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Real Algebraic Geometry I – Exercise Sheet 8

Exercise 1 (4P). Let (K, P) be an ordered field with real closure R. Fix a class $\mathscr{R}$ of real closed extension fields of R. Show that all R-semialgebraic classes in $\mathscr{R}_n$ are K-semialgebraic classes as well.

Exercise 2 (6P). Which of the following statements are valid for all real closed fields R? Give a proof or a counterexample!

(a) Let $n \in \mathbb{N}$. Every polynomial in $R[X_1, \dots, X_n]$ attains a minimum on

$$\left\{ x \in R^n \mid \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

(b) In R we have $\lim_{n\to\infty} \sqrt[n]{n} = 1$, where for every $n \in \mathbb{N}$ the notation $\sqrt[n]{n}$ stands for the $x \in R_{>0}$ with $x^n = n$ which exists and is unique by Descartes' rule of signs.

(c) Let $f \in R[X]$ and $a, b \in R$. Then there are $c, d \in R$ with $\{f(x) \mid x \in [a, b]_R\} = [c, d]_R$.

(d) $\forall n \in \mathbb{N} : \forall x \in R : \forall \varepsilon \in R_{>0} : \exists \delta \in R_{>0} : \forall y \in R \setminus \{x\} :$

$$\left(|x - y| < \delta \implies \left| \frac{x^n - y^n}{x - y} - nx^{n-1} \right| < \varepsilon \right)$$

Exercise 3 (6P+3BP). Let R be a real closed field and $n \in \mathbb{N}_0$. Consider the"distance function" $d\colon R^n \to R, x \mapsto \sum_{i=1}^{n} x_i^2$ and set $B(x, \varepsilon) = \{y \in R^n \mid d(x, y) < \varepsilon\}$ for $x \in R^n$ and $\varepsilon \in R$. Which of the following sets are semialgebraic for all semialgebraic $S \subseteq R^n$?

(a) The *interior* $\mathring{S} := \{x \in R^n \mid \exists \varepsilon \in R_{>0} : \; B(x, \varepsilon) \subseteq S\}$ of S.

(b) The affine hull

$$\operatorname{aff}(S) := \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N}, \; x_1, \dots, x_k \in S, \; \lambda_1, \dots, \lambda_k \in R, \; \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

(c) The set

$$\Sigma(S) := \left\{ \sum_{i=1}^{k} x_i \mid k \in \mathbb{N}, \; x_1, \dots, x_k \in S \right\}.$$

(d) Every *spectrahedron* $\{x \in \mathbb{R}^n \mid L(x) \text{ has signature } m\}$ where $L \in S(R[X_1, ..., X_n]_1)^{m\times m}$ (i.e. $L \in R[X_1, ..., X_n]_1^{m\times m}$ and $L = L^T$)

(e) (Bonus) The R-Zariski closure of S in R^n

(f) (Bonus) The convex hull of S

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ x_1, \ldots, x_k \in S,\ \lambda_1, \ldots, \lambda_k \in R_{\geq 0},\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Please submit until Thursday, December 22, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.