Q}$ such that

$$(\alpha_1 b_1 + \ldots + \alpha_{j-1} b_{j-1} + b_j) s \in \mathrm{span}_{\mathbb{Q}}(BT_0).$$

Since $b_1, \dots, b_j$ are $\mathbb{Q}$-linearly independent, we have that $(\alpha_1 b_1 + \ldots + \alpha_{j-1} b_{j-1} + b_j) \neq 0$, whence

$$s = (\alpha_1 b_1 + \ldots + \alpha_{j-1} b_{j-1} + b_j)^{-1} \cdot a,$$

for some $a \in \mathrm{span}_{\mathbb{Q}}(BT_0)$. We obtain that $s \in \mathbb{Q}(BT_0 \cup B)$. Hence, $S \setminus T_0 \subset \mathbb{Q}(BT_0 \cup B)$. Clearly, also $T_0 \subset \mathbb{Q}(BT_0 \cup B)$, and thus,

$$\mathbb{Q}(BS) \subseteq \mathbb{Q}(BT_0 \cup B).$$

Note that $B \subset \mathbb{Q}(\overline{\mu}, \overline{\sigma})$, and that $|T_0| \leq \tau_1 + \tau_2$, by Lemma 3.11. Hence,

$$\begin{aligned} \mathrm{td}_{\mathbb{Q}}(BS) &\leq \mathrm{td}_{\mathbb{Q}}(BT_0 \cup B) \\ &\leq \mathrm{td}_{\mathbb{Q}}(B) + \mathrm{td}_{\mathbb{Q}(B)}(BT_0) \\ &\leq \mathrm{td}_{\mathbb{Q}}(\overline{\mu}, \overline{\sigma}) + |T_0| \\ &\leq \tau_2 + \tau_1 + \tau_2 \\ &= \tau_1 + 2\tau_2. \end{aligned}$$

Q. E. D.

In order to proceed to our main lemma, we need to use a main result from Bombieri, Masser and Zannier [3] on anomalous subvarieties.

Definition 3.14 (See [3] p. 3.)**.** Let X be an irreducible subvariety of $(K^\times)^n$. An irreducible subvariety Y of X is X-*anomalous* if it is contained in a coset of an algebraic subgroup Γ of $(K^\times)^n$ satisfying

$$\dim Y > \max\{0, \dim X - \mathrm{codim}\,\Gamma\},$$

where $\mathrm{codim}\,\Gamma = n - \dim \Gamma$. Moreover, if X is not contained in any strictly larger anomalous subvariety of V, then it is *maximal.*

The following result is proved in [3] over the field $\mathbb{C}$. However, as [6] notes, it also holds for any algebraically closed field of characteristic 0.

Theorem 3.15 (See [3] Theorem 1.4 and [6] Theorem 4.4)**.** *Let X be an irreducible variety in $(K^{\times})^n$ of positive dimension defined over K. Then there exists a finite collection Φ_X of proper algebraic tori A such that for all $A \in \Phi_X$,*

$$1 \leq n - \dim A \leq X.$$

Moreover, every maximal X-anomalous subvariety Y of X is a component of $V \cap A\theta$, the intersection of V with a coset $A\theta$, for some $A \in \Phi_X$ and $\theta \in (K^{\times})^n$.

The following lemma is the key step in the proof of Shapiro's Conjecture. We will show the existence of an infinite set S' of common zeros of f and g whose $\mathbb{Q}$-linear dimension is finite. This will eventually lead to a contradiction. We will not give as much detail as in other proofs in this paper, but we will highlight the main steps and mention in particular when Schanuel's Conjecture and the lemmas based on Schanuel's Conjecture are applied. A detailed proof can be found in [6], p. 608 ff.

Lemma 3.16 (SC)**.** *There exists a finite subset S' of S such that the $\mathbb{Q}$-vector space spanned by S' is finite dimensional.*

Proof. Consider an irreducible component C of the variety $V(F, G)$ defined over H, the algebraic closure of $\mathbb{Q}(\overline{\lambda}, \overline{\rho})$, such that for infinitely many $s \in S$, the point $\left(e^{b_1 s}, \dots, e^{b_L s}\right)$ is contained in C. We now only consider the infinite subset

$$\left\{ s \in S \mid \left(e^{\bar{b}s}\right) \in C \right\}$$

of S and continue to call it S. By Remark 3.10, the dimension of C over H is at most $L - 2$.

Let $\overline{s} = (s_1, \dots, s_k)$ be a tuple of finitely many distinct elements of S, and let T be the set of entries of $\overline{s}$. Since the $\mathbb{Q}$-linear relations amongst $\bar{b}s_1, \dots, \bar{b}s_k$ can be transformed into $\mathbb{Z}$-linear ones, they induce multiplicative relations between $e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}$. This determines an algebraic subgroup Γ_k of $(K^{\times})^{Lk}$, generated by $e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}$, with codimension $Lk - D(T)$ and dimension $D(T)$ over $\mathbb{Q}$. The (Lk)-tuple $\left(e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}\right)$ lies in C^k. We have the upper bound

$$\operatorname{td}_{\mathbb{Q}}\left(e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}\right) \leq k(L-2) + \tau_1.$$

Since $\left(e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}\right)$ also lies in Γ_k, we will work with the irreducible component $W_{\overline{s}} \subseteq C^k \cap \Gamma_k$ containing the point $\left(e^{\bar{b}s_1}, \dots, e^{\bar{b}s_k}\right)$ over H.

Claim 1. For $k > \tau_1 + \tau_2$, either $W_{\overline{s}}$ is anomalous or of dimension 0 over H.

We can use the upper bounds established earlier to prove this claim. If $W_{\overline{s}}$ is not anomalous, then

$$\dim(W_{\overline{s}}) \leq \dim\left(C^{Lk}\right) - \operatorname{codim}\left(\Gamma_k\right)$$

$$\leq k(L-2)+\tau_1-(Lk-D(T)).$$

By Schanuel's Conjecture and Lemma 3.13,

$$D(T)\leq \dim(W_{\overline{s}})+\tau_1+2\tau_2.$$

Hence,

$$k\leq \tau_1+\tau_2,$$

proving the claim.

Suppose that $\overline{s^*}$ is obtained from $\overline{s}$ by rearranging the entries. Using automorphisms of affine (Lk)-spaces, one can show that $W_{\overline{s}}$ has dimension 0 over L if and only if $W_{\overline{s^*}}$ has dimension 0, and $W_{\overline{s}}$ is anomalous if and only if $W_{\overline{s^*}}$ is anomalous.

Claim 2. If $\dim W_{\overline{s}}=0$, then $D(T)\leq 2\tau_1+2\tau_2$.

If $\dim W_{\overline{s}}=0$, then the components of all points in $\dim W_{\overline{s}}$ are algebraic over H. Hence,

$$\mathrm{td}_{\mathbb{Q}}\left(\mathrm{e}^{\overline{b}s_1},\ldots,\mathrm{e}^{\overline{b}s_k}\right)\leq \tau_1.$$

By Lemma 3.13,

$$\mathrm{td}_{\mathbb{Q}}\left(\overline{b}s_1,\ldots,\overline{b}s_k\right)\leq \tau_1+2\tau_2.$$

Hence, by Schanuel's Conjecture,

$$D(T)\leq 2\tau_1+2\tau_2.$$

As a result, if $D(T)>2\tau_1+2\tau_2$, then $W_{\overline{s^*}}$ is anomalous for any arrangement $\overline{s^*}$ of the elements in T.

Now we work with a countably infinite subset $\{s_1,s_2\ldots\}$ of S and continue to call it S. This enumeration naturally induces an order $(<)$ on S. Let $T_k=\{s_1,\ldots,s_k\}$, and $W_k=W_{\overline{s}}$ for some tuple $\overline{s}$ enumerating T_k.

If infinitely many W_k are of dimension 0, then by Claim 2, for infinitely many k,

$$D(T_k)\leq 2(\tau_1+\tau_2).$$

Hence, $\mathrm{ldim}_{\mathbb{Q}}(S)$ is finite, as required.

Consider the case that there exists k' such that for all $k\geq k'$, the variety W_k has dimension 0 and is therefore anomalous, by Claim 1. The remaining steps of this proof include results from [3] as well as one instance of Ramsey's Theorem on colourings of sufficiently large complete graphs.

Let k_1 be the least integer with the property that for any (k_1+1)-tuple $\overline{s}$ in S, the corresponding variety $W_{\overline{s}}$ is C^{k_1+1}-anomalous. Fix such a variety W. By

Theorem 3.15, or more generally the Structure Theorem from [3], there exist proper algebraic tori $A_1, \ldots, A_r$ of $(K^\times)^{(k_1+1)L}$ such that each maximal anomalous subvariety of C^{k_1+1} is a component of $C^{k_1+1} \cap A_j\theta$ for some $1 \le j \le r$ and $\theta \in (K^\times)^{(k_1+1)L}$. For each $1 \le j \le r$, pick a single relation

$$(x_1)^{\alpha_{j,1}} \cdot \ldots \cdot \left(x_{(k_1+1)L}\right)^{\alpha_{j,(k_1+1)L}} = 1$$

which is a condition of the torus A_j, and let Q_j be the multiplicative subgroup of $(K^\times)^{(k_1+1)L}$ of codimension 1 defined by this relation. Now every anomalous subvariety of $(K^\times)^{(k_1+1)L}$ is contained in a coset of one of $Q_1, \ldots, Q_r$. This coset can be defined over H. Hence, there exists $\theta_W \in H$ such that for all $\overline{w} \in W$,

$$\overline{w}^{\overline{\alpha_j}} = \prod_{i=1}^{(k_1+1)L} w_i^{\alpha_{j,i}} = \theta_W,$$

for some $1 \le j \le (k_1+1)L$.

