POSITIVE POLYNOMIALS LECTURE NOTES (17: 15/06/10)

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Contents

1. HAVILAND'S THEOREM (continued)

Recall Theorem 4.3 of last lecture:

Theorem 1.1. Riesz Representation Theorem:
Let χ be a locally compact Hausdorff space and $L : \mathrm{Cont}_c(\chi, \mathbb{R}) \to \mathbb{R}$ be a positive linear functional i.e. $L(f) \geq 0 \; \forall f \geq 0$ on χ. Then there exists a unique (positive regular) Borel measure μ on χ such that $L(f) = \int\limits_{\chi} f d\mu \quad \forall \; f \in \mathrm{Cont}_c(\chi, \mathbb{R})$, where $\mathrm{Cont}_c(\chi, \mathbb{R}) :=$ the ring ($\mathbb{R}$-algebra) of all continuous functions $f : \chi \to \mathbb{R}$ (addition and multiplication defined pointwise) with compact support i.e. such that the set $\mathrm{supp}(f) := \{x \in \chi : f(x) \neq 0\}$ is compact. □

We shall use theorem 1.1 to prove the following general result. Haviland's theorem (2.5 of lecture 15) will follow as a special case.

Theorem 1.2. Let A be an $\mathbb{R}$-algebra, χ a Hausdorff space and $\hat{} : A \to \mathrm{Cont}_c(\chi, \mathbb{R})$ an $\mathbb{R}$ algebra homomorphism. Assume $\exists \; p \in A$ such that $\hat{p} \geq 0$ on χ and $\forall \; k \in \mathbb{N} : \; \chi_k := \{\alpha \in \chi \mid \hat{p}(\alpha) \leq k\}$ is compact. . . . ($\star$)
Then for any linear functional $L : A \to \mathbb{R}$ satisfying $\forall \; a \in A : \hat{a} \geq 0$ on $\chi \Rightarrow L(a) \geq 0$, $\exists$ a Borel measure μ on χ such that $L(a) = \int\limits_{\chi} \hat{a} \; d\mu \; \forall \; a \in A$.

1.3. Remarks before proof.

1. ($\star$) implies in particular that χ is locally compact (i.e. $\forall \; x \in \chi : \; \exists$ an open neighbourhood $\mathcal{U} \ni x$ such that $\overline{\mathcal{U}}$ is compact).

Proof. Let $x \in \chi$, fix $k \geq 1$ s.t. $\hat{p}(x) < k$

Set $\mathcal{U}_k := \{y \in \chi \mid \hat{p}(y) < k\}$

$\subseteq \{y \in \chi \mid \hat{p}(y) \leq k\} = \chi_k$

$\mathcal{U}_k$ is open , $x \in \mathcal{U}_k$; $\overline{\mathcal{U}_k} \subseteq \chi_k$; so $\overline{\mathcal{U}_k}$ is compact.

[$\chi_k = \hat{p}^{-1}((-\infty, k])$ being inverse image of closed set under continuous map is closed but not necessarily compact, and $\mathcal{U}_k = \hat{p}^{-1}((-\infty, k))$ being inverse image of open set under continuous map is open.] □

2. Haviland's Theorem is a corollary (to Theorem 1.2) if we set $\chi = K$ closed subset of $\mathbb{R}^n, A = \mathbb{R}[\underline{X}]$, and

$$\hat{}: \mathbb{R}[\underline{X}] \to \text{Cont}(K, \mathbb{R});$$

$$f \mapsto \hat{f} \text{ (restriction of the polynomial function f to } K)$$

$\hat{p}(x) = \sum x_i^2 = ||\underline{x}||^2, \chi_k$ compact.

1.4. *Proof of Theorem 1.2.* Set $C(\chi) = \text{Cont}(\chi, \mathbb{R})$ and $C_c(\chi) = \text{Cont}_c(\chi, \mathbb{R})$.
Let $\hat{A} := \{\hat{a} \mid a \in A\}$ (the image under the $\mathbb{R}$-algebra homomorphism $\hat{}$ is a subalgebra).
Define $\mathcal{B}(\chi) \subseteq C(\chi)$ to be the following subalgebra of $C(\chi)$:

$$\mathcal{B}(\chi) := \{f \in C(\chi) \mid \exists\, a \in A : |f| \leq |\hat{a}| \text{ on } \chi\}.$$

Observe that $\mathcal{B}(\chi)$ is a subalgebra of $C(\chi)$ and $\hat{A} \subseteq \mathcal{B}(\chi) \subseteq C(\chi)$.

