POSITIVE POLYNOMIALS LECTURE NOTES (14: 01/06/10)

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Contents

1. Rings of bounded elements

Let A be a commutative ring with 1, $\mathbb{Q} \subseteq A$ and M be a quadratic module $\subseteq A$.

Definition 1.1. Consider

$$B_M = \{a \in A \mid \exists\, n \in \mathbb{N} \text{ s.t. } n + a \text{ and } n - a \in M\},$$

B_M is called the **ring of bounded elements**, which are bounded by M.

Proposition 1.2.

(1) M is an archimedean module of A iff $B_M = A$.

(2) B_M is a subring of A.

(3) $\forall\, a \in A$, $a^2 \in B_M \Rightarrow a \in B_M$.

(4) More generally, $\forall\, a_1, \ldots, a_k \in A$, $\sum_{i=1}^{k} a_i^2 \in B_M \Rightarrow a_i \in B_M\ \forall\, i = 1, \ldots, k$.

Proof. (1) Clear.

(2) Clearly $\mathbb{Q} \subseteq B_M$ and B_M is an additive subgroup of A.

To show: $a, b \in B_M \Rightarrow ab \in B_M$

Using the identity

$$ab = \frac{1}{4}[(a+b)^2 - (a-b)^2],$$

we see that in order to show that B_M is closed under multiplication it is sufficient to show that: $\forall\, a \in A:\ a \in B_M \Rightarrow a^2 \in B_M$.

Let $a \in B_M$. Then $n \pm a \in M$ for some $n \in \mathbb{N}$. Now $n^2 + a^2 \in M$.

Also $2n(n^2 - a^2) = (n^2 - a^2)[(n + a) + (n - a)]$

$$\text{So, } (n^2 - a^2) = \frac{1}{2n}\Big[(n + a)(n^2 - a^2) + (n - a)(n^2 - a^2)\Big]$$
$$= \frac{1}{2n}\Big[(n + a)^2(n - a) + (n - a)^2(n + a)\Big] \in M.$$

So $(n^2 + a^2)$ and $(n^2 - a^2)$ both $\in M$. So by definition $a^2 \in B_M$. □ (2)

(3) Assume $a^2 \in B_M$. Say $n - a^2 \in M$, for $n \geq 1, n \in \mathbb{N}$, then use the identity:

$$(n \pm a) = \frac{1}{2}[(n - 1) + (n - a^2) + (a \pm 1)^2] \in M.$$

So, $a \in B_M$. □ (3)

(4) If $\sum a_i^2 \in B_M$. Say $(n - \sum a_i^2) \in M$, then

$$(n - a_i^2) = \Big(n - \sum a_i^2\Big) + \sum_{j \neq i} a_j^2 \in M.$$

So, $a_i^2 \in B_M$ and so by (3), $a_i \in B_M$. □ (4)

□

Corollary 1.3. Let M be a quadratic module of $\mathbb{R}[\underline{X}]$. Then M is archimedean iff there exists $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^{n} X_i^2 \in M$$

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) First note that $\mathbb{R}_+ \subseteq M$ so, $\mathbb{R} \subseteq B_M$ (B_M subring).

Also $N - \sum_{i=1}^{n} X_i^2$ and $N + \sum_{i=1}^{n} X_i^2 \in M$. Therefore by definition $\sum_{i=1}^{n} X_i^2 \in B_M$.

So (by Proposition 1.2) $X_1, \ldots, X_n \in B_M$. This implies $\mathbb{R}[X_1, \ldots, X_n] \subseteq B_M$ and so M is archimedean. □

2. SCHMÜDGEN'S POSITIVSTELLENSATZ

Theorem 2.1. Let $S = \{g_1, \ldots g_s\} \subseteq \mathbb{R}[\underline{X}]$. Assume that $K = K_S = \{\underline{x} \mid g_i(\underline{x}) \geq 0\}$ is compact. Then there exists $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^{n} X_i^2 \in T_S = T.$$

In particular T_S is an archimedean preordering (by Corollary 1.3) and thus $\forall f \in \mathbb{R}[\underline{X}]$: $f > 0$ on $K_S \Rightarrow f \in T_S$.