We define a colouring on $[S]^{k_1+1}$, the set of all subsets of S of cardinality k_1+1, as follows:

$$\varphi : [S]^{k_1+1} \to \{\overline{\alpha_1}, \ldots, \overline{\alpha_r}\},$$

where a set in $[S]^{k_1+1}$ with corresponding tuple $\overline{s}$ is mapped to $\overline{\alpha_j}$ for the minimum j such that $W_{\overline{s}}$ is contained in a coset over H of Q_j.

By Ramsey's Theorem, there exist an infinite set $R \subseteq S$ and a fixed j_0 such that φ takes the constant value $\overline{\alpha_{j_0}}$ on $[R]^{k_1+1}$. Rewrite $\overline{\alpha_{j_0}} \in \mathbb{Z}^{(k_1+1)L}$ as the concatenation of the tuples $\overline{\beta} \in \mathbb{Z}^{k_1 L}$ and $\overline{\gamma} \in \mathbb{Z}^L$, i. e.

$$\overline{\alpha_{j_0}} = \overline{\beta}\overline{\gamma}.$$

Suppose that $\overline{\gamma} \neq \overline{0}$. Let $T = \{\eta_1, \ldots, \eta_{k_1}\} \subset R$ with ordering $\eta_1 < \ldots < \eta_{k_1}$ inherited from S. For each $s \in R \setminus T$ such that $\eta_{k_1} < s$, the set

$$T \cup \{s\}$$

lies in $[R]^{k_1+1}$. Hence, there exists $\theta_s \in H$ such that

$$\left(\mathrm{e}^{\bar{b}\eta_1} \ldots \mathrm{e}^{\bar{b}\eta_{k_1}}\right)^{\overline{\beta}} \left(\mathrm{e}^{\bar{b}s}\right)^{\overline{\gamma}} = \theta_s.$$

Let $E = \left(\mathrm{e}^{\bar{b}\eta_1} \ldots \mathrm{e}^{\bar{b}\eta_{k_1}}\right)^{\overline{\beta}}$. Then

$$\left(\mathrm{e}^{\bar{b}s}\right)^{\overline{\gamma}} = \frac{\theta_s}{E} \in H(E).$$

Hence,

$$\mathrm{td}_{\mathbb{Q}}\left(\left\{\mathrm{e}^{(\bar{b}\cdot\overline{\gamma})s} \mid s \in T \setminus R\right\}\right) \le \mathrm{td}_{\mathbb{Q}}(H(E)).$$

Since $\mathrm{td}_{\mathbb{Q}}(H(E))$ is finite, Schanuel's Conjecture implies that

$$\mathrm{ldim}_{\mathbb{Q}}\left(\left\{\left(\overline{b}\cdot\overline{\gamma}\right)s \mid s \in T \setminus R\right\}\right)$$

is finite. Since $\overline{\gamma} \neq \overline{0}$ and $\overline{b}$ is linearly independent, the inner product $\overline{b}\cdot\overline{\gamma}$ is non-zero. Hence, $S' = T \setminus R$ is an infinite subset of S such that the $\mathbb{Q}$-vector space spanned by S' is finite dimensional.

In the case that $\overline{\gamma} = \overline{0}$, we can shift to the next block of length L in $\overline{\beta}$ which is non-zero, then complete the tuple and argue similarly.

Q. E. D.

Now we can assume that S, the set of infinitely common non-zero zeros of f and g, spans a finite dimensional vector space over $\mathbb{Q}$. Let Γ be the divisible hull of the multiplicative group generated by

$$\{\mathrm{e}^{\mu_j s} \mid 1 \leq j \leq n, s \in S\},$$

that is, for every $\gamma \in \Gamma$ and any non-zero integer ℓ, there exists $\zeta \in \Gamma$ such that $\zeta^\ell = \gamma$, and Γ is the smallest such group containing $\{\mathrm{e}^{\mu_j s} \mid 1 \leq j \leq n, s \in S\}$. Note that all multiplicative dependencies between the elements of Γ correspond to additive dependencies of elements in

$$\mathrm{span}_{\mathbb{Q}}(\{\mu_j s \mid 1 \leq j \leq n, s \in S\}) = \mathrm{span}_{\mathbb{Q}}(\mu_1 S \cup \ldots \cup \mu_n S).$$

Since $\mathrm{span}_{\mathbb{Q}}(S)$ is finite dimensional, Γ has finite rank. We will use this fact to apply a result from Evertse, Schlickewei and Schmidt [7] on linear functions on finite rank groups.

Definition 3.17. A solution $(v_1, \ldots, v_N)$ of the linear equation

$$a_1x_1 + \ldots + a_Nx_N = 1$$

over K is *non-degenerate* if for every proper non-empty subset J of $\{1, \ldots, N\}$,

$$\sum_{j\in J} a_j v_j \neq 0.$$

Theorem 3.18 (See [7] Theorem 1.1)**.** *Let N be a positive integer, and let Λ be a subgroup of $(K^\times)^N$ with finite rank r. Then any linear equation*

$$a_1x_1 + \ldots + a_Nx_N = 1 \tag{3.1}$$

over K with $a_1, \ldots, a_N \neq 0$ has at most

$$\exp\left((6N)^{3N}(r+1)\right)$$

many non-degenerate solutions in Λ, where exp *denotes the standard exponential function on $\mathbb{R}$.*

Later we will only use the fact that there exists a finite upper bound on the number of non-degenerate solutions in Λ. In order to apply this result, we need to find an equation of the form (3.1) corresponding to $f(z) = 0$.

Let $q = \operatorname{ldim}_{\mathbb{Q}}(S)$ and fix a $\mathbb{Q}$-basis $\{s_1, \ldots, s_q\}$ of $\operatorname{span}_{\mathbb{Q}}(S)$. Let $s \in S$. Then there exist $c_1, \ldots, c_q \in \mathbb{Q}$ such that

$$s = \sum_{i=1}^{q} c_i s_i.$$

Hence,

$$0 = f(s) = \lambda_1 \prod_{i=1}^{q} \mathrm{e}^{\mu_1 c_i s_i} + \ldots + \lambda_n \prod_{i=1}^{q} \mathrm{e}^{\mu_n c_i s_i},$$

and

$$\overline{\omega_s} = \left(\prod_{i=1}^{q} \mathrm{e}^{\mu_1 c_i s_i}, \ldots, \prod_{i=1}^{q} \mathrm{e}^{\mu_n c_i s_i} \right) \in \Gamma$$

is a solution of the equation

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0. \tag{3.2}$$

Since $\lambda_n \neq 0$, we can set

$$\lambda'_j = \left(-\lambda_n \prod_{i=1}^{q} \mathrm{e}^{\mu_n c_i s_i} \right)^{-1} \lambda_j$$

for $1 \leq j \leq n-1$. Then

$$\lambda'_1 \prod_{i=1}^{q} \mathrm{e}^{\mu_1 c_i s_i} + \ldots + \lambda'_{n-1} \prod_{i=1}^{q} \mathrm{e}^{\mu_{n-1} c_i s_i} = 1$$

and so

$$\overline{\omega_s^*} = \left(\prod_{i=1}^{q} \mathrm{e}^{\mu_1 c_i s_i}, \ldots, \prod_{i=1}^{q} \mathrm{e}^{\mu_{n-1} c_i s_i} \right)$$

is a solution of the equation

$$\lambda'_1 y_1 + \ldots + \lambda'_{n-1} y_{n-1} = 1, \tag{3.3}$$

which is of the required form in (3.1). Note that all solutions of (3.3) lie in some group Γ^*, a subgroup of Γ of finite rank. We can therefore apply Theorem 3.18 to obtain that there are only finitely many non-degenerate solutions of (3.3) in Γ^*.

For the proof of Theorem 3.8, we will state a last lemma which does not depend on Schanuel's Conjecture.

Lemma 3.19 (See [6] Lemma 5.6)**.** *Let $h \in \mathcal{E}$ be an exponential polynomial which is not simple, and let s_1 and s_2 be two distinct non-zero zeros of h. Suppose that*

$$\alpha_1 x_1 + \ldots + \alpha_N x_N = 0$$

is the equation of the form (3.2) *corresponding to h. Then the solutions corresponding to s_1 and s_2 are different.*

Proof of Theorem 3.8. Let $f, g \in \mathcal{E}$ be distinct irreducibles, and assume that they have infinitely many common zeros in K. These are the same assumptions that we made throughout this section. We will therefore be able to apply all results and to use the same notation.

We will show by induction on the length of f that g divides f. Since f and g are distinct irreducibles, this will lead to the required contradiction.

Suppose that $f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \lambda_2 \mathrm{e}^{\mu_2 z}$. Then

$$f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} \left(1 + \lambda_1^{-1} \lambda_2 \mathrm{e}^{(\mu_2 - \mu_1) z}\right),$$

and $g(z)$ has infinitely many common zeros with $\left(1 + \lambda_1^{-1} \lambda_2 \mathrm{e}^{(\mu_2 - \mu_1) z}\right)$. Arguing as in the proof of Theorem 3.7, we obtain an exponential polynomial of the form $\sin(T(z))$ dividing both $f(z)$ and $g(z)$. Since g is irreducible, this implies that g divides f.

