POSITIVE POLYNOMIALS LECTURE NOTES (20: 24/06/10)

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1. Topology on finite and countable dimensional $\mathbb{R}$-vectorspace 1

1. TOPOLOGY ON FINITE AND COUNTABLE DIMENSIONAL $\mathbb{R}-$VECTOR SPACE (continued)

We want to prove Theorem 2.4 of Lecture 18, i.e.

Theorem 1.1. V is a topological vector space with finite topology τ. Moreover (V, τ) is a topological $\mathbb{R}$-algebra if V is endowed with $\mathbb{R}$ algebra structure.

We still need more helping lemmas (towards proof of 1.1):

Lemma 1.2. (About finite dimensional spaces with ET)

1. Finite dimensional $\mathbb{R}$-vector spaces V with ET are topological spaces.
2. Linear functionals $L : V \to \mathbb{R}$ are continuous. More generally, let V_i, V_j be finite dimensional vectorpaces with ET and $L : V_i \times V_i \to V_j$ bilinear map, then L is continuous. □

1.3. Helping lemma III. Let $V = \bigcup_i V_i$ be a countable dimensional vector space with (finite topology) $\tau_{\text{fin}}(V)$, where V_i's are finite dimensional. Let (χ, x) be a topological space and $f : V \to \chi$ be a map. Then f is continuous (with respect to $\tau_{\text{fin}}(V)$ and χ) iff $f|_{V_i}$ is continuous (with respect to ET on V_i and χ) for each $i \in \mathbb{N}$.

Proof. (⇒) Clear.
(⇐) Let $X \subseteq (\chi, x)$ be open. To show: $f^{-1}(X)$ is open in V. Using Hilfslemma I (4) it is enough to show that $f^{-1}(X) \cap V_i$ is open in V_i $\forall$ i. But $f^{-1}(X) \cap V_i = (f|_{V_I})^{-1}(X)$ which is open in V_i $\forall$ i since $f|_{V_i}$ is assumed to be continuous $\forall$ i. □

Corollary 1.4. Let V be countable dimensional with finite topology $\tau_{\text{fin}}(V)$ and $L : V \to \mathbb{R}$ be a linear functional. Then L is continuous. □

1.5. *Proof of the theorem 1.1.* Helping lemma $\underbrace{\text{I} + \text{II}}_{\text{(last lecture)}}$ + III implies the proof as follows:

(i) We need to verify that $+ : (V \times V, \underbrace{\tau_{\text{fin}}(V) \times \tau_{\text{fin}}(V))}_{\text{(product topology)}} \to (V, \tau_{\text{fin}}(V))$ is continuous.

Using Helping lemma II, it is enough to verify that

$$+ : (V \times V, \tau_{\text{fin}}(V \times V)) \to (V, \tau_{\text{fin}}(V)) \text{ is continuous.}$$

Proof. Let $V = \bigcup_{i \in \mathbb{N}} V_i$, then $V \times V = \bigcup_i (V_i \times V_i)$. By Hilfslemma III, enough to verify that

$$+ : (V_i \times V_i, ET) \to (V, \tau_{\text{fin}}(V)) \text{ is continuous.}$$

Let $U \subseteq V$ open in $\tau_{\text{fin}}(V)$. We show that $(+)^{-1}(U) \subseteq V_i \times V_i$ is open in ET.
But V_i is a subspace so $(+)^{-1}(U) = (+)^{-1}(U \cap V_i)$. Now $U \cap V_i$ is open in V_i and by lemma 1.2 we know that V_i is a topological vector space so $(+)^{-1}(U \cap V_i)$ is open. □

(ii) Scalar multiplication:

$$. : \mathbb{R} \times V \to V; (r, v) \mapsto rv \text{ is continuous.}$$

Proof. Analogous. □

(iii) Multiplication: Let V be a $\mathbb{R}$-algebra. Then

$$\times : (V \times V, \underbrace{\tau_{\text{fin}}(V) \times \tau_{\text{fin}}(V))}_{\text{(product topology)}} \to (V, \tau_{\text{fin}}(V) \text{ is continuous.}$$

Proof. Observe that restriction of multiplication to the finite dimensional subspaces V_i is not well defined i.e. V_i need not be a sub algebra, but

Claim 1: $\exists\, j$ large enough so that

$$\times : V_i \times V_i \to V_j \text{ is well defined.}$$

Proof of claim 1: Let $\{v_1, \ldots, v_i\}$ be a basis of V_i. Let j be large enough so that the product vectors $v_l v_k \in V_j$ for all $1 \le l, k \le i$.