Proof. [Reference: Dissertation, Thorsten Wörmann]

- K compact $\Rightarrow$ K bounded $\Rightarrow$ $\exists\, k \in \mathbb{N}$ such that $\left(k - \sum_{i=1}^{n} X_i^2\right) > 0$ on K.
- By applying the Positivstellensatz to above we get: $\exists\, p, q \in T_S$ such that $p\left(k - \sum_{i=1}^{n} X_i^2\right) = 1 + q$. So, $p\left(k - \sum_{i=1}^{n} X_i^2\right)^2 = (1+q)(k - \sum_{i=1}^{n} X_i^2\big)$. So, $(1+q)\left(k - \sum_{i=1}^{n} X_i^2\right) \in T_S$.
- Set $T' = T + \left(k - \sum_{i=1}^{n} X_i^2\right)T$. By Corollary 1.3, T' is an archimedean preordering. Therefore $\exists\, m \in \mathbb{N}$ such that $(m-q) \in T'$; say: $m - q = t_1 + t_2\left(k - \sum_{i=1}^{n} X_i^2\right)$ for some $t_1, t_2 \in T$.
- So, $(m-q)(1+q) = t_1(1+q) + t_2\left(k - \sum_{i=1}^{n} X_i^2\right)(1+q) \in T_S$. So $(m-q)(1+q) \in T_S$.
- Adding

$$(m-q)(1+q) = mq - q^2 + m - q \in T_S, \tag{1}$$

$$\left(\frac{m}{2} - q\right)^2 = \frac{m^2}{4} + q^2 - mq \in T_S. \tag{2}$$

yields

$$\left(m + \frac{m^2}{4} - q\right) \in T_S. \tag{3}$$

- Multiplying L.H.S. of (3) by $k \in T_S$, and adding $\left(k - \sum_{i=1}^{n} X_i^2\right)(1+q) \in T_S$ and $q\left(\sum_{i=1}^{n} X_i^2\right) \in T_S$, yields

$$k\left(m + \frac{m^2}{4} - q\right) + \left(k - \sum_{i=1}^{n} X_i^2\right)(1+q) + q\left(\sum_{i=1}^{n} X_i^2\right) \in T_S$$

i.e. $km + k\frac{m^2}{4} + k - \sum_{i=1}^{n} X_i^2 \in T_S$

i.e. $k\left(\frac{m}{2} + 1\right)^2 - \sum_{i=1}^{n} X_i^2 \in T_S$

Set $N := k\left(\frac{m}{2} + 1\right)^2$. □

(End of Schmüdgen's Positivstellensatz)

2.2. Final Remarks on Schmüdgen's Positivstellensatz (SPSS):

1. **Corollary (Schmüdgen's Nichtnegativstellensatz):**
   $f \geq 0$ on $K_S \Rightarrow \forall \epsilon$ real, $\epsilon > 0 : f + \epsilon \in T_S$.

2. SPSS fails in general if we drop the assumption that "K is compact".

   For example:

   (i) Consider $n = 1$, $S = \{X^3\}$, then $K_S = [0, \infty)$ (noncompact). Take $f = X + 1$. Then $f > 0$ on K_S. Claim: $f \notin T_S$, indeed elements of T_S have the form $t_0 + t_1 X^3$, where $t_0, t_1 \in \sum \mathbb{R}[X]^2$. We have shown before at the beginning of this course (in 2.4 of lecture 2) that non zero elements of this preordering either have even degree or odd degree ≥ 3.