Now suppose that for every exponential polynomial h distinct from g and of length strictly less than n, if h and g have infinitely many common zeros, then g divides h.

Let $n > 2$, and let

$$\lambda'_1 y_1 + \ldots + \lambda'_{n-1} y_{n-1} = 1 \tag{3.4}$$

be the linear equation associated to $f(z) = \lambda_1 \mathrm{e}^{\mu_1 z} + \ldots + \lambda_n \mathrm{e}^{\mu_n z} = 0$ as in (3.3). As we noted earlier, we can apply Theorem 3.18 to show that Γ^* only contains finitely many non-degenerate solutions

$$\overline{\omega_s^*} = \left(\omega_1^{(s)}, \ldots, \omega_{n-1}^{(s)}\right)$$

of this linear equation. Consider the equation as in (3.2) associated to $f(z) = 0$,

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0. \tag{3.5}$$

By Lemma 3.19, this has infinitely many distinct solutions

$$\overline{\omega_s} = \left(\omega_1^{(s)}, \ldots, \omega_n^{(s)}\right) \in \Gamma,$$

each of which corresponds to some $s \in S$. Without loss of generality, we can assume that there are infinitely many solutions $\overline{\omega_s}$ which differ in the first component, as otherwise we can just relabel the equations and the components of the solution. Each solution $\overline{\omega_s}$ of (3.5) can be turned into a solution $\overline{\omega_s^*}$ of (3.4) by removing the last component of $\overline{\omega_s}$. Hence, there are infinitely many distinct solutions $\overline{\omega_s^*}$ of (3.4), each of which is determined by some $s \in S$. By Theorem 3.18, all but finitely many

$\overline{\omega_s^*}$ are degenerate. Hence, for infinitely many solutions $\overline{\omega_s} = \left(\omega_1^{(s)}, \ldots, \omega_n^{(s)}\right)$ of (3.5) there exists a proper non-empty $J_s \subset \{1, \ldots, n\}$ such that

$$\sum_{j \in J_s} \lambda_j \omega_j^{(s)} = 0.$$

Since $\{1, \ldots, n\}$ only has a finite number of distinct subsets, and there are infinitely many subsets J_s, there exists a proper non-empty

$$J' = \{j_1, \ldots, j_t\} \subset \{1, \ldots, n\}$$

such that for infinitely many $s \in S$ we have

$$\sum_{j \in J'} \lambda_j \omega_j^{(s)} = 0.$$

Hence, the linear equation

$$\lambda_{j_1} x_{j_1} + \ldots + \lambda_{j_t} x_{j_t} = 0$$

has infinitely many solutions corresponding to common zeros of $f(z)$ and $g(z)$.

Let

$$f_1(z) = \lambda_{j_1} \mathrm{e}^{\mu_{j_1} z} + \ldots + \lambda_{j_t} \mathrm{e}^{\mu_{j_t} z},$$

and

$$f_2(z) = f(z) - f_1(z).$$

It follows that the exponential polynomial $g(z)$ has infinitely many common zeros with $f_1(z)$ which are also zeros of $f(z)$ and thus also zeros of $f_2(z)$.

Since J is a proper non-empty subset of $\{1, \ldots, n\}$, both f_1 and f_2 are exponential polynomials of length strictly less than n. By Ritt's factorisation theorem (Theorem 3.4), g has infinitely many common zeros with either an irreducible or a simple factor of f_1 in $\mathcal{E}$. Call this factor $h_1(z)$. If h_1 is simple, we are in Case 1 of Shapiro's Conjecture, and g and h_1 must have a common divisor. Since g is irreducible, it then divides h_1. If h_1 is irreducible, then it is either a unit multiple of g, in which case g divides h_1, or g and h_1 are distinct irreducibles, in which case g divides h_1, by our inductive hypothesis. Hence, in all cases g divides h_1 and thus it also divides f_1. We can argue similarly to show that g divides f_2.

Hence, g divides f. This completes the induction and thus the proof.

Q. E. D.

3.2 Generic solutions of polynomial exponential equations

Generic points played an important role in the advancement of Algebraic Geometry in the second third of the last century, e. g. in Weil's foundational approach to

Algebraic Geometry [32]. They have some useful properties in the context of algebraically closed fields, such as the complex numbers $\mathbb{C}$, which we are dealing with. We will only give a brief outline of the background in Algebraic Geometry which we will use. Further properties of generic points are assumed to be known to the reader.

A *variety* V in $\mathbb{C}^2$ is a set of common zeros of a collection of polynomial equations in $\mathbb{C}[X, Y]$. It is a well-known fact that in $\mathbb{C}^2$ any variety can be expressed as the set of zeros of a single polynomial. Thus, for every variety V,

$$V = \{(X, Y) \in \mathbb{C} \times \mathbb{C} \mid p(X, Y) = 0\}$$

for some $p(X, Y) \in \mathbb{C}[X, Y]$. The variety V is then also called a *curve*. It is *irreducible* if it cannot be expressed as the union of two proper subvarieties. In $\mathbb{C}$ this is the case when $p(X, Y)$ is an irreducible polynomial.

In general fields F, a point of a variety V defined over F is generic over F if it does not lie in any proper subvariety of V defined over F, or equivalently if every polynomial which is satisfied by the point is satisfied by the whole variety (see Marker [20] p. 227). Intuitively, generic points have properties similar to variables. For instance, if an algebraic function maps a generic point of a curve C_1 to a generic point of a curve C_2, then it maps the whole curve C_1 onto C_2. In our context, Definition 2.9 suffices to identify generic points of the curves we consider.

Recall Theorem 2.8:

Theorem 2.8 (SC)**.** *Suppose that $p(X, Y) \in \overline{\mathbb{Q}}[X, Y]$ is irreducible and depends on both X and Y. Then there are infinitely many algebraically independent zeros of $f(z) = p(z, \mathrm{e}^z)$.*

The proof of this theorem consists of two parts: Firstly we will show in a slightly more general case that f has infinitely many zeros, and secondly we establish algebraic independence of the zeros in our specific case.

Our first lemma is a direct application of Hadamard's Factorisation Theorem of entire functions (see e. g. [4] §3).

Lemma 3.20. *Let $p(X, Y) \in \mathbb{C}[X, Y] \setminus \mathbb{C}[X]$ and $f(z) = (z, \mathrm{e}^z)$. Suppose that f only has finitely many zeros. Then there exist a constant $a \in \mathbb{C}$ and $q(X) \in \mathbb{C}[X]$ such that*

$$f(z) = \mathrm{e}^{az} q(z).$$

Since Hadamard's Factorisation Theorem is only applicable to entire functions, by using this result we restrict all further conclusions to the setting $\mathbb{C}_{\exp}$.

The next theorem relates the expressions in the *ring of exponential terms*

$$\mathbb{C}[X_1, \dots, X_n]^{\exp}$$

with the corresponding functions from $\mathbb{C}^n$ to $\mathbb{C}$. Following Henson and Rubel [9] (Definition 5.1) we can introduce $\mathbb{C}[X_1, \dots, X_n]^{\exp}$ formally as the collection of the following terms:

- every variable $X_1, \dots, X_n$ is a term;
- every complex number is a constant term;
- if s and t are terms, so are their sum $(s + t)$, their product $(s \cdot t)$ and exponentiation $\exp(s)$.

The identities stating when two terms are equal in $\mathbb{C}[X_1, \dots, X_n]^{\exp}$ are the ones making $(\mathbb{C}[X_1, \dots, X_n]^{\exp}, +, \cdot)$ a commutative ring with identity, all rules relating the constant terms in $\mathbb{C}$ and the identity $\exp(s + t) = (\exp(s) \cdot \exp(t))$ (for a full description see [9] Theorem 5.2). To every term in $\mathbb{C}[X_1, \dots, X_n]^{\exp}$ we can assign a holomorphic function on $\mathbb{C}^n$ in the trivial way, i. e. by interpreting the variable terms as variables of the function, the constant terms as the corresponding complex numbers and $+$, $\cdot$ and exp as the corresponding operations in $\mathbb{C}_{\exp}$.

Theorem 3.21. *The natural homomorphism from $\mathbb{C}[X_1, \dots, X_n]^{\exp}$ to the ring of holomorphic functions from $\mathbb{C}^n$ to $\mathbb{C}$ is injective.*

This theorem was not only proved by [9] but independently by van den Dries in [29], Corollary 4.2.

The existence of infinitely zeros of f is a corollary of this theorem for the case $n = 1$.