Claim 1: $C_c(\chi) \subseteq \mathcal{B}(\chi)$

Proof of Claim 1. Let $f \in C_c(\chi)$, f continuous and $\overline{\{x \in \chi : f(x) \neq 0\}}$ compact subset. Then $|f| \leq k$, for some $k \in \mathbb{N}; k \in A$, i.e. $|f| \leq \hat{k}$ on χ.
So $C_c(\chi) \subseteq \mathcal{B}(\chi)$ as claimed i.e. $C_c(\chi)$ is a subalgebra of $\mathcal{B}(\chi)$. □(Claim 1)

Let now as in the hypothesis of the theorem:
$L : A \to \mathbb{R}$ with $L(a) \geq 0$ if $\hat{a} \geq 0$ on $\chi, \forall\, a \in A$.

We define $\bar{L} : \hat{A} \to \mathbb{R}$, by $\bar{L}(\hat{a}) := L(a)$.

Claim 2: $\bar{L}$ is a well defined linear function.

Proof of Claim 2. Since $\bar{L}(\hat{a} + \hat{b}) = \bar{L}(\widehat{a+b}) = L(a + b)$, so it is enough to prove that: $\hat{a} = 0 \Rightarrow L(a) = 0$

Now $\hat{a} \geq 0 \Rightarrow L(a) \geq 0$, and $-\hat{a} \geq 0 \Rightarrow -L(a) = L(-a) \geq 0$; (together) $\Rightarrow L(a) = 0$. □(Claim 2)

Claim 3: $\bar{L}$ extends to a linear map:

$$\bar{L} : \mathcal{B}(\chi) \to \mathbb{R} \text{ with } \bar{L}(f) \geq 0 \text{ for } f \geq 0 \text{ on } \chi.$$

Proof of Claim 3. We use Zorn's lemma to prove this:
Consider the collection of all pairs $(B, \bar{L})$, where B is a $\mathbb{R}-$subspace of $\mathcal{B}(\chi)$ containing $\hat{A}$ and $\bar{L}$ is an extension of $\bar{L}$ (on A) with the property:

$$\forall f \in B : f \geq 0 \text{ on } \chi \ \Rightarrow \bar{L}(f) \geq 0 \qquad \ldots (\dagger)$$

and consider a partial order: $(B_1, \bar{L}_1) \subseteq (B_2, \bar{L}_2) :\Leftrightarrow B_1 \subseteq B_2$ and $\bar{L}_2|_{B_1} := \bar{L}_1$.

