POSITIVE POLYNOMIALS LECTURE NOTES (18: 17/06/10)

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Contents

1. HAVILAND'S THEOREM (continued)

We will continue the proof of the following theorem from last lecture. Havilands theorem will follow as a special case.

Theorem 1.1. (Recall 1.2 of last lecture) Let A be an $\mathbb{R}$-algebra, χ a Hausdorff space and $\hat{}: A \to \mathrm{Cont}_c(\chi, \mathbb{R})$ an $\mathbb{R}$ algebra homomorphism. Assume $\exists\ p \in A$ such that $\hat{p} \geq 0$ on χ and $\forall\ k \in \mathbb{N} :\ \chi_k := \{\alpha \in \chi \mid \hat{p}(\alpha) \leq k\}$ is compact.
Then for any linear functional $L : A \to \mathbb{R}$ satisfying $\forall\ a \in A : \hat{a} \geq 0$ on $\chi \Rightarrow L(a) \geq 0$, $\exists$ a Borel measure μ on χ such that $L(a) = \int_\chi \hat{a}\, d\mu\ \forall\ a \in A$.

Proof. We have $C_c(\chi) \subseteq \mathcal{B}(\chi) := \{f \in C(\chi) \mid \exists\ a \in A : |f| \leq |\hat{a}| \text{ on } \chi\}$; $\hat{A} \subseteq \mathcal{B}(\chi)$; $\bar{L} : \hat{A} \to \mathbb{R}$, defined by $\bar{L}(\hat{a}) := L(a)$.
In particular we got (as in claim 3 in 1.4 of last lecture) $\bar{L}$ is a positive linear functional on $C_c(\chi)$ s.t.

$$\bar{L}(f) = \int_\chi f d\mu\ \forall\ f \in C_c(\chi) \subseteq \mathcal{B}(\chi).$$

We **claim** that this holds also $\forall\ f \in \mathcal{B}(\chi)$, i.e. $\bar{L}(f) = \int_\chi f d\mu\ \forall\ f \in \mathcal{B}(\chi)$.

[In particular the proof of the theorem will be completed after proving this since $\hat{A} \subseteq \mathcal{B}(\chi)$, and so for $f = a \in \hat{A} : L(a) \underbrace{=}_{\text{(definition)}} \bar{L}(\hat{a}) = \int_\chi \hat{a} d\mu$.]

Let $f \in \mathcal{B}(\chi)$. Set $q := f + \hat{p}$; for $q \in \mathcal{B}(\chi)$.
For each $k \geq 1$, consider $\chi_k' := \{\alpha \in \chi \mid q(\alpha) \leq k\}$

- $\forall\ k : \chi_k^{'} \subseteq \chi_k$ and $\chi_k^{'}$ is closed. So $\chi_k^{'}$ is compact.
- $\chi_k^{'} \subseteq \chi_{k+1}^{'}$ and $\chi = \bigcup_k \chi_k^{'}$.

Subclaim 1: For each $k \in \mathbb{N}\ \exists\ f_k \in C_c(\chi)$ such that $0 \le f_k \le f$; $f_k = f$ on χ_k' and $f_k = 0$ outside χ_{k+1}'.

Proof of subclaim 1. For this we need Urysohn's lemma, which states that
Let X be a topological space and $A, B \subseteq X$ be closed sets such that $A \cap B = \phi$. Then $\exists\ g \in C(\chi) : g : X \to [0, 1]$ such that $g(a) = 0\ \forall\ a \in A$ and $g(b) = 1\ \forall\ b \in B$.
Applying it with $X = \chi_{k+1}', A = Y_k' = \{\alpha \in \chi_{k+1}' \mid k + \frac{1}{2} \le q(\alpha) \le k + 1\}$, and $B = \chi_k'$, we get $g_k : \chi_{k+1}' \to [0, 1]$ continuous such that $g_k = 0$ on Y_k' and $g_k = 1$ on χ_k'.
Extend g_k to χ by setting $g_k = 0$ on complement of χ_{k+1}'. Set $f_k := f g_k$
Then indeed $0 \le f_k \le f$ on χ , $f_k = f$ on χ_k' and $f_k = 0$ off χ_{k+1}'. In particular Supp$(f) \subseteq \chi_{k+1}'$ is compact (because closed subset of a compact set is compact), so indeed $f_k \in C_c(\chi)$. □(Subclaim 1)

Subclaim 2: $\bar{L}(f) = \lim_{k\to\infty} \bar{L}(f_k)$

Note that once the subclaim 2 is proved we are done because:

$$\int f d\mu = \lim_{k\to\infty} \int f_k d\mu = \lim_{k\to\infty} \bar{L}(f_k) = \bar{L}(f).$$

Proof of subclaim 2. Observe that the inequality

$$\frac{q^2}{k} \ge f - f_k \ge 0 \text{ on } \chi, \text{ holds } \forall\ k \in \mathbb{N}.$$

To see this first of all $f = f_k$ on χ_k', so clearly $\frac{q^2}{k} \ge f - f_k \ge 0$ on χ_k'.