Corollary 3.22. *Let $p(X, Y) \in \mathbb{C}[X, Y]$ be an irreducible polynomial which depends on X and Y. Then $f(z) = p(z, \mathrm{e}^z)$ has infinitely many zeros.*

Proof. Assume that $p(X, Y) \in \mathbb{C}[X, Y]$ is an irreducible polynomial depending on X and Y such that $f(z) = p(z, \mathrm{e}^z)$ only has finitely many zeros. By Lemma 3.20 there exist $a \in \mathbb{C}$ and $q(X) \in \mathbb{C}[X]$ such that

$$f(z) - \mathrm{e}^{az} q(z) = 0.$$

$f(z) - \mathrm{e}^{az} q(z)$ corresponds to a term in the ring of exponential terms $\mathbb{C}[X]^{\exp}$. By Theorem 3.21 this term must be the zero term in $\mathbb{C}[X]^{\exp}$. This forces a to be a non-negative integer. We obtain

$$p(X, Y) = Y^a q(X).$$

Since p depends on both X and Y, neither Y^a nor $q(X)$ are equal to 1. Hence, $p(X,Y)$ is reducible, which contradicts our assumption of $p(X,Y)$ being irreducible.

Q. E. D.

Schanuel's Conjecture first comes into play when we go one step further, that is, when we show the existence of algebraically independent zeros. For the remaining parts of the proof of Theorem 2.8 we restrict $p(X,Y)$ to lie in $\overline{\mathbb{Q}}[X,Y]$, the set of polynomials in two variables over the algebraic closure of $\mathbb{Q}$, rather than $\mathbb{C}[X,Y]$. Moreover, $p(X,Y)$ is assumed to be irreducible and dependent on both X and Y. We fix $f(z) = p(z, \mathrm{e}^z)$.

First we consider a single non-zero zero v of f and show its transcendence over $\mathbb{Q}$ by using the Lindemann-Weierstrass Theorem, an important result in Transcendence Theory.

Theorem 3.23 (Lindemann-Weierstrass Theorem)**.** *If $\alpha_1, \dots, \alpha_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $\mathrm{e}^{\alpha_1}, \dots, \mathrm{e}^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

Proposition 3.24. *If $f(v) = 0$ and $v \neq 0$, then v is transcendental over $\mathbb{Q}$ and generic, i. e. $\mathrm{td}_{\mathbb{Q}}(v, \mathrm{e}^v) = 1$.*

Proof. Suppose that $f(v) = 0$ and $v \neq 0$. Assume that v were algebraic over $\mathbb{Q}$. Then $p(v,Y) \in \overline{\mathbb{Q}}[Y]$. Since $p(v, \mathrm{e}^v) = 0$, also e^v is algebraic over $\mathbb{Q}$. This contradicts the Lindemann-Weierstrass Theorem. Hence, v is transcendental over $\mathbb{Q}$, yielding $\mathrm{td}_{\mathbb{Q}}(v) = 1$. Now $p(v,Y) \in \overline{\mathbb{Q}}(v)[Y]$, whence e^v is algebraic over $\mathbb{Q}(v)$, and we obtain $\mathrm{td}_{\mathbb{Q}}(v, \mathrm{e}^v) = 1$. **Q. E. D.**

Next we want to establish algebraic independence of a pair of zeros of f. When we consider two distinct zeros v, w of f, we need to exclude the case $w = -v$, as for example

$$p(X,Y) = 1 + X^2 Y + Y^2$$

would give us $p(-v, \mathrm{e}^{-v}) = 0$ whenever $p(v, \mathrm{e}^v) = 0$.

Theorem 3.25 (SC)**.** *Suppose that v and w are non-zero such that $f(v) = f(w) = 0$ and $v \neq \pm w$, then v and w are algebraically independent.*

Proof. Since $v, w \neq 0$, they are transcendental and generic over $\mathbb{Q}$ by Proposition 3.24.

Assume that v and w were algebraically dependent. Then

$$\mathrm{td}_{\mathbb{Q}}(v, w, \mathrm{e}^v, \mathrm{e}^w) = \mathrm{td}_{\mathbb{Q}}(v, w) = \mathrm{td}_{\mathbb{Q}}(v) = 1.$$

By Schanuel's Conjecture, v and w are $\mathbb{Q}$-linearly dependent. So there exist coprime integers m, n such that

$$mv = nw.$$

Without loss of generality we can assume that $n > 0$.

Let $u = \frac{v}{n}$. We obtain

$$\mathrm{e}^v = (\mathrm{e}^u)^n \text{ and } \mathrm{e}^w = (\mathrm{e}^u)^m.$$

For every positive integer j let $C_j \subset \mathbb{C} \times \mathbb{C}^\times$ be the curve given by $p(jX, Y^j) = 0$. Since

$$p(mu, (\mathrm{e}^u)^m) = p(w, \mathrm{e}^w) = 0$$

and

$$p(nu, (\mathrm{e}^u)^n) = p(v, \mathrm{e}^v) = 0,$$

(u, e^u) lies in $C_m \cap C_n$. Since the varieties C_m and C_n have non-empty intersection, they have a common irreducible component. Thus $p(nX, Y^n)$ and $p(mX, Y^m)$ have a common irreducible factor.

[21] uses an argument from algebraic geometry on generic points to show that the n^{th} roots of unity act transitively on the irreducible components of C_n in the following sense:

If we factorise $p(nX, Y^n)$ into irreducibles

$$p(nX, Y^n) = \prod_{j=1}^{\ell} q_j(X, Y)^{s_j},$$

then each $q_j(X, Y)$ is of the form $q_1(X, \omega Y)$ for some n^{th} root of unity ω, and $s_1 = \ldots = s_\ell = s$, say. Note for a later proof using a similar argument that each irreducible factor depends on both X and Y.

We obtain

$$\deg_X p = \ell s \deg_X q_1,$$

where $\deg_X$ describes the degree of $p(X, Y)$ considered as a polynomial in $\mathbb{C}[Y][X]$, and

$$n \deg_Y p = \ell s \deg_Y q_1.$$

In the case $m > 0$ we can similarly factorise $p(mX, Y^m)$ into irreducibles

$$p(mX, Y^m) = \prod_{j=1}^{k} r_j(X, Y)^t,$$

such that $r_1(X, Y) = q_1(X, Y)$, the common irreducible factor of $p(nX, Y^n)$ and $p(mX, Y^m)$.

Considering the degrees as before,

$$\ell s \deg_X q_1 = \deg p = kt \deg_X q_1.$$

Since $\deg_X p \neq 0$, we obtain

$$\ell s = kt \neq 0.$$

Thus, by the second equality,

$$n \deg_Y p = \ell s \deg_Y q_1 = kt \deg_Y q_1 = m \deg_Y p.$$

Since $\deg_Y p \neq 0$, we finally obtain $n = m$ contradicting the fact that m and n are coprime.

In the case $m < 0$, consider the polynomial

$$g(X, Y) = Y^{-m \deg_Y p} p(mX, Y^m)$$

instead of $p(mX, Y^m)$. Note that $\deg_X g = \deg_X p$ and $\deg_Y g = -m \deg_Y p$. By a similar argument and analysis of the degrees as before, we obtain that $-n = m$, also contradicting coprimality of m and n. **Q. E. D.**

We finally proceed to proving the existence of infinitely many algebraically independent zeros. Note that an infinite collection of complex numbers is algebraically independent if and only if every finite subcollection is algebraically independent. It thus suffices to prove algebraic independence for finite collections of zeros of f. We will first do this under the additional assumption on p that the curve defined by $p(mX, Y^m) = 0$ is irreducible for every non-zero $m \in \mathbb{Z}$ and then show the result for the general case. Following the terminology of [21], such a polynomial p is called *primitive.*

Theorem 3.26 (SC)**.** *Let p be primitive and let $v_1, \dots, v_n$ be non-zero zeros of $f(z) = p(z, \mathrm{e}^z)$ with $v_i \neq \pm v_j$ for all $i \neq j$. Then $v_1, \dots, v_n$ are algebraically independent.*

Proof. Suppose that p is primitive. Assume that there exists an algebraically dependent collection $v_1, \dots, v_n, v_{n+1}$ of $n+1$ non-zero zeros of f such that $v_i \neq \pm v_j$ for all $i \neq j$. Assume further that n is minimal with this property.

By Theorem 3.25, two such zeros are always algebraically independent. Thus we must have $n \geq 2$. Moreover, by minimality of n, the collection $v_1, \dots, v_n$ must be algebraically independent. Hence,

$$\operatorname{td}_{\mathbb{Q}}(v_1, \dots, v_{n+1}, \mathrm{e}^{v_1}, \dots, \mathrm{e}^{v_{n+1}}) = n < n + 1.$$

By Schanuel's Conjecture, $v_1, \ldots, v_{n+1}$ are $\mathbb{Q}$-linearly independent. Thus there exist integers $k_1, \ldots, k_n, m$ with no common divisor such that

$$\sum_{j=1}^{n} k_j v_j = m v_{n+1}.$$

Let $u_j = \frac{v_j}{m}$. Let $C \subset \mathbb{C} \times \mathbb{C}^\times$ be the curve defined by $p(X, Y) = 0$ and $C_m \subset \mathbb{C} \times \mathbb{C}^\times$ be the curve defined by $p(mX, Y^m) = 0$. As we took p to be primitive, C_m is irreducible. Since $v_1, \ldots, v_n$ are algebraically independent, so are $u_1, \ldots, u_n$. Hence, $(u_1, \mathrm{e}^{u_1}), \ldots, (u_n, \mathrm{e}^{u_n})$ are generic points on C_m with $\mathrm{td}_{\mathbb{Q}}(u_1, \mathrm{e}^{u_1}, \ldots, u_n, \mathrm{e}^{u_n}) = n$. In particular, $(u_1, \mathrm{e}^{u_1}, \ldots, u_n, \mathrm{e}^{u_n})$ is a generic point in C_m^n.

