POSITIVE POLYNOMIALS LECTURE NOTES (19: 22/06/10)

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1. Topology on finite and countable dimensional $\mathbb{R}$-vectorspace 1

1. TOPOLOGY ON FINITE AND COUNTABLE DIMENSIONAL $\mathbb{R}-$VECTOR SPACE

1.1. Helping lemma I. Let V be a countable dimensional $\mathbb{R}$-vectorspace. Let W be a finite dimensional subspace. Fix a basis $w_1, \ldots, w_n$ of W. The map

$$\Phi : \sum r_i w_i \mapsto (r_1, \ldots r_n)$$

defines a vector space isomorphism $W \cong \mathbb{R}^n$.
Let τ the pullback (induced by Φ) topology on W, i.e. a set in (W, τ) is open if it is of the form $\Phi^{-1}(U)$ with $U \subseteq \mathbb{R}^n$ open in the Euclidean topology.
(For simplicity we will write ET for Euclidean topology from now on.)

1. Note that the ET is convex because the open balls form a subbasis for the topology. So τ is locally convex.
2. τ does not depend on the choice of the basis (Hint: a basis change produces a linear change of coordinates i.e. a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ which is continuous in the ET).
3. In particular if $W_1 \subseteq W_2$ are finite dimensional subspace of V, the ET on W_1 is the same as the topology induced by the topology on W_2, i.e. the same as the relative topology.
   ($U_1 \subset W_1$ is open in the ET iff $U_1 \subset W_1$ is open in the relative topology, i.e. U_1 is of the form $U_1 = W_1 \cap U_2$ with U_2 open in W_2.)
   Now define the **finite topology** on V:
   $U \subseteq V$ open iff $U \cap W$ in W is open for any finite dimensional subspace W.

4. Fix a basis $\{v_1, \ldots, v_n \ldots\}$, and set $V_n = \text{Span}\{v_1, \ldots, v_n\}$ a sequence of finite dimensional subspaces such that $V = \cup_i V_i$. We have $V_1 \subseteq \ldots \subseteq V_n \subseteq \ldots$. Then:

   $U \subseteq V$ is open in the finite topology iff $U \cap V_i$ is open in V_i for every i.

   Proof. Clear (Hint: Use the fact that every finite dimensional subspace is contained in a V_i and use 3. in particular.) □

Theorem 1.2. (Theorem 2.3 of last lecture) The open sets in V which are convex form a basis for the topology (i.e. the finite topology is locally convex).

Proof. If V is finite dimensional $\Rightarrow$ ET is convex, so nothing to prove.
So assume without loss of generality V is infinite dimensional. Let $\{v_1, \ldots, v_n, \ldots\}$ be an $\mathbb{R}$ basis for V.
Set V_n =Span$\{v_1, \ldots, v_n\}$. Now let $U \subset V$ be open and $x_0 \in U$.
We show that there exists convex and open $U' \subset U$ such that $x_0 \in U'$.
Since

$$T_{x_0} : V \to V$$
$$v \mapsto v - x_0 \quad \text{are continuous translations,}$$

it suffices to find a convex neighbourhood U'' of 0 with $U'' \subseteq U - x_0$. Then $U' = U'' + x_0$ is the required convex neighbourhood of x_0. In other words we are reduced to the case when $x_0 = 0$.
We proceed (by induction on $n \in \mathbb{N}$) to construct an increasing sequence $C_n \subseteq U \cap V_n$ of convex subsets as follows:

- For $n = 1$: $U \cap V_1$ is open in $V_1 = \mathbb{R}v_1$ and $0 \in U \cap V_1$. So there exists $a_1 \in \mathbb{R}, a_1 > 0$ such that $C_1 := \{y_1 v_1 \mid -a_1 \leq y_1 \leq a_1\} := [-a_1, a_1] \subseteq U \cap V_1$.
- By induction on $n \in \mathbb{N}$: We assume we have found $a_1, \ldots, a_n \in \mathbb{R}_+$ such that

  $$C_n := \{y_1 v_1 + \ldots + y_n v_n \mid -a_i \leq y_i \leq a_i\,; i \in \{1, \ldots, n\}\} := \prod_{i=1}^{n} [-a_i, a_i] \subseteq U \cap V_n.$$

  Note that C_n is closed (in V_n, as well as) in V_{n+1}; $C_n \subseteq U \cap V_{n+1}$ and $V_{n+1} \setminus U$ is closed in V_{n+1} (because $V_{n+1} \cap U$ is open in V_{n+1}).
- For $n + 1$: We claim $\exists\, a_{n+1} > 0, a_{n+1} \in \mathbb{R}$ such that

  $$C_{n+1} := \{y_1 v_1 + \ldots + y_n v_n + y_{n+1} v_{n+1} \mid -a_i \leq y_i \leq a_i\,; i \in \{1, \ldots, n+1\}\}$$
  $$= \prod_{i=1}^{n+1} [-a_i, a_i] \subseteq U \cap V_{n+1}.$$

  Proof of claim by contradiction: If not, then $\forall\, N\ \exists\, x^N \in V_{n+1}$ such that

$x^N = y_1v_1 + \dots y_nv_n + y_{n+1}v_{n+1}$ with $-a_i \leq y_i \leq a_i$; $i \in \{1, \dots, n\}$ and $-\frac{1}{N} \leq y_{n+1} \leq \frac{1}{N}$; but $x^N \notin U$.

