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Geometry of Linear Matrix Inequalities – Exercise Sheet 2

Exercise 1 (2P) For a matrix $A \in \mathbb{R}^{k\times k}$, we write $A \succeq 0$ to denote that A is psd, i.e., $A \in S\mathbb{R}^{k\times k}$ [$\to$ 1.6.1(c)] and $x^T Ax \geq 0$ for all $x \in \mathbb{R}^k$ [$\to$ 2.3.1(b)]. A *spectrahedron* in $\mathbb{R}^n$ is a set of the form

$$S = \left\{ x \in \mathbb{R}^n \mid A_0 + \sum_{i=1}^n x_i A_i \succeq 0 \right\}$$

for some $k \in \mathbb{N}_0$ and $A_0, \dots, A_n \in S\mathbb{R}^{k\times k}$ [$\to$ 1.6.1(c)]. A cone in $\mathbb{R}^n$ that is a spectrahedron is called a *spectrahedral cone*. Fix $S \subseteq \mathbb{R}^n$. Show that the following are equivalent:

(a) S is a spectrahedral cone.

(b) There is $k \in \mathbb{N}_0$ and $A_1, \dots, A_n \in S\mathbb{R}^{k\times k}$ such that

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i A_i \succeq 0 \right\}$$

Exercise 2 (4P) Show that the map $S\mathbb{R}^{k\times k} \to \mathbb{R}^k$ sending $A \in S\mathbb{R}^{k\times k}$ to $(\lambda_1, \dots, \lambda_k)$ whenever $\det(XI_k - A) = \prod_{i=1}^k (X - \lambda_i)$ with $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ with $\lambda_1 \geq \dots \geq \lambda_k$ is continuous.

Exercise 3 (4P) Suppose $S \subseteq \mathbb{R}^n$ is compact, $x_1, \dots, x_k \in S^\circ$ are pairwise distinct, $u \in \mathbb{R}[\underline{X}]$ is defined as in Theorem 9.1.12, $f \in \mathbb{R}[\underline{X}]$ and $f(x_1) = \dots = f(x_k) = 0$. Show again [$\to$ 9.2.5] that the following are equivalent, this time only using basic multivariate analysis:

(a) $f > 0$ on $S \setminus \{x_1, \dots, x_k\}$ and $\operatorname{Hess} f(x_1), \dots, \operatorname{Hess} f(x_k)$ are pd.

(b) There is some $\varepsilon \in \mathbb{R}_{>0}$ such that $f \geq \varepsilon u$ on S.

Exercise 4 (4P) Let $g_1, \dots, g_m \in \mathbb{R}[\underline{X}]$ such that $M(g_1, \dots, g_m)$ is Archimedean. Set $S := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$. Fix $\varepsilon > 0$ and set $B := \{x \in \mathbb{R}^n \mid \|x\| \leq \varepsilon\}$. Using only Putinar's Positivstellensatz (Example 8.2.14) but not the degree bounds for it (Corollary 9.2.4), show that there is $N \in \mathbb{N}$ such that

$$S \subseteq \{x \in \mathbb{R}^n \mid \forall f \in \mathbb{R}[\underline{X}]_1 \cap M_N(g_1, \dots, g_m) : \; f(x) \geq 0\} \subseteq S + B.$$

Exercise 5 (6P)

(a) Suppose $k, n \in \mathbb{N}_0$, $A_1, \dots, A_n \in S\mathbb{R}[\underline{X}]^{k \times k}$ and

$$S := \{x \in \mathbb{R}^n \mid I_n + A_1 x_1 + \ldots + A_n x_n \succeq 0\}$$

is compact. Show that $A_1, \dots, A_n$ are linearly independent.

(b) For fixed $k \in \mathbb{N}$, determine the largest $n \in \mathbb{N}_0$ for which there exist $A_1, \dots, A_n \in S\mathbb{R}[\underline{X}]^{k \times k}$ such that $\{x \in \mathbb{R}^n \mid I_n + A_1 x_1 + \ldots + A_n x_n \succeq 0\}$ is compact.

Please submit until Tuesday, July 18, 2017, 9:55 in the box named RAG II near to the room F411.