POSITIVE POLYNOMIALS LECTURE NOTES (21: 29/06/10)

SALMA KUHLMANN

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1. K-MOMENT PROBLEM (continuation to Lecture 17)

1.1. Framework

$$A = \mathbb{R}[\underline{X}]$$
$$S = \{g_1, \dots, g_s\}$$
$$K = K_S\text{; b.c.s.a.set}$$
$$T_S\text{: f.g. preordering.}$$

We have the containment (recall 3.5 of Lecture 16)

$$T_S \subseteq \overline{T}_S \subseteq \operatorname{Psd}(K_S) \tag{1}$$

Remark 1.2. We have an interesting comparison between $\operatorname{Psd}(K_S)$ and $\overline{T}_S$. One can show:

$$\operatorname{Psd}(K_S) = \bigcap_{\alpha:\mathbb{R}[\underline{X}]\to\mathbb{R} \text{ homomorphism of } \mathbb{R}-\text{algebra with } \alpha(T_S)\geq 0} \alpha^{-1}(\mathbb{R}_+)$$

$$= \bigcap_{\alpha:\mathbb{R}[\underline{X}]\to\mathbb{R},\ \alpha=ev_{\underline{x}},\ \underline{x}\in K_S} \alpha^{-1}(\mathbb{R}_+)$$

whereas

$$\overline{T}_S = T_S^{\vee\vee} = \bigcap_{L:\mathbb{R}[\underline{X}]\to\mathbb{R} \text{ linear homomorphism of } \mathbb{R}-\text{vector spaces with } L(T_S)\geq 0} L^{-1}(\mathbb{R}_+).$$

We Shall study the containment in (1).

1.3. Recall.

(a) If T_S =Psd(K_S), then T_S is saturated.

(b) If $\overline{T}_S$ = Psd(K_S), then "S solves the K_S-MP ".

Proposition 1.4. If $T_S \subseteq \mathbb{R}[\underline{X}]$ is closed then S solves the KMP if and only if T_S is saturated.

Proof. Immediate from (a) and (b) (of 1.3 above) and $T_S = \overline{T}_S$ if T_S is closed. □

We shall therefore study closed preorderings now:

2. CLOSED FINITELY GENERATED PREORDERINGS

Proposition 2.1. Let $A = \mathbb{R}[\underline{X}]$ endowed with finite topology and $A_d = \mathbb{R}[\underline{X}]_d = \{f \in A \mid \deg f \leq d\}$; $d \in \mathbb{Z}_+$. This subspace is finite dimensional generated by $\underline{X}^{\underline{\alpha}}$ of degree $|\underline{\alpha}| := \alpha_1 + \ldots + \alpha_n \leq d$.
$\mathrm{Dim}(A_d) = \binom{n+d}{d}$; $\{A_d\}_{d\in\mathbb{N}}; A_d \subseteq A_{d+1}; A = \bigcup_d A_d$.
So $T \subseteq A$ is closed in A if and only if $T_d := T \cap A_d$ is closed in A_d for ET; for all $d \in \mathbb{Z}_+$.

Theorem 2.2. Let $\mathbb{R}[\underline{X}] = \mathbb{R}[X_1, \ldots, X_n]$. Then

(i) $\sum \mathbb{R}[\underline{X}]^2$ is closed in $(\mathbb{R}[\underline{X}], \tau_{\text{fin}})$ (Berg et al; 1970).

(ii) Let $S = \{g_1, \ldots, g_s\}$ and $K = K_S \subseteq \mathbb{R}^n$ ba a b.c.s.a. set.

(K-M) If K_S contains a cone with nonempty interior (equivalently a cone of dimension n, equivalently just a non empty generating Cone C), then T_S is closed.

The proof of (i) will follow from a series of lemma:

Lemma 2.3. It is enough to show that $\sum_d := (\sum \mathbb{R}[\underline{X}]^2) \cap A_d$ is closed in A_d $\forall\, d \in 2\mathbb{Z}_+$. □

Lemma 2.4. Let $f \in \sum_d$, d even.

1. if $f = \sum_{i=1}^{m} h_i^2$ then $\deg(f) = \max_{i=1,\ldots,m} \{\deg h_i^2\}$

2. therefore for any representation $\sum_{i=1}^{m} h_i^2$ of f we must have $\deg(h_i) \leq \frac{d}{2}$ for all $i = 1, \ldots, m$.