Now we consider the complement of χ_k', there $q(\alpha) > k$ for $\alpha \in$ complement of χ_k'. So

$$\begin{aligned} q^2(\alpha) > kq(\alpha) = k(f(\alpha) + \hat{p}(\alpha)) &\ge kf(\alpha) \\ &\ge k(f(\alpha) - f_k(\alpha)) \text{ [Since } f_k(\alpha) \ge 0\ \forall\ \alpha \in \chi] \end{aligned}$$

Hence $\frac{q^2(\alpha)}{k} \ge (f - f_k)(\alpha)$ for all $\alpha \in (\chi_k')^{\text{compliment}}$.
So,

$$\frac{q^2}{k} \ge f - f_k \ge 0 \text{ on } \chi\ \ \forall\ k \in \mathbb{N}.$$

So,

$$\bar{L}(\frac{q^2}{k}) \ge \bar{L}(f - f_k) \ge 0.$$

Now let $k \to \infty$ to get

$$\lim_{k\to\infty} \bar{L}(\frac{q^2}{k}) = 0 \Rightarrow \lim_{k\to\infty} \bar{L}(f_k) = \bar{L}(f).$$ □(Subclaim 2)

□

2. $\mathbb{R}[\underline{X}]$ AS TOPOLOGICAL $\mathbb{R}$-VECTOR SPACE

Let $A = \mathbb{R}[\underline{X}]$ be a countable dimensional $\mathbb{R}$-algebra.

Every finite dimensional subspace has the Euclidean Topology (ET on $\mathbb{R}^N$: open balls are a basis. If W is a finite dimensional subspace, fix $B = \{w_1, \dots, w_N\}$ basis and get an isomorphism $W \cong \mathbb{R}^N$; pullback the ET from $\mathbb{R}^N$ to W. This topology on W is uniquely determined and does not depend on the choice of the basis because a change of basis results in a linear change of coordinates and linear transformations $\underline{x} \mapsto a\underline{x}$; $\det(A) \neq 0$ are continuous).

Definition 2.1. Define a topology on $A := \mathbb{R}[\underline{X}]$ as:
$U \subseteq A$ is **open** (respectively **closed**) iff $U \cap W$ is open (respectively closed) in W, for every finite dimensional subspace W of A.
This is called **direct limit topology** on A.
Equivalently, take $A_d = \{f \in A | \deg f \leq d\}, d \in \mathbb{Z}_+$. Then $A = \cup_{d\geq 1} A_d$, ask for:
$U \subseteq A$ is open (respectively closed) iff $U \cap A_d$ is open (respectively closed) in A_d for all $d \geq 1$.

We now list the important properties of this topology. We first need to recall the following definitions:

Definition 2.2. (i) $C \subseteq A$ is called a **cone** if C is closed under addition and scalar multiplication by (nonnegetive) positive real numbers.
(ii) $C \subseteq A$ is **convex** if $\forall\, a, b \in C; \forall\, \lambda \in [0, 1] : \lambda a + (1 - \lambda)b \in C$.
Note that a cone is automatically convex.

Theorem 2.3.
1. The open convex sets of A form a basis for the topology, i.e. A is with locally convex topology, i.e. $x \in U$ and U open subset of A $\Longrightarrow$ there is a convex neighbourhood U' of x such that $U' \subseteq U$.
2. This topology is the finest non-trivial locally convex topology on A.

Proof. Later (in next lecture as theorem 1.2). □

Theorem 2.4.
1. A endowed with this topology is a topological $\mathbb{R}$−algebra, i.e. the topology is (Hausdorff) comparable with addition, scalar multiplication and multiplication, i.e.

$+ : A \times A \to A$,
$\times : A \times A \to A$, and
$. \ : \mathbb{R} \times A \to A$
are all continuous.

2. Every linear functional is continuous in this finest locally convex topology.

Proof. Later (1.5 of Lecture 20). □

Theorem 2.5. (Separation Theorem) Let $C \subseteq A$ be a closed cone in A and let $a_0 \in A \setminus C$. Then there is a linear functional $L : A \to \mathbb{R}$ such that $L(C) \geq 0$ but $L(a_0) < 0$.
(Equivalent statement: Let $C \subseteq A$ be a cone and $U \subseteq A$ be an open convex set such that $U \cap C = \phi; U, C \neq \phi$. Then $\exists$ a linear functional $L : A \to \mathbb{R}$ such that $L(U) < 0$ and $L(C) \geq 0$).

Proof. Later (1.8 of Lecture 20). □

Corollary 2.6. For any cone $C \subseteq A$ with $C \neq \phi$, we have

$$\overline{C} = C^{\vee\vee} := \{a \in A \mid L(a) \geq 0 \text{ for any linear functional } L \text{ such that } L(C) \geq 0\}$$
$$= \{a \in A \mid L(a) \geq 0 \ \forall \ L \in C^{\vee}\}.$$

Proof. Clearly $\overline{C} \subseteq C^{\vee\vee}$: since $C \subseteq C^{\vee\vee}$ (from definition), and $C^{\vee\vee}$ is closed (because $L \in C^{\vee}$ is continuous), so $\overline{C} \subseteq C^{\vee\vee}$.
Conversely apply separation theorem (theorem 2.5): if $a_0 \notin \overline{C}$, there exists $L \in C^{\vee}$ (i.e. $L(C) \geq 0$) with $L(a_0) < 0$. So, $a_0 \notin C^{\vee\vee}$. □

Corollary 2.7. Let $A = \mathbb{R}[\underline{X}]$, $M \subseteq A$ be a quadratic module. Then $\overline{M} = M^{\vee\vee}$ and $\overline{M}$ is a quadratic module.

Proposition 2.8. (i) Every cone C is convex.

(ii) Every quadratic module M is a cone.

(iii) If C is a cone, then $\overline{C}$ is a cone.