Claim 2: The mapping $\times : V_i \times V_i \to V_j$ is bilinear and hence continuous by lemma 1.2. □

□□(proof of theorem 1.1)

Theorem 1.6. (Separation Theorem) (Theorem 2.5 of Lecture 18) Let V be a countable dimensional vector space, $U \subseteq V$ be open and convex, $C \subseteq V$ be a cone such that $U, C \neq \phi$ and $U \cap C = \phi$. Then there exists a linear functional $L : V \to \mathbb{R}$ such that $L(U) < 0$ and $L(C) \geq 0$.

Corollary 1.7. If $C \subseteq V$ is closed cone and $x_0 \notin C$ then there exists $L : V \to \mathbb{R}$ such that $L(x_0) < 0$ and $L(C) \geq 0$.

Proof. $\exists\, U' \ni x_0 : U'$ open and $U' \cap C = \phi$. By theorem 2.3 of Lecture 18, let U be an open convex subset of V with $x_0 \in U \subseteq U'$ and $U \cap C = \phi$. □

1.8. *Proof of the theorem 1.6.*
Consider $\{D \mid D \text{ is a cone in } V, D \supseteq C; D \cap U = \phi\}$. This family is nonempty. By Zorn's lemma let D be the maximal element (with these properties).
Claim 1: $-U \subseteq D$.
If not let $x \in -U, x \notin D$. By maximality: $(D + x\mathbb{R}_+) \cap U \neq \phi$.
So $\exists\, y \in D; r \geq 0; u \in U$ with $y + rx = u$. So $y = r(-x) + u$.

$$\text{So} \quad \underbrace{\frac{y}{1+r}}_{\in D \text{ since } D \text{ is a cone}} = \underbrace{\frac{r}{1+r}(-x) + \frac{1}{1+r}u}_{\in U \text{ by convexity of } U} \in D \cap U, \text{ a contradiction.}$$

□ (claim 1)

Claim 2: $D \cup -D = V$.
Let $x \in V$ and $x \notin D$. Then $(D + \mathbb{R}_+x) \cap U \neq \phi$. So $\exists\, u = d + rx$ such that $u \in U, r > 0, d \in D$. Then $-x = \frac{1}{r}(d-u) \in \frac{1}{r}(D-U) \underbrace{\subseteq}_{\text{(by claim 1)}} \frac{1}{r}(D+D) \subseteq D$.

□ (claim 2)

Claim 3: D is closed.
If not, let $d_i \in D$ such that $\lim_{i\to\infty} d_i \to x$ and $x \notin D$. Then $(D + \mathbb{R}_+x) \cap U \neq \phi$. So $\exists\, u = d + rx$; $u \in U, r > 0, d \in D$. Then $u = d + r\lim_{i\to\infty} d_i = \lim_{i\to\infty}(d + rd_i)$. So $d + rd_i \in U$ for i sufficiently large (since U is open so complement of U is closed), but also $d + rd_i \in D$ (since D is a cone). This contradicts $U \cap D = \phi$. □ (claim 3)

Now let $W := D \cap -D$. Fix $x_0 \in U$. By previous claims we see that W is a subspace. Further $x_0 \in U \Rightarrow x_0 \notin D \Rightarrow x_0 \notin W$.
Now consider the subspace $W \oplus \mathbb{R}x_0$.
Claim 4: $V = W \oplus \mathbb{R}x_0$ (i.e. W is a hyperplane in V i.e. has codimension 1 in V).
Let $y \in V$, w.l.o.g. $y \in D$ (if $y \notin D$; $-y \in D$ same argument).
Consider $\{\lambda x_0 + (1-\lambda)y \mid 0 \leq \lambda \leq 1\}$ and the largest λ in the interval $[0,1]$ such that $z = \lambda x_0 + (1-\lambda)y \in D$. Then $\lambda < 1; z \in D \cap -D = W$.

So $y = \frac{1}{1-\lambda}z + \frac{-\lambda}{1-\lambda}x_0 \in W + \mathbb{R}x_0$. □ (claim 4)

Now let $L : V \to \mathbb{R}$ be the uniquely determined functional defined by $L(W) = 0$ and $L(x_0) = -1$.
Claim 5: $L \geq 0$ on D.
Let $y \in D$. If $y \in W$ then $L(y) = 0$, so done. If $y \notin W$ then for some λ :

$\lambda x_0 + (1-\lambda)y \in W;\ 0 < \lambda < 1$. Applying L :
$\lambda L(x_0) + (1-\lambda)L(y) = -\lambda + (1-\lambda)L(y) = 0$.
So $L(y) = \dfrac{\lambda}{1-\lambda} > 0$. □ (claim 4)

□□ (proof of theorem 1.6)