Define the map

$$\varphi : (\mathbb{C} \times \mathbb{C}^\times)^n \to \mathbb{C} \times \mathbb{C}^\times, (x_1, y_1, \ldots, x_n, y_n) \mapsto \left(\sum_{j=1}^{n} k_j x_j, \prod_{j=1}^{n} y_j^{k_j} \right).$$

Since φ maps $(u_1, \mathrm{e}^{u_1}, \ldots, u_n, \mathrm{e}^{u_n})$ in C_m^n to $(v_{n+1}, \mathrm{e}^{v_{n+1}})$, which is a generic point in C, and C_m is irreducible, φ actually maps C_m^n to C. Hence, for any points $(x_1, \mathrm{e}^{x_1}), \ldots, (x_n, \mathrm{e}^{x_n})$ in C_m,

$$\varphi(x_1, \mathrm{e}^{x_1}, \ldots, x_n, \mathrm{e}^{x_n}) = \left(\sum_{j=1}^{n} k_j x_j, \prod_{j=1}^{n} \mathrm{e}^{k_j x_j} \right)$$

lies in C. Note that for any zeros $w_1, \ldots, w_n$ of f, the pairs

$$\left(\frac{w_1}{m}, \mathrm{e}^{\frac{w_1}{m}} \right), \ldots, \left(\frac{w_n}{m}, \mathrm{e}^{\frac{w_n}{m}} \right)$$

lie in C_m. It follows that the sum

$$\sum_{j=1}^{n} \frac{k_j}{m} w_j \tag{3.6}$$

is also a zero of f. Setting $w_1 = w_2 = v_1$ and $w_j = v_j$ for all $j > 2$, we obtain that

$$w = \frac{k_1 + k_2}{m} v_1 + \frac{k_3}{m} v_3 + \ldots + \frac{k_n}{m} v_n \tag{3.7}$$

is also a zero of f. Consider the case $n > 2$. Since $v_1, \ldots, v_n$ are algebraically independent and $k_1, \ldots, k_n$ are non-zero, w can neither be 0 nor $\pm v_j$ for any j. Thus, we have constructed an algebraically dependent collection $v_1, v_3 \ldots, v_n, w$ of n non-zero zeros of f, contradicting the minimality of n. Hence, $n = 2$ remains as the only possible case.

In this case, v_1, v_2, v_3 is our collection of algebraically dependent non-zero zeros of f, satisfying

$$m v_3 = k_1 v_1 + k_2 v_2.$$

Using (3.7), we can also find another zero w of f with

$$w = \frac{k_1 + k_2}{m} v_1.$$

By Proposition 3.24, if v_1 and w are two algebraically dependent zeros of f, either one of them is 0 or $w = \pm v_1$. Since $v_1 \neq 0$, only the cases $w = 0$ or $k_1 + k_2 = \pm m$ are possible.

We will consider each case separately and apply individual arguments leading to contradictions.

Suppose that $w = 0$. Then $k_1 + k_2 = 0$. Setting $w_1 = v_1$ and $w_2 = 0$ in (3.6), $\frac{k_1}{m} v_1$ is also a zero of f. Similarly $\frac{k_2}{m} v_1$ is a zero of f. Since $v_1 \neq 0$, this is again, by Proposition 3.24, possible if and only if $k_1 = -k_2 = \pm m$. As the integers k_1, k_2, m have no common divisor, we can, without loss of generality, assume that $v_3 = v_1 - v_2$. Then $v_1 = v_2 + v_3$, and by changing the roles of the variables and setting $w_1 = w_2 = v_2$ in (3.6), $2v_2$ is also a non-zero zero of f. This contradicts Proposition 3.24.

Suppose that $k_1 + k_2 = m$. Recall that $mv_3 = k_1 v_1 + k_2 v_2$. By switching the roles of the variables and multiplying by -1 if necessary, we may assume, without loss of generality, that $|m| \geq |k_1|, |k_2|$ and $k_1, m > 0$. In particular $0 < \frac{k_1}{m} < 1$. Construct a sequence (z_j) of zeros of f by $z_1 := v_1$ and

$$z_{j+1} := \frac{k_1}{m} z_j + \frac{k_2}{m} v_2$$

for all $j \geq 1$. Using the fact that $\frac{k_2}{m} = 1 - \frac{k_1}{m}$, a simple induction on k shows that for all $j \geq 0$,

$$z_{j+1} = \left(\frac{k_1}{m}\right)^j v_1 + \left(1 - \left(\frac{k_1}{m}\right)^j\right) v_2.$$

Since $0 < \frac{k_1}{m} < 1$, the coefficient $\left(\frac{k_1}{m}\right)^j$ of v_1 takes a different value for each j. By algebraic independence of v_1 and v_2, the sequence (z_j) takes infinitely many distinct values. Finally, let $M = \max\{v_1, v_2\}$. Then

$$|z_{j+1}| \leq \left|\left(\frac{k_1}{m}\right)^j\right| M + \left|1 - \left(\frac{k_1}{m}\right)^j\right| M \leq 2M.$$

So there are infinitely many zeros of f lying in the disc centred at 0 with radius $2M$. Hence, there exists an accumulation point of zeros of f, which must also be a zero. Since f is an entire function, this implies that $f = 0$, a contradiction.

Finally, suppose that $k_1 + k_2 = -m$. Then $\frac{k_1}{m} + \frac{k_2}{m} = -1$. Set $s = \frac{k_1}{m}$. Then $\frac{k_2}{m} = -(1 + s)$, and

$$v_3 = sv_1 - (1 + s)v_2.$$

With $w_1 = v_3$ and $w_2 = v_2$ in (3.6), we see that

$$\tilde{w} = sv_3 - (1+s)v_2$$

is a zero of f. Substituting the first equation into the second, we obtain

$$\tilde{w} = s^2 v_1 - (1+s)^2 v_2.$$

Treating $v_1, v_2, \tilde{w}$ as our algebraically dependent collection of zeros of f and arguing as before, we obtain that $s^2 - (1+s)^2 = -2s - 1$ must equal 0 or ± 1. The cases $s^2 - (1+s)^2 = 0$ and $s^2 - (1+s)^2 = 1$ would lead to contradiction as already shown.[5] In the remaining case $-2s - 1 = -1$, we obtain that $s = 0$ and hence that $k_1 = 0$, a contradiction. **Q. E. D.**

Using this theorem, the proof of the main result of this section can by done by strong induction on the X-degree of p.

Proof of Theorem 2.8. Let $p(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ be irreducible and dependent on both X and Y, and let $f(z) = p(z, \mathrm{e}^z)$, an entire function. By Corollary 3.22, f has infinitely many zeros.

If p is primitive, then by Theorem 3.26 there exist infinitely many algebraically independent zeros of f, namely any collection of non-zero zeros $(v_j)_{j \in J}$, for some index set J, satisfying $v_i \neq \pm v_j$ for all $i \neq j$ in J.

If p is not primitive, then let $C_m \subset \mathbb{C} \times \mathbb{C}^\times$ be a curve defined by $p(mX, Y^m)$ for some m such that C_m is reducible. Then C_m has an irreducible component defined by some polynomial $r(X,Y) \in \overline{\mathbb{Q}}[X,Y]$. Arguing as in the proof of Theorem 3.25, we obtain that r depends on both X and Y, and

$$0 < \deg_X r < \deg_X p.$$

Now we argue by induction on $\deg_X p$.

Suppose that $\deg_X p = 1$. Then p must be primitive, as otherwise there would be a polynomial r such that $0 < \deg_X r < 1$. Hence, f has infinitely many algebraically independent zeros.

As inductive hypothesis, suppose that for all irreducible polynomials $q(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ depending on both X and Y such that $\deg_X q < \deg_X p$, the entire function $g(z) = q(z, \mathrm{e}^z)$ has infinitely many algebraically independent zeros.

If p is primitive, then we are already done. Otherwise there exists $r(X,Y) \in \overline{\mathbb{Q}}[X,Y]$, as defined before, such that $\deg_X r < \deg_X p$ and r is irreducible and

[5] In the case $s^2 - (1+s)^2 = 1$, we could alternatively argue that then $-2s - 1 = 1$, whence $s = -1$. This would give us $k_2 = 0$ and $k_1 = -m$, thus the contradiction $v_3 = -v_1$.

depends on both X and Y. By our hypothesis, the entire function $h(z) = r(z, \mathrm{e}^z)$ has infinitely many algebraically independent zeros $u_1, u_2, \ldots$ say. Since $r(X, Y)$ is a factor of $p(mX, Y^m)$, we have $f(mu_j) = p(mu_j, \mathrm{e}^{mu_j}) = 0$ for all j. Hence, $mu_1, mu_2, \ldots$ is an infinite collection of algebraically independent zeros of f.

Q. E. D.

When considering the irreducible polynomial $p(X, Y) = X - q(Y)$ for some polynomial in one variable $q(Y) \in \overline{\mathbb{Q}}[Y]$, this result yields that, assuming Schanuel's Conjecture, the entire function $f(z) = z - q(\mathrm{e}^z)$ has infinitely many algebraically independent zeros, and thus the exponential polynomial $q(\mathrm{e}^z)$ has infinitely many fixed points. Choosing $q(Y) = Y$, this implies that the exponential function e^z has infinitely many algebraically independent fixed points.