   (ii) Consider $n \geq 2, S = \emptyset$, then $K_S = \mathbb{R}^n$. Take strictly positive versions of the Motzkin polynomial

   $$m(X_1, X_2) := 1 - X_1^2 X_2^2 + X_1^2 X_2^4 + X_1^4 X_2^2,$$

   i.e. $m_\epsilon := m(X_1, X_2) + \epsilon$; $\epsilon \in \mathbb{R}_+$. Then $m_\epsilon > 0$ on $K_S = \mathbb{R}^2$, and it is easy to show that $m_\epsilon \notin T_S = \sum \mathbb{R}[\underline{X}]^2 \ \forall \epsilon \in \mathbb{R}_+$.

3. SPSS fails in general for a quadratic module instead of a preordering. [Mihai Putinar's question answered by Jacobi + Prestel in Dissertation of T. Jacobi (Konstanz)]

4. SPSS fails in general if the condition "$f > 0$ on K_S" is replaced by "$f \geq 0$ on K_S".

   Example (Stengle): Consider $n = 1, S = \{(1 - X^2)^3\}$, $K_S = [-1, 1]$ compact. Take $f := 1 - X^2 \geq 0$ on K_S but $1 - X^2 \notin T_S$. (This example has already been considered at the beginning of this course in 2.4 of lecture 2).

5. PSS holds for any real closed field but not SPSS:

   Example: Let R be a non archimedean real closed field. Take $n = 1, S = \{(1-X^2)^3\}$, then $K_S = [-1,1]_R = \{x \in R \mid -1 \leq x \leq 1\}$. Take $f = 1 + t - X^2$, where $t \in R^{>0}$ is an infinitesimal element (i.e. $0 < t < \epsilon$, for every positive rational ϵ). Then $f > 0$ on K_S. We claim that $f \notin T_S$:

   Let v be the natural valuation on R. So $v(t) > 0$ for $t > 0$. Now suppose for a contradiction that $f \in T_S$. Then

   $$1 + t - X^2 = f = t_0 + t_1(1-X^2)^3;\ t_0, t_1 \in \sum R[X]^2 \qquad (1)$$

   Let $t_i = \sum f_{ij}^2$; for $i = 0, 1$ and $f_{ij} \in R[X]$.

   Let $s \in R$ be the coefficient of the lowest value appearing in the f_{ij}, i.e. $v(s) = \min\{v(a) \mid a \text{ is coefficient of some } f_{ij}\}$.

   Case I. if $v(s) \geq 0$, then applying the residue map $\left(\theta_v \longrightarrow \overline{R} := \overline{\dfrac{\theta_v}{\mathcal{I}_v}};\right.$ defined by $x \longmapsto \overline{x}$, where θ_v is the valuation ring $\Big)$ to (1), we obtain

   $$1 - X^2 = \overline{t_0} + \overline{t_1}(1-X^2)^3$$

   and since $\overline{t_i} = \sum \overline{f_{ij}}^2 \in \sum \mathbb{R}[X]^2; i = 0, 1$; we get a contradiction to Example 2.4 (ii) of Lecture 2.

   Case II. if $v(s) < 0$. Dividing f by s^2 and applying the residue map we obtain

   $$0 = \overline{\frac{t_0}{s^2}} + \overline{\frac{t_1}{s^2}}(1-X^2)^3$$

   (Note that $v(s^2) = 2v(s)$ is $\min\{v(a)\}$; a is coefficient of some f_{ij}^2, i.e. $v(s^2) \leq v(a)$ for any coefficient a, so $\dfrac{f_{ij}^2}{s^2}$ has coefficients with value ≥ 0.)

   So we obtain

   $$0 = t_0^{'} + t_1^{'}(1-X^2)^3, \text{ with } t_0^{'}, t_1^{'} \in \sum \mathbb{R}[X]^2 \text{ not both zero.}$$

   Since $t_0^{'}, t_1^{'}$ have only finitely many common roots in $\mathbb{R}$ and $1 - X^2 > 0$ on the finite set $(-1, 1)$, this is impossible. □(claim)

6. SPSS holds over archimedean real closed fields.