But x^N has form $x^N = \underbrace{y_1v_1 + \dots + y_nv_n}_{\in C_n} + y_{n+1}v_{n+1}$, $(\star)$

i.e. the sequence $\{x^N\}_{n\in\mathbb{N}} \subseteq V_{n+1} \setminus U$.

Now for each $i \in \{1, \dots, n\}$, since x^N has form $(\star)$:

the i^{th} coordinates of $\{x^N\}$ are bounded $\forall\ N \in \mathbb{N}$, i.e. $\{x^N\}$ is a bounded sequence of reals.

So we can find a convergent sequence of i^{th} coordinate $\forall\ i \in \{1, \dots, n\}$, i.e. there is a subsequence $\{x^{N_j}\}_{j\in\mathbb{N}} \subseteq V_{n+1} \setminus U$ such that

(1) the first $i = 1, \dots, n$ coordinates sequences converge, and
(2) the $(n + 1)^{th}$ coordinate sequence converges to 0.

So $\{x^{N_j}\}$ converges (in V_{n+1}) as $j \to \infty$ to $x \in C_n \subseteq U$ (since C_n is closed in V_{n+1}). So the sequence $\{x^{N_j}\}_{j\in\mathbb{N}} \subseteq V_{n+1} \setminus U$ converges to $x \in U$. This contradicts the fact that $V_{n+1} \setminus U$ is closed in V_{n+1}. Hence the claim is established.

Now consider $D_n := \prod_{i=1}^{n}(-a_i, a_i) = \{y_1v_1 + \dots + y_nv_n \mid -a_i < y_i < a_i\ ; i \in \{1, \dots, n\}\}$, then $D_n \subset C_n \subseteq U \cap V_n$ is open convex in V_n. Set $U' := \cup_{n\in\mathbb{N}} D_n := \prod_{n=1}^{\infty}(-a_n, a_n)$. Finally (verify that) $0 \in U'$. Then U' is open, convex and $U' \subseteq U$. □

Moreover, let V be a finite dimensional $\mathbb{R}$ vector space, τ be a locally convex topology on V and Z open in this locally convex topology. Then Z is open in the finite topology.

Theorem 1.3. (Theorem 2.4 of last lecture) V is a topological vector space with finite topology τ. Moreover (V, τ) is a topological $\mathbb{R}$-algebra if V is a $\mathbb{R}$-algebra.

1.4. Helping lemma II. Let V and V' be vector spaces of countable dimension each endowed with the corresponding locally convex (finite) topology. Then the finite topology on $V \times V'$ coincides with the product topology, i.e. $\tau_{\text{fin}}(V \times V') = \tau_{\text{fin}}(V) \times \tau_{\text{fin}}(V')$.

Proof. $(\Leftarrow)$ First observe that if a set is open in the product topology on $V \times V'$, then it is open in finite topology on $V \times V'$:
Fix a basis $\{v_1, \dots, v_n, \dots\}$ of V and $\{v'_1, \dots, v'_n, \dots\}$ of V'. Set $V_n = \text{span}\{v_1, \dots, v_n\}$ and $V'_n = \text{span}\{v'_1, \dots, v'_n\}$. Then $V \times V' = \cup_n(V_n \times V'_n)$.

Let $U \times U' \subseteq V \times V'$ be open in the product topology, where U open in finite topology on V and U' open in finite topology on V'.
We show $U \times U'$ is open in the finite topology on $V \times V'$.
It is enough to verify that $(U \times U') \cap (V_n \times V_n')$ is open in ET on $V_n \times V_n'$.
But $(U \times U') \cap (V_n \times V_n') := (U \cap V_n) \times (U' \cap V_n')$, where $U \cap V_n$ is open in ET on V_n and $U' \times V_n'$ is open in ET on V_n'. $\square(\Leftarrow)$

$(\Rightarrow)$ Conversely we show that open set in the finite topology on $V \times V'$ implies open in the product topology.
Wlog let $\mathcal{U}''$ be a convex open neighbourhood of zero in $V \times V'$.
Set $\mathcal{U} := \{x \in V \mid (2x, 0) \in \mathcal{U}''\}$ and $\mathcal{U}' := \{y \in V' \mid (0, 2y) \in \mathcal{U}''\}$. $\mathcal{U}$ and $\mathcal{U}'$ are convex open neighbourhoods of zero in V and V' respectively. So $\mathcal{U} \times \mathcal{U}'$ is open in product topology. Also $\mathcal{U} \times \mathcal{U}' \subseteq \mathcal{U}''$ because if $(x, y) \in \mathcal{U} \times \mathcal{U}'$ then $(x, y) = \frac{1}{2}(2x, 0) + \frac{1}{2}(0, 2y) \in \mathcal{U}''$, since $\mathcal{U}''$ is convex. $\square$