Marker asked in [21] whether this property of e^z can be proved without assuming Schanuel's Conjecture – so far this remains an open question.

Another question asked by Marker is whether one could prove Theorem 2.8 for the case that p is not defined over a number field, i. e. a finite or equivalently algebraic field extension of $\mathbb{Q}$. In 2014 Mantova managed to find a generalisation and thus an answer to this question presented in [19]. In the following, we will consider Mantova's results and outline the structure of his proof.

Recall the statement of the main theorem of [19]:

Theorem 2.10 (SC)**.** *For any finitely generated field $k \subset \mathbb{C}$, and for any irreducible polynomial $p(X, Y) \in k[X, Y]$ depending on both X and Y, the equation*

$$p(z, \mathrm{e}^z) = 0$$

has a solution generic over k.

Mantova actually proves this theorem not only for $\mathbb{C}_{\exp}$ but for any exponential field (satisfying Schanuel's Conjecture) and where $p(z, \mathrm{e}^z)$ has infinitely many zeros. As we saw earlier in this section, our proof that $p(z, \mathrm{e}^z)$ has infinitely many zeros requires Hadamard's Factorisation Theorem and therefore restricts $p(z, \mathrm{e}^z)$ to $\mathbb{C}_{\exp}$. We will therefore only state the results leading to the proof of Theorem 2.10 over $\mathbb{C}_{\exp}$.

The general idea of the proof is to show that $p(z, \mathrm{e}^z)$ only has finitely many zeros in $\overline{k}$, the algebraic closure of k. This forces the other zeros to be generic over k. We will briefly summarise the four steps leading to the proof.

In the first step, the setting is changed to a finite dimensional $\mathbb{Q}$-vector space by the following result:

Proposition 3.27 (SC)**.** *There exists a finite dimensional $\mathbb{Q}$-vector space $L \subset \mathbb{C}$ containing all zeros of $p(z, \mathrm{e}^z)$ in $\overline{k}$.*

As a corollary, if $\mathbf{b} \in \mathbb{C}^\ell$ is a basis for L, where $\ell = \dim_{\mathbb{Q}}(L)$, then $p(z, \mathrm{e}^z)$ has finitely many zeros in $\overline{k}$ if and only if there are only finitely many $\mathbf{q} \in \mathbb{Q}^\ell$ such that

$$p\left(\mathbf{q} \cdot \mathbf{b}, \mathrm{e}^{\mathbf{q} \cdot \mathbf{b}}\right) = 0.$$

In the second step, Mantova uses the main result from Günaydin [8] (Theorem 1.1) to reduce the problem to counting integer solutions:

Proposition 3.28. *There exists a $\mathbb{Q}$-linearly independent $\boldsymbol{b}' \in \mathbb{C}^\ell$ such that*

$$p\left(\mathbf{q} \cdot \mathbf{b}, \mathrm{e}^{\mathbf{q} \cdot \mathbf{b}}\right) = 0$$

has only finitely many rational solutions $\mathbf{q} \in \mathbb{Q}^\ell$ if and only if

$$p\left(\mathbf{m} \cdot \mathbf{b}', \mathrm{e}^{\mathbf{m} \cdot \mathbf{b}'}\right) = 0$$

has only finitely many integer solutions $\mathbf{m} \in \mathbb{Z}^\ell$.

A further reduction to counting solutions only involving algebraic exponential expressions is made in the third step. Here, Mantova uses the main result from Zannier [34] (Theorem 1) to prove the following proposition:

Proposition 3.29. *There are finitely many irreducible polynomials $r(X, Y) \in \mathbb{C}[X, Y]$ dependent on both X and Y, and there exists a $\mathbb{Q}$-linearly independent $\mathbf{c} \subset L$ with $\mathrm{e}^{\mathbf{c}} \subset \overline{\mathbb{Q}}$ such that*

$$p\left(\mathbf{m} \cdot \mathbf{b}', \mathrm{e}^{\mathbf{m} \cdot \mathbf{b}'}\right) = 0$$

has only finitely many integer solutions $\mathbf{m} \in \mathbb{Z}^\ell$ if and only if each equation

$$r\left(\mathbf{m}' \cdot \mathbf{c}, \mathrm{e}^{\mathbf{m}' \cdot \mathbf{c}}\right) = 0$$

has only finitely many integer solutions $\mathbf{m}' \in \mathbb{Z}^{\ell'}$, where ℓ' denotes the length of $\mathbf{c}$.

Finally, in the fourth step the existence of only finitely many integer solutions of each equation $r\left(\mathbf{m}' \cdot \mathbf{c}, \mathrm{e}^{\mathbf{m}' \cdot \mathbf{c}}\right) = 0$ is proved by considering the cases $2\pi\mathrm{i} \in L$ and $2\pi\mathrm{i} \notin L$ and using some arithmetic on $\overline{\mathbb{Q}}$. By the previous propositions, this shows that $p(z, \mathrm{e}^z)$ only has finitely many zeros in $\overline{k}$ and hence proves Theorem 2.10.

Mantova's argument does not make use of any of the results from Marker [21]. Thus, Marker's result (Theorem 2.8) is a consequence of Mantova's theorem (Theorem 2.10) as follows:

Aussume Schanuel's Conjecture, and suppose that $p(X,Y) \in \overline{\mathbb{Q}}[X,Y]$ is irreducible and depends on both X and Y. Let k be the field obtained by adjoining the coefficients of p to $\mathbb{Q}$. By Theorem 2.10, there exists a zero $v \in \mathbb{C}$ of $p(z, \mathrm{e}^z)$ such that (v, e^v) is generic over k. Since $p(X,Y) \in k(v)$, the argument can be repeated to show the existence of infinitely many algebraically independent zeros of $p(z, \mathrm{e}^z)$ in $\mathbb{C}$.

We do, however, note that Mantova uses other strong results from (Transcendental) Number Theory whereas Marker's argument only involves basic results from Algebraic Geometry. This also makes the study of the proof of Marker's theorem worthwhile.

4 Zilber's pseudo-exponential fields

As we mentioned previously, some of the questions related to Schanuel's Conjecture arise from Zilber [36]. In this paper Zilber constructs a non-first-order sentence axiomatising a class of structures imitating $\mathbb{C}_{\exp}$ and satisfying Schanuel's Conjecture. Those structures are called pseudo-exponential fields. Moreover, he proves that this class has a unique model in every uncountable cardinality. As a result, if one could show that the unique model of cardinality continuum, denoted by $\mathbb{B}_{\exp}$, is isomorphic to $\mathbb{C}_{\exp}$, Schanuel's Conjecture would be proved.

4.1 Infinitary Logic

Since Zilber uses infinitary logic, as opposed to first-order logic, for the axiomatisation of his pseudo-exponential fields, we will give a very short introduction to the basics and terminology of this area of Logic. This can, for example, be found in Keisler and Knight [11].

Definition 4.1. Let $\mathcal{L}$ be a language of first-order logic with equality whose alphabet contains a collection of non-logical symbols τ. For infinite cardinals $\mu \leq \kappa$ we define the *infinitary logic* $\mathcal{L}_{\kappa,\mu}$ as the language with κ variables, equality and all non-logical symbols of τ, such that the following formulas are admitted:

- all formulas of $\mathcal{L}$;
- conjunctions and disjunctions of sets of $\mathcal{L}_{\kappa,\mu}$-formulas of size strictly less than κ;
- formulas obtained by applying existential and universal quantifiers over sets of variables of size strictly less than μ.

Remark 4.2. $\mathcal{L}_{\omega,\omega}$ is the usual first-order logic. We call formulas in $\mathcal{L}_{\omega,\omega}$ *finitary.*

Since no further non-logical symbols are added to $\mathcal{L}$ to obtain an infinitary logic, the structures satisfying $\mathcal{L}_{\kappa,\mu}$-formulas are $\mathcal{L}$-structures.

Definition 4.3 (Satisfaction for $\mathcal{L}_{\kappa,\mu}$-formulas)**.** Let $\mu \leq \kappa$ be infinite cardinals, $\mathcal{A}$ be an $\mathcal{L}$-structure and s be an assignment on the set of variables $\{v_\alpha \mid \alpha < \kappa\}$. Denote the restriction of s to $\{v_\alpha \mid \alpha < \omega\}$, the set of variables in first-order logic, by $\tilde{s}$.

