POSITIVE POLYNOMIALS LECTURE NOTES (13: 27/05/10)

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Contents

1. ARCHIMEDEAN MODULES

Let A be a commutative ring, $Q \subseteq A$, T a preprime.

Definition 1.1. Let M a T-module. M is **archimedean** if:

$$\forall\, a \in A, \exists\, N \geq 1, N \in \mathbb{Z}_+ \text{ s.t. } N + a, N - a \in M\ .$$

Proposition 1.2. Let T be a generating preprime, M a maximal proper T-module. Assume that M is archimedean. Then $\exists$ a uniquely determined $\alpha \in \mathrm{Hom}(A, \mathbb{R})$ s.t. $M = \alpha^{-1}(\mathbb{R}_+) = P_\alpha$.
(In particular, M is an ordering, not just a semi-ordering.)

Proof. Let $a \in A$, define:

cut $(a) = \{r \in \mathbb{Q} \mid r - a \in M\}$, this is an **upper cut** in $\mathbb{Q}$ (i.e. final segment of $\mathbb{Q}$) .

Claim 1: $\mathrm{cut}(a) \neq \emptyset$ and $\mathbb{Q}\backslash(\mathrm{cut}(a)) := \mathrm{L}(a) \neq \emptyset$, where $\mathrm{L}(a)$ is a **lower cut** in $\mathbb{Q}$.

Proof of claim 1. Since M is archimedean $\exists\, n \geq 1$ s.t. $n - a \in M$, so $\mathrm{cut}(a) \neq \emptyset$.

Also $\exists\, m \geq 1$ s.t. $(m + a) \in M$.

If $-(m + 1) - a \in M$, then adding we get $-1 \in M$, a contradiction (since M is proper). So we have $-(m + 1) - a \notin M$, which $\Rightarrow -(m + 1) \in Q\backslash(\mathrm{cut}(a)) = \mathrm{L}(a)$.
□(claim 1)

Now define a map $\alpha : A \longrightarrow \mathbb{R}$ by

$$\alpha(a) := \inf\,(\mathrm{cut}(a))$$

α is well-defined.

Claim 2: $\alpha(1) = 1,\ \alpha(M) \subseteq \mathbb{R}_+;\ \alpha(a \pm b) = \alpha(a) \pm \alpha(b)\ \forall\ a, b \in A$ and $\alpha(tb) = \alpha(t)\ \alpha(b)\ \forall\ t \in T, b \in A$.

This is left as an exercise.

Claim 3: $\alpha(ab) = \alpha(a)\ \alpha(b)\ \forall\ a, b \in A$

Proof of claim 3. T generating $\Rightarrow a = t_1 - t_2,\ t_1, t_2 \in T$

so, $\alpha(ab) = \alpha(t_1 b - t_2 b) = \alpha(t_1 a) - \alpha(t_2 b)$

$$= \alpha(t_1)\alpha(b) - \alpha(t_2)\alpha(b) \text{ [by claim 2]}$$
$$= (\alpha(t_1) - \alpha(t_2))\alpha(b) = \alpha(t_1 - t_2)\alpha(b) = \alpha(a)\alpha(b)\ .$$

□(claim 3)

Claim 4: $\alpha^{-1}(\mathbb{R}_+) = M$

Proof of claim 4. By Claim 2, $M \subseteq \alpha^{-1}(\mathbb{R}_+)$

so, by maximality of M and since $P_\alpha = \alpha^{-1}(\mathbb{R}_+)$ is an ordering it follows that $M = \alpha^{-1}(\mathbb{R}_+)$. □

Corollary 1.3. Let A be a commutative ring with 1, T an archimedean preprime, M a T-module, $-1 \notin M$ (i.e. M proper T-module). Then $\chi_M \neq \emptyset$.

Proof. Since T is archimedean, T is generating (because $a = (n + a) - n$, for $a \in A$) and M is a proper archimedean module (archimedean module because for an archimedean preprime T, every T-module is also archimedean). By Zorn's lemma extend M to a maximal proper archimedean module Q. Apply Proposition 1.2 to Q to get $\alpha \in \mathrm{Hom}(A, \mathbb{R})$ such that $Q = \alpha^{-1}(\mathbb{R}_+)$. This implies $M \subseteq \alpha^{-1}(\mathbb{R}_+)$. So, $\alpha \in \chi_M$, which implies $\Rightarrow\ \chi_M \neq \emptyset$. □

2. REPRESENTATION THEOREM (STONE-KRIVINE, KADISON-DUBOIS)

The following corollary (to Proposition 1.2 and Corollary 1.3) answers the question raised in 2.4 of lecture 12:

Corollary 2.1. (Stone-Krivine, Kadison-Dubois) Let A be a commutative ring, T an archimedean preprime in A, M a proper T-module. Let $a \in A$ and

$$\hat{a} : \chi \to \mathbb{R} \quad \text{defined by}$$
$$\hat{a}(\alpha) := \alpha(a)$$

If $\hat{a} > 0$ on χ_M, then $a \in M$.