- this collection is nonempty since $(\hat{A}, \bar{L})$ belongs to it : $\hat{a} \geq 0$ on $\chi \Rightarrow \bar{L}(\hat{a}) = L(a) \geq 0$ (by definition)
- every chain has an upper bound
- Let $(B, \bar{L})$ be a maximal element.
  Subclaim: we claim that $B = \mathcal{B}(\chi)$
  Otherwise let $g \in \mathcal{B}(\chi) \setminus B$.
  If $f_1, f_2 \in B$ s.t. $f_1 \leq g$ and $g \leq f_2$ on χ, then $f_1 \leq f_2$ on χ so $\bar{L}(f_1) \leq \bar{L}(f_2)$.
  So we consider the following sets of reals
  $\mathcal{U} := \{\bar{L}(f_1) \mid f_1 \in B, f_1 \leq g \text{ on } \chi\} \leq \{\bar{L}(f_2) \mid f_2 \in B, g \leq f_2 \text{ on } \chi\} =: \theta$
  Note that these sets $\mathcal{U}, \theta$ are nonempty, i.e. f_1, f_2 exist.
  [e.g. let $a \in A$ s.t. $|g| \leq |\hat{a}|$ on χ
  now $(\hat{a} \pm 1)^2 \geq 0$, so $|\hat{a}| \leq \dfrac{\hat{a}^2 + 1}{2} \in \hat{A}$
  so take $f_1 := -\dfrac{\hat{a}^2 + 1}{2} \in \hat{A}$; $f_2 := \dfrac{\hat{a}^2 + 1}{2} \in \hat{A}$]
  By completeness of $\mathbb{R}$, let $e \in \mathbb{R}$ s.t.
  $\sup \{\bar{L}(f_1) \mid f_1 \in B, f_1 \leq g\} \leq e \leq \inf \{\bar{L}(f_2) \mid f_2 \in B, g \leq f_2\}$.
  Extend $\bar{L}$ to $B + \mathbb{R}g \subseteq \mathcal{B}(\chi)$ by setting
  $\bar{L}(g) := e$ and $\bar{L}(f + dg) := \bar{L}(f) + de; d \in \mathbb{R}$
  To verify: $\forall f + dg \in B + \mathbb{R}g : f + dg \geq 0 \Rightarrow \bar{L}(f + dg) \geq 0$. (Exercise)
  This will contradict the maximal choice of B and will complete subclaim that $B = \mathcal{B}(\chi)$, and so complete the proof of claim 3. □(Claim 3)

Thus $\bar{L}$ is defined on $\mathcal{B}(\chi)$ and satisfies:

$$\forall f \in \mathcal{B}(\chi) : f \geq 0 \text{ on } \chi \ \Rightarrow \bar{L}(f) \geq 0. \qquad \ldots (\dagger\dagger)$$

In particular $\bar{L}$ is defined on $C_c(\chi)$ and satisfies $(\dagger\dagger)$, i.e. $\bar{L}$ is a positive linear functional on $C_c(\chi)$. So we can apply Riesz Representation Theorem (theorem 1.1) on $\bar{L}$:

$$\exists \ \mu \text{ on } \chi \text{ such that } \bar{L}(f) = \int_{\chi} f d\mu \ \forall f \in C_c(\chi) \subseteq \mathcal{B}(\chi). \qquad \ldots (\dagger\dagger\dagger)$$

Main claim: († † †) holds also $\forall f \in \mathcal{B}(\chi)$, i.e. $\bar{L}(f) = \int\limits_{\chi} f d\mu \ \forall f \in \mathcal{B}(\chi)$.

In particular the proof of the theorem will be completed after proving this since $\hat{A} \subseteq \mathcal{B}(\chi)$, and so for $f = a \in \hat{A} : L(a) \underbrace{=}_{\text{(definition)}} \bar{L}(\hat{a}) = \int\limits_{\chi} \hat{a} d\mu$.

Proof of main claim. Let $f \in \mathcal{B}(\chi)$

Set $f_+ := \max\{f, 0\}, f_- := -\min\{f, 0\}; f = f_+ - f_-$

So, w.l.o.g. we are reduced to the case $f \geq 0$ on χ, $f \in \mathcal{B}(\chi)$.

Set $q := f + \hat{p}$; for $q \in \mathcal{B}(\chi)$.

For each $k \geq 1$, consider $\chi_k^{'} := \{\alpha \in \chi \mid q(\alpha) \leq k\}$

- $\forall k : \chi_k^{'} \subseteq \chi_k$ and $\chi_k^{'}$ is closed. So $\chi_k^{'}$ is compact.
- $\chi_k^{'} \subseteq \chi_{k+1}^{'}$ and $\chi = \bigcup\limits_k \chi_k^{'}$.

Subclaim 1: For each $k \in \mathbb{N} \ \exists f_k \in C_c(\chi)$ such that $0 \leq f_k \leq f$; $f_k = f$ on χ_k' and $f_k = 0$ outside χ_{k+1}'.

Subclaim 2: $\bar{L}(f) = \lim\limits_{k \to \infty} \bar{L}(f_k)$

Note that once they are proved we are done because:

$$\int f d\mu = \lim_{k \to \infty} \int f_k d\mu = \lim_{k \to \infty} \bar{L}(f_k) = \bar{L}(f).$$

We will prove subclaim 1 and 2 in next lecture.

□