For an $\mathcal{L}$-formula φ,

$$\mathcal{A} \models_s \varphi \ :\Longleftrightarrow\ \mathcal{A} \models_{\tilde{s}} \varphi.$$

For a set Σ of $\mathcal{L}_{\kappa,\mu}$-formulas such that $|\Sigma| < \kappa$, we define

$$\mathcal{A} \models_s \bigwedge_{\varphi \in \Sigma} \varphi \ :\Longleftrightarrow\ \mathcal{A} \models_s \varphi \text{ for all } \varphi \in \Sigma,$$

and

$$\mathcal{A} \models_s \bigvee_{\varphi \in \Sigma} \varphi \ :\Longleftrightarrow\ \mathcal{A} \models_s \varphi \text{ for some } \varphi \in \Sigma.$$

For a family of variables $\overline{v} = (v_i)_{i \in I}$ with some index set I of cardinality less than μ and an $\mathcal{L}_{\kappa,\mu}$-formula φ, we define

$$\mathcal{A} \models_s \forall \overline{v}\ \varphi \ :\Longleftrightarrow\ \mathcal{A} \models_{s[\overline{v}|\overline{a}]} \varphi \text{ for all families } \overline{a} = (a_i)_{i \in I} \text{ in } \mathrm{dom}(\mathcal{A}),$$

where $s[\overline{v}|\overline{a}]$ denotes the assignment obtained from s by substituting, for all $i \in I$, the image of v_i with a_i in the domain of $\mathcal{A}$, and

$$\mathcal{A} \models_s \exists \overline{v}\ \varphi \ :\Longleftrightarrow\ \mathcal{A} \models_{s[\overline{v}|\overline{a}]} \varphi \text{ for some family } \overline{a} \text{ in } \mathrm{dom}(\mathcal{A}).$$

In infinitary logic, some classes of structures can be axiomatised with a single sentence. For instance, Archimedean ordered fields are the models of the conjuction of the usual axioms for ordered fields together with

$$\forall x \bigvee_{n \in \omega} \underbrace{1 + \ldots + 1}_{n \text{ times}} > x.$$

This is a $\mathcal{L}_{\omega_1,\omega}$-sentence.

The infinitary logic $\mathcal{L}_{\omega_1,\omega}$ has ω_1 variables, countable conjunctions and disjunctions and finite quantifiers. It becomes important in the following part, in which we look at Zilber's sentence axiomatising his pseudo-exponential fields.

4.2 Summary of axioms and properties

We will briefly summarise some of the most important results of [36]. Zilber constructs an $\mathcal{L}_{\omega_1,\omega}(Q)$-sentence axiomatising the so-called *strongly exponentially-algebraically closed fields with pseudo-exponentiation*, which are very similar to $\mathbb{C}_{\exp}$.

Here Q stands for an additional quantifier in the language expressing "there exist uncountably many". He proceeds by showing that there is a unique model, up to isomorphism, of a given uncountable cardinality. Finally he gives two conditions under which $\mathbb{C}_{\exp}$ is exactly that unique model. Those two conditions can be re-stated as two conjectures about $\mathbb{C}_{\exp}$, the first being Schanuel's Conjecture and the second that certain systems of exponential equations in $\mathbb{C}_{\exp}$ have complex solutions. A simple version of the latter one was considered in Section 3.2.

Since Zilber presents a very technical construction of his pseudo-exponential fields, a detailed description would go beyond the scope of this dissertation. Instead we will use the summaries in Marker [21], Kirby [13] and Shkop [27] to explain the properties of Zilber's pseudo-exponential fields and how they are related to the applications of Schanuel's Conjecture in Sections 3.1 & 3.2.

There are five axioms which Zilber's fields satisfy. All of them can be stated in the language $\mathcal{L}_{\omega,\omega_1}(Q)$ using non-logical symbols $(+, \cdot, \exp)$. We will state them, explain the terminology, clarify in which logic they are expressible (finitary or infinitary), and comment on their connection with $\mathbb{C}_{\exp}$. For a detailed analysis and discussion of these axioms, see [13], section 2.

A strongly exponentially-algebraically closed field with pseudo-exponentiation $(K, +, \cdot, 0, 1, \exp)$, as introduced by Zilber, satisfies the following five axioms:

(Z1) **ELA-field:** The abbreviation ELA stands for the following properties of K:

- Exponential: The exponential map exp on K is a homomorphism from its additive group $(K, +)$ to its multiplicative group $(K^{\times}, \cdot)$. (K is an exponential field.)
- Logarithm: $\exp : (K, +) \to (K^{\times}, \cdot)$ is surjective. (A logarithm function can be defined.)
- Algebraically closed: K is of characteristic 0, and every non-constant polynomial in $K[X]$ has a root in K. (K is algebraically closed.)

(Z2) **Standard kernel:** The kernel of the exponential map

$$\ker(\exp) = \{x \in K \mid \exp(x) = 1\}$$

is an infinite cyclic group generated by a transcendental element α, that is, an element which is not the root of a non-zero polynomial with coefficients in the copy of $\mathbb{Q}$ in K (or equivalently with coefficients in the copy of $\mathbb{Z}$ in K).

These first two axioms imply that Zilber's fields satisfy the conditions of the exponential fields considered in Section 3.1. Hence, Shapiro's Conjecture holds in K.

(Z3) **Schanuel property:** Schanuel's Conjecture holds in K. Using Version 2, this means that for the predimension function δ and any tuple $\overline{a}$ from K

$$\delta(\overline{a}) \geq 0.$$

Again, we can consider $\mathbb{Q}$ as a subfield of K, since K has characteristic 0.

The next axiom builds on several yet undefined terms. As they will not be considered specifically thereafter, we will not explain them for general n but only for the case $n = 1$ in $\mathbb{C}_{\exp}$, which was treated in Section 3.2.

(Z4) **Strong exponential-algebraic closure:** For all finite $A \subset K$, if $V \subseteq K^n \times (K^\times)^n$ is an irreducible, free and normal variety, then there exists a point $(\overline{x}, \exp(\overline{x})) \in V$ which is generic over A.

For $n = 1$ and in $\mathbb{C}_{\exp}$ the conditions on V translate as follows: Let V be a *variety* in $\mathbb{C}^2$. As we explained at the beginning of Section 3.2, V is a curve defined by

$$V = \{(X, Y) \in \mathbb{C} \times \mathbb{C} \mid p(X, Y) = 0\}$$

for some $p(X, Y) \in \mathbb{C}[X, Y]$, and it is irreducible if and only if $p(X, Y)$ is an irreducible polynomial. We only consider the points of V lying in $\mathbb{C} \times \mathbb{C}^\times$, since we are only interested in solutions of $p(z, \mathrm{e}^z)$ not allowing us zero in the second component. V is *normal* if its dimension is non-zero. Since $\mathbb{C}$ is algebraically closed, this holds whenever $p(X, Y)$ is non-constant. Finally, V is *free* if for all non-zero $\ell \in \mathbb{Z}$ and non-zero $b \in \mathbb{C}$,

$$V \not\subseteq \{(X, Y) \in \mathbb{C} \times \mathbb{C}^\times \mid \ell X = b\} \text{ and}$$
$$V \not\subseteq \left\{(X, Y) \in \mathbb{C} \times \mathbb{C}^\times \mid Y^\ell = b\right\}.$$

Since restrictions for these sets only depend on one variable, this yields that V is free whenever $p(X, Y)$ depends on both X and Y.

The conclusion under these conditions is that for every finite set $A \subset \mathbb{C}$ there exists a point $(z, \mathrm{e}^z) \in V$ which is generic over A, meaning that (z, e^z) has transcendence degree 1 over $\mathbb{Q}(A)$. In our special case this means that there is an infinite set of algebraically independent zeros of the entire function $f(z) = p(z, \mathrm{e}^z)$ on $\mathbb{C}$, as finitely many could lie in such a set A.

Theorem 2.8 makes the further restriction that the polynomial p must lie in $\overline{\mathbb{Q}}[X, Y]$ rather than $\mathbb{C}[X, Y]$ and hence only proves a very specific case of (Z3) in $\mathbb{C}_{\exp}$.

The generalisation showing that property (Z4) holds in $\mathbb{C}_{\exp}$ for the simplest case $n = 1$ with no further restrictions is given in Theorem 2.10.

Algebraic closure of a field means that every polynomial in one variable with coefficients from that field has a zero which lies in the field. Interpreting (Z4) in terms of exponential polynomials and zeros thereof makes it clear why this property is called strong exponential-algebraic closure, where the attribute "strong" stands for the additional property of zeros being generic points.

For the last axiom, we will give one more definition. If K has the Schanuel property (Z3), then for any finite set $A \subset K$ the predimension satisfies $\delta(A) \geq 0$ (see Version 2 of Schanuel's Conjecture, considering the finite set A as a tuple).

Definition 4.4. For finite $A \subset K$, we define

$$d(A) := \inf \{\delta(B) \mid A \subseteq B \subset A \text{ and } B \text{ is finite}\}.$$

The *Schanuel closure* scl of A is defined by

$$\mathrm{scl}(A) := \{a \in K \mid d(A \cup \{a\}) = d(A)\}.$$

(Z5) **Countable closure property:** For any finite set $A \subset K$, the Schanuel closure $\mathrm{scl}(A)$ is countable.

Zilber's sentence is the conjunction of these five axioms.

(Z1): This axiom is finitary, i. e. first-order expressible, and $\mathbb{C}_{\exp}$ satisfies all conditions.

(Z2): This axiom can be split into three parts, two of which are finitary. The third part states that the *multiplicative stabiliser* of ker(exp),

$$Z(K) = \{a \in K \mid \forall x \ (y \in \ker(\exp) \ \rightarrow \ ax \in \ker(\exp))\},$$

is in fact the copy of $\mathbb{Z}$ in K. This is expressed as

$$(\forall a \in Z(K)) \bigvee_{n \in \omega} (r \doteq \underbrace{1 + \ldots + 1}_{n \text{ times}} \vee r + \underbrace{1 + \ldots + 1}_{n \text{ times}} \doteq 0.)$$

(Note that $(\forall a \in Z(K))$, 0 and 1 are all standard abbreviations and could be embedded as finitary formulas.) This sentence is not finitary but a single $\mathcal{L}_{\omega,\omega_1}$-sentence. In $\mathbb{C}_{\exp}$ this axiom is satisfied. The kernel of exp is the infinite cyclic group generated by the transcendental element $2\pi\mathrm{i}$, since $\mathrm{e}^z = 1$ if and only if z is an integer multiple of $2\pi\mathrm{i}$.

