Universität Konstanz
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Real Algebraic Geometry I – Exercise Sheet 11

Exercise 1 (4P). Let $S \subseteq \mathbb{R}^n$ and $y \in S$. The set S is called *star-shaped* relative to y if for all points $x \in S$ the straight line segment between x and y lies in S (i.e., $\operatorname{conv}\{x, y\} \subseteq S$). Prove the following:

(a) Let $K \subseteq \mathbb{R}^n$ be an unbounded set which is closed and star-shaped relative to y. Then K contains a half-line starting from y (i.e., $y + \mathbb{R}_{\geq 0} u \subseteq K$ for some $u \in \mathbb{R}^n \setminus \{0\}$).

(b) Let $f \in \mathbb{R}[\underline{X}]$ be a polynomial with Newton polytope $N(f)$ and $\{\alpha_1, ..., \alpha_m\} = \frac{1}{2}N(f) \cap \mathbb{N}_0^n$. Set $v = (\underline{X}^{\alpha_1} \dots \underline{X}^{\alpha_m})^T$. Show that the *Gram spectrahedron*

$$\{G \in S\mathbb{R}^{m \times m} \mid G \text{ psd}, \ f = v^T G v\}$$

of f is a convex compact subset of $\mathbb{R}^{m \times m} \cong \mathbb{R}^{m^2}$.

Exercise 2 (4P). Let A be a commutative ring and $P \subseteq A$. Show that the following are equivalent:

(a) P is a prime cone of A.

(b) P is a proper preorder of A and for all $a, b \in A$

$$ab \in P \implies (a \in P \text{ or } -b \in P).$$

(c) P is a proper preorder and for all $a, b \in A$

$$ab \in P \implies (a, b \in P \text{ or } -a, -b \in P).$$

Exercise 3 (4P). A commutative ring A is called *real* if $a_1^2 + ... + a_n^2 = 0$ implies $a_1 = 0$ for all $n \in \mathbb{N}$ and $a_1, ..., a_n \in A$. Prove:

(a) If A is real, then so is $S^{-1}A$ for any multiplicative set $S \subseteq A$.

(b) Show that A is real if and only if A is reduced and $A_{\mathfrak{p}}$ is real for all minimal prime ideals $\mathfrak{p}$ of A.

(c) Show that if A is Noetherian, then every ascending sequence of prime cones gets eventually constant.

Exercise 4 (4P). Determine what the maximal prime cones of $A = C([0,1], \mathbb{R})$ are.

Hint: Show first that if $I \subseteq A$ is a prime ideal, the set $\{x \in [0,1] \mid \forall f \in I : f(x) = 0\}$ contains exactly one element.

Please submit until Thursday, January 26, 2017, 11:44 in the box named RAG I, Number 10, near to the room F411.