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Real Algebraic Geometry II – Exercise Sheet 5

Exercise 1 (4P) Let $f :=X^4Y^2 + X^2Y^4 - 3X^2Y^2 + 2 \in \mathbb{R}[X, Y]$ (cf. Example 2.4.15).

(a) Show that for all $R \in \mathbb{R}$ there exist $s, t \in \sum \mathbb{R}[X, Y]^2$ with

$$(*) \qquad f = s + t(R^2 - (X^2 + Y^2)).$$

(b) Show that there is no $D \in \mathbb{N}$ such that for every $R \in \mathbb{R}$ there exist $s, t \in \sum \mathbb{R}[X, Y]_D^2$ with $(*)$.

Exercise 2 (4P) ("intermediate value theorem for semialgebraic functions")
Suppose R is a real closed field, $f : R \to R$ a semialgebraic function and $a, b \in R$ such that $a \leq b$ and $\operatorname{sgn}(f(a)) \neq \operatorname{sgn}(f(b))$. Show that there is $c \in [a, b]_R$ with $f(c) = 0$.

Exercise 3 (4P+4BP) Let R be a real closed field and $f : R \to R$ semialgebraic. Show that there is a finite set $E \subseteq R$ such that the restriction of f onto $R \setminus E$ is continuous. Find a proof by using one of the two strategies suggested in the following hint (for the bonus points follow both strategies).

Hint: Justify that if is enough to consider the case $R = \mathbb{R}$. Argue that it suffices to show that every nonempty open set $A \subseteq R$ contains a point x where f is continuous (in the usual sense, i.e., the preimage of every neighborhood of $f(x)$ is a neighborhood of x).

Strategy 1: First consider the easy case in which where exists a non-empty open set $B \subseteq A$ such that $\#f(B) < \infty$. In the remaining case find an appropriate point x via nested intervals.

Strategy 2: Find an infinite compact interval $I \subseteq A$ and $g, h_1, \ldots, h_m \in \mathbb{R}[X, Y]$ such that

$$\Gamma_f \cap (I \times \mathbb{R}) = \{(x, y) \in I \times \mathbb{R} \mid g(x, y) = 0, h_1(x, y) > 0, ..., h_m(x, y) > 0\}.$$

Argue that we can shrink I in such a way that there is an $\varepsilon > 0$ such that $\Gamma_f \cap (I \times \mathbb{R})$ is bounded and such that there is $\varepsilon \in \mathbb{R}_{>0}$ with

$$\Gamma_f \cap (I \times \mathbb{R}) = \{(x, y) \in I \times \mathbb{R} \mid g(x, y) = 0, h_1(x, y) \geq \varepsilon, ..., h_m(x, y) \geq \varepsilon\}$$

so that $\Gamma_f \cap (I \times \mathbb{R})$ is compact. Now every point in the interior of I can be taken as the desired point x.

Exercise 4 (4P) Let R be a real closed field and $f : R \to R$ a semialgebraic function. Show that there exists a finite set $E \subseteq R$ such that for all $a, b \in R$ with $a < b$ for which $I := (a, b)_R \subseteq R \setminus E$, we have that $f|_I$ is continuous and exactly one of the following holds:

(a) $f|_I$ is constant,

(b) $f|_I$ injective and monotonic,

(c) $f|_I$ is injective and anti-monotonic

Hint: Let $a, b, c, d \in R$ with $a < b$ and $c < d$ and suppose that $g\colon [a, b] \to R$ is a continuous semialgebraic function with $[c, d] \subseteq g([a, b])$. Let $e = f(a)$ and show that

$$h\colon [c, d] \to [a, b], y \mapsto \min\{x \in [a, b] \mid g(x) = y\}$$

is monotonic or anti-monotonic on both $[c, e]$ and $[e, d]$. What does this information tell us about monotonicity properties of g?

Exercise 5 (4P) Let

$$A := \{(x, y) \in \mathbb{R}^2 \mid x > 1, y > 1\} \qquad \text{and}$$
$$B := \{(x, y) \in \mathbb{R}^2 \mid x < -1 \text{ or } y < -1\}.$$

Show that there exists no $f \in \mathbb{R}[X, Y]$ with $f \geq 1$ on A and $f \leq -1$ on B.

Please submit until Tuesday, May 30, 2017, 9:55 in the box named RAG II near to the room F411.