(Z1) and (Z2) were chosen such that they hold in $\mathbb{C}_{\exp}$.

(Z3): Provided (Z2) holds, it turns out that Schanuel's Conjecture is equivalent to an axiom scheme consisting of countably many finitary sentences and thus expressible as a single $\mathcal{L}_{\omega,\omega_1}$-sentence (see [13] section 2.2).

(Z4): Provided (Z1), (Z2) and (Z3) hold, this is finitary (see [13] section 2.3). It is yet to be shown that $\mathbb{C}_{\exp}$ satisfies this condition.

(Z5): This is expressible as an $\mathcal{L}(Q)$-scheme and therefore as a $\mathcal{L}_{\omega,\omega_1}(Q)$-sentence (see [13] section 2.4). Also this condition is satisfied by $\mathbb{C}_{\exp}$, as proved in [36], Lemma 5.12.

Since $\mathbb{C}_{\exp}$ satisfies (Z1), (Z2) and (Z5), if one showed that (Z3) and (Z4) also hold in $\mathbb{C}_{\exp}$, it would already follow that $\mathbb{C}_{\exp}$ is the unique model of Zilber's sentence of cardinality continuum $\mathbb{B}_{\exp}$. One further step was made by [13], Theorem 1. It says that if $\mathbb{C}_{\exp}$ and $\mathbb{B}_{\exp}$ are elementarily equivalent, then they are isomorphic. We will state the equivalence between those conjectures as a theorem.

Theorem 4.5. *The following conjectures about Zilber's exponential field $\mathbb{B}_{\exp}$ of cardinality continuum are equivalent:*

1. $\mathbb{C}_{\exp}$ *and* $\mathbb{B}_{\exp}$ *are isomorphic.*
2. $\mathbb{C}_{\exp}$ *and* $\mathbb{B}_{\exp}$ *are elementarily equivalent.*
3. $\mathbb{C}_{\exp}$ *satisfies (Z3) and (Z4), i. e. Schanuel's Conjecture holds in* $\mathbb{C}_{\exp}$ *and* $\mathbb{C}_{\exp}$ *is strongly exponentially-algebraically closed.*

It is hard to see how conjecture 1 (or 2) can be proved without proving Schanuel's Conjecture in $\mathbb{C}_{\exp}$. An attempt has been made by Bleybel [2] but seems not to have succeeded. On the other hand, [21] and [19] managed to prove a simple case of the strong exponential-algebraical closure property (Z3) (see section 3.2), only assuming that (Z4), Schanuel's Conjecture, holds in $\mathbb{C}_{\exp}$, and thus possibly made another step towards the proof of conjecture 3. Since these results on $\mathbb{C}_{\exp}$ are quite recent, it is possible that a proof of property (Z3) in $\mathbb{C}_{\exp}$ under the assumption of (Z4) is not too far away. If this could be shown, then $\mathbb{C}_{\exp}$ would be isomorphic to $\mathbb{B}_{\exp}$ if and only if Schanuel's Conjecture holds in $\mathbb{C}_{\exp}$. Moreover, [19] showed that the simplest case of (Z4) holds in any exponential field which satisfies (Z3) and with infinitely many zeros of the corresponding polynomial exponential equations. If this could be shown for the general case, (Z4) could be replaced by a weaker condition.

Nevertheless, all results in both Model Theory and Transcendental Number Theory which may have made progress towards the proof of Schanuel's Conjecture, although being intrinsicly interesting and somewhat powerful, still leave us with the impression that a proof of this significant conjecture stays out of our reach.

5 Conclusion

5.1 Summary of recent results on Schanuel's Conjecture

Although the proof of Schanuel's Conjecture still seems to be out of reach, we can see that research in exponential fields has made some progress since Zilber's model theoretical approach in [36]. We have seen two instances of how Schanuel's Conjecture implies major results on $\mathbb{C}_{\exp}$: The first one is Shapiro's Conjecture (Section 3.1), also applicable to more general exponential fields which include Zilber's fields. The second one is the existence of generic solutions of polynomial exponential equations (Section 3.2). There are many more applications in other areas, such as exponential rings (e. g. Terzo [28] and Macintyre [17] in the exponential subring of $\mathbb{R}$).

Since all applications of Schanuel's Conjecture in $\mathbb{C}_{\exp}$ yield results which will only become valid once the conjecture has been proved, one may think that conclusions in this context do not make any real progress in the analysis of $\mathbb{C}_{\exp}$. However, every application of Schanuel's Conjecture which can also be applied to Zilber's field gives us results which are already valid, as those fields are constructed such that Schanuel's Conjecture holds in them. Shapiro's Conjecture is one such example. On the other hand, Schanuel's Conjecture can be used to advance the study of the strong exponential-algebraical closure property in $\mathbb{C}_{\exp}$, as shown in Section 3.2. This might subsequently lead towards the result that $\mathbb{C}_{\exp}$ and $\mathbb{B}_{\exp}$ are isomorphic if and only if Schanuel's Conjecture holds in $\mathbb{C}_{\exp}$.

D'Aquino, Macintyre and Terzo emphasise in [6], p. 598 f., the importance of Schanuel's Conjecture:

"A very distinguished number theorist has remarked that if one assumes Schanuel's Conjecture one can prove anything. The sense of this is clear if one restricts 'anything' to refer to statements in transcendence theory."

We will give a short overview of recent results on Schanuel's Conjecture with references to the corresponding papers.

Bays, Kirby and Wilkie prove in [1] an analogue of Schanuel's conjecture for raising to the power of an exponentially transcendental real number.

Theorem 5.1 ([1] Theorem 1.1)**.** *Let $\lambda \in \mathbb{R}$ be exponentially transcendental, and let $\overline{y} \in (\mathbb{R}_{>0})^n$ be multiplicatively independent. Then*

$$\operatorname{td}\left(\overline{y}, \overline{y}^{\lambda}/\lambda\right) \geq n.$$

$\operatorname{td}(X/Y)$ is short for $\operatorname{td}(\mathbb{Q}(X,Y)/\mathbb{Q}(Y))$.

More generally:

Theorem 5.2 ([1] Theorem 1.2). *Let K be an exponential field, let $\lambda \in K$ be exponentially transcendental, and let $\overline{x} \in K^n$ such that $\exp(\overline{x})$ is multiplicatively independent. Then*

$$\operatorname{td}(\exp(\overline{x}), \exp(\lambda\overline{x})/\lambda) \geq n.$$

On the other hand, assuming Schanuel's Conjecture yields some interesting results. For instance, D'Aquino, Macintyre and Terzo [5] prove the Schanuel Nullstellensatz in $\mathbb{B}_{\exp}$:

Theorem 5.3. *Let $p(X_1, \ldots, X_n)$ be an exponential polynomial with coefficients in $\mathbb{B}_{\exp}$. If $p \neq \exp(q(X_1, \ldots, X_n))$ for any exponential polynomial $q(X_1, \ldots, X_n)$, then p has a root in $\mathbb{B}_{\exp}$*

Here, exponential polynomials allow multiple instances of exponentiation. Shkop presents an alternative proof of this theorem only using basic exponential algebra and idependent of Schanuel's Conjecture in [25] and [26]. In [25], she also proves a special case of Shapiro's Conjecture in pseudo-exponential fields:

Theorem 5.4. *Let K be an algebraically closed field satisfying Schanuel's Conjecture. Suppose that*

$$p(x) = \sum_{j=1}^{n} a_j \exp(b_j x) \text{ and } q(x) = \sum_{j=1}^{m} c_j \exp(d_j x),$$

where $a_j, b_j, c_j, d_j \in \overline{\mathbb{Q}}$, have no common factors aside from units in the exponential subring of $K[x]^{\exp}$ generated by $\overline{\mathbb{Q}}[x]$. Then p and q have only finitely many common zeros in K.

5.2 Open questions

To complete this survey, we will collect a few open questions and problems related to Schanuel's Conjecture and the complex exponential field $\mathbb{C}_{\exp}$.

- (Mycielski [22] p. 308) Does $\mathbb{C}_{\exp}$ have an automorphism other than the identity and complex conjugation?

This questions is closely related to the following (see Zilber [35]):

- ([21] p. 791) Is $\mathbb{R}$ definable in $\mathbb{C}_{\exp}$? And more generally:
- ([35] p. 226, [21] p. 791) Does $\mathbb{C}_{\exp}$ have the property of quasi-minimality, that is, is every definable set in $\mathbb{C}_{\exp}$ countable or co-countable?

Zilber [36] showed that the answer to the second question is positive in $\mathbb{B}_{\exp}$. As a consequence, $\mathbb{R}$ would not be definable in $\mathbb{C}_{\exp}$ if $\mathbb{B}_{\exp}$ and $\mathbb{C}_{\exp}$ were shown to be isomorphic.

- ([19]) Does Schanuel's Conjecture imply the strong exponential-algebraic closure property in $\mathbb{C}_{\exp}$, and more generally, in Zilber's fields of any uncountable cardinality?
- ([21] p. 797) Can we find infinitely many algebraically independent fixed points of e^z without using Schanuel's Conjecture?

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