Proof. Assume $\hat{a} > 0$ on χ_M, i.e. $\hat{a}(\alpha) > 0 \;\forall\; \alpha \in \chi_M$.

<u>To show</u>: $a \in M$

- Consider $M_1 := M = aT$

  Since $\alpha(a) > 0 \;\forall\; \alpha \in \chi_M$, we have $\chi_{M_1} = \emptyset$ [because if $\alpha \in \chi_{M_1}$, then $\alpha(M_1) \subseteq \mathbb{R}_+$. So, $\alpha(-a) = -\alpha(a) \geq 0$. So, $\alpha(a) \leq 0$, but $\alpha \in \chi_M$ so $\alpha(a) > 0$, a contradiction].

  So (since M_1 is an archimedean T-module), we can apply Corollary 1.3 to M_1 to deduce that $-1 \in M_1$.

  Write $-1 = s - at$, $s \in M, t \in T$

  $\Rightarrow \; at - 1 = s \in M$ $\qquad (\star)$

- Consider $\sum := \{r \in \mathbb{Q} \mid r + a \in M\}$

  We **claim** that: $\exists\, \rho \in \sum;\ \rho < 0$

  Once the claim is established we are done (with the proof of corollary) because

  $a = \underbrace{(a+\rho)}_{\in M} + \underbrace{(-\rho)}_{\in M} \in M$.

  <u>Proof of the claim</u>: First observe that $\sum \neq \emptyset$ (since $\exists\, n \geq 1$ s.t $n + a \in T \subseteq M$, so $n \in \sum$).

  Now fix $r \in \sum$, $r \geq 0$ and fix an integer $k \geq 1$ s.t $(k - t) \in T$

  Write: $kr - 1 + ka = \underbrace{(k-t)}_{\in T}\underbrace{(r+a)}_{\in M} + \underbrace{(at-1)}_{\in M} + \underbrace{rt}_{\in M} \in M$

  Multiplying by $\dfrac{1}{k}$, we get

  $\left(r - \dfrac{1}{k}\right) + a \in M$, i.e. $\left(r - \dfrac{1}{k}\right) \in \sum$

  Repeating we eventually find $\rho \in \sum$, $\rho < 0$. $\square$

Note 2.2. For a quadratic module $M \subseteq \mathbb{R}[\underline{X}]$, set

$$K_M := \{x \in \mathbb{R}^n \mid g(x) \geq 0 \;\forall\; g \in M\}.$$

Note that if $M = M_S$ with $S = \{g_1, \ldots, g_s\}$, then $K_S = K_M$.

We have the following corollaries to Corollary 2.1. (Stone-Krivine, Kadison-Dubois):

Corollary 2.3. (Putinar's Archimedean Positivstellensatz) Let $M \subseteq \mathbb{R}[\underline{X}]$ be an archimedean quadratic module. Then for each $f \in \mathbb{R}[\underline{X}]$:

$$f > 0 \text{ on } K_M \Rightarrow f \in M\,.$$

Corollary 2.4. Let $A = \mathbb{R}[\underline{X}]$ and $S = \{g_1, \ldots, g_s\}$. Assume that the finitely generated preordering T_S is archimedean. Then for all $f \in A$:

$$f > 0 \ \text{ on } K_S \Rightarrow f \in T_S\,.$$

Remark 2.5.

1. To apply the corollary we need a criterion to determine when a preordering (quadratic module) is archimedean.
2. T_S is archimedean $\Rightarrow$ for $f = \sum X_i^2$: $\exists\, N$ s.t. $N - f = N - \sum X_i^2 \in T_S$

   $\Rightarrow N - \sum X_i^2 \geq 0$ on K_S.

   $\Rightarrow K_S$ is bounded. Also K_S is closed.

   So T_S is archimedean implies K_S is compact.