Universität Konstanz
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Real Algebraic Geometry II – Exercise Sheet 8

Exercise 1 (8P) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a semialgebraic function and R be a real closed extension field of $\mathbb{R}$.

(a) Show that $\operatorname{Transfer}_{\mathbb{R},R}(\Gamma_f) \subseteq R^2$ equals the graph Γ_g of an $\mathbb{R}$-semialgebraic function $g\colon R \to R$.

(b) Let $x \in \mathbb{R}$ and $\delta \in \mathfrak{m}_R \setminus \{0\}$. Show that f is continuous at x if and only if $g(x-\delta), g(x+\delta) \in \mathcal{O}_R$ and

$$\operatorname{st}(g(x-\delta)) = f(x) = \operatorname{st}(g(x+\delta)).$$

(c) Let $x \in \mathbb{R}$, $\delta \in \mathfrak{m}_R \setminus \{0\}$ and $a \in \mathbb{R}$. Show that f is differentiable at x with $f'(x) = a$ if and only if $\frac{g(x-\delta)-g(x)}{-\delta}, \frac{g(x+\delta)-g(x)}{\delta} \in \mathcal{O}_R$ and

$$\operatorname{st}\left(\frac{g(x-\delta)-g(x)}{-\delta}\right) = a = \operatorname{st}\left(\frac{g(x+\delta)-g(x)}{\delta}\right).$$

Exercise 2 (4P) Let $A \subseteq \mathbb{R}^n$ be closed and convex. Show that the following are equivalent for $x \in A$:

(a) x is an extreme point of A.

(b) For every $\varepsilon > 0$, there exists a linear function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that $\varphi(x) > c$ and $\forall y \in A : (\varphi(y) > c \implies \|x-y\| < \varepsilon)$.

Hint: Consider the case where A is bounded first.

Exercise 3 (4P) Let $A \subseteq \mathbb{R}^n$ be convex. An *exposed extreme point* of A is a point $x \in A$ such that $\{x\}$ is an exposed face of A. Now suppose that A is compact. Show that the closure of the set of exposed extreme points of A contains all extreme points of A.

Hint: Consider $x \in \operatorname{extr} A$. Let $\varepsilon > 0$ be arbitrary. We want to find an exposed extreme point z of A with $\|z-x\| < \varepsilon$. For this purpose, choose $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $w^T x > c$ and $\forall y \in A : (w^T y > c \implies \|x-y\| < \varepsilon)$. Now show that for $\lambda \in \mathbb{R}$ big enough, every point $z \in A$ maximizing $\|z-(x-\lambda w)\|$ does the job.

Please submit until Tuesday, June 20, 2017, 9:55 in the box named RAG II near to the room F411.