3. w.l.o.g. we may assume that $m \leq N := \dim A_{d/2} = \binom{n+\frac{d}{2}}{\frac{d}{2}}$.

4. Therefore (for d even) $f \in \sum_d$ can be written as: $f = \sum_{i=1}^{N} h_i^2$ with $\deg(h_i) \leq \frac{d}{2}$ $\forall\ i = 1, \ldots, n$.

Proof. (1) and (2): clear.
Proof of (3): Let $f \in \mathbb{R}[\underline{X}]$, $d = \deg f = 2q$. Set $N = \binom{n+q}{q}$.
Claim: $f \in \mathbb{R}[\underline{X}^2]$ iff there exists an $N \times N$ psd symmetric matrix $M \in S_{N\times N}(\mathbb{R})$ such that $f(\underline{x}) = Y^T M Y$, where $Y = \begin{pmatrix} Y_1 \\ \vdots \\ y_N \end{pmatrix}$ where $\{Y_1, \ldots, Y_N\}$ is an enumeration of all possible monomials $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $|\underline{\alpha}| := \alpha_1 + \ldots + \alpha_n \leq q$.
In particular: $f \in \sum \mathbb{R}[\underline{X}]^2$ iff $f = \sum_{i=1}^{N} h_i^2$
Proof of the claim:
($\Rightarrow$) Assume $f \in \mathbb{R}[\underline{X}^2]$ and $f = \sum h_i^2$ where $h_i \in A_q$. Write $h_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{iN} \end{pmatrix} \in \mathbb{R}^N$ and define $M_{\alpha\beta} := \sum_i a_{i\alpha} a_{i\beta}$ the $\alpha\beta^{th}$ coefficient of the matrix M for $\alpha, \beta \in \{1, \ldots, N\}$.
Obviously it is symmetric. Check that M is PSD and that $f = Y^T M Y$.
($\Leftarrow$) Conversely if $f = Y^T M Y$ with M symmetric and psd; i.e. $M \in S_{N\times N}(\mathbb{R})$. By spectral theorem write

$$M = B^T B, \text{ where } B \in M_{N\times N}$$

So $f = (Y^T B^T)(BY) = (BY)^T(BY) = \sum_{\alpha=1}^{N} (BY)_\alpha^2$. □

Lemma 2.5. Fix a set D of $d+1$ distinct real numbers and set $\Delta := D^n \subseteq \mathbb{R}^n$. Consider the map

$$\begin{aligned} \Psi : A_d &\to \mathbb{R}^\Delta \\ g(\underline{X}) &\mapsto (g^{(\underline{a})})_{\underline{a}\in\Delta} \end{aligned}$$

Then Ψ is linear and $\Psi(g) = \underline{0}$ iff $g \equiv 0$ (i.e. $Ker(\Psi) = \{0\}$). So Ψ is homomorphism onto a closed subspace of $\mathbb{R}^\Delta$.

Proof. The only thing to verify is $Ker(\Psi) = \{0\}$.
By induction on n.
If $n = 1$ and g is a polynomial of degree $\leq d$ that has $d + 1$ roots is identically the zero polynomial i.e. $g \equiv 0$. So on it follows for all n. □

Corollary 2.6. Let $\{f_j\}_j \subseteq A_d$; $f \in A_d$. Then

1. $f_j \to f$ in A_d if and only if $f_j(\underline{a}) \to f(\underline{a})$ in $\mathbb{R}$ for each $\underline{a} \in \Delta$ (i.e. point wise convergence on Δ).
2. More generally $\{f_j\}_j \subseteq A_d$ is a convergent in A_d iff $\{f_j(\underline{a})\}_j$ is convergent sequence in $\mathbb{R}$ for each $\underline{a} \in \Delta$.

Proof. Proof of 2:
($\Leftarrow$) From assumption $\Psi(f_j)$ converges to a point $\gamma \in \mathbb{R}^\Delta$. But since Im Ψ is a subspace of $\mathbb{R}^\Delta$ it is closed so $\gamma \in \mathrm{Im}\Psi$. So $\lim\limits_{j\to\infty} f_j = \Psi^{-1}(\gamma) \in A_d$. □

2.7. *Proof of Theorem 2.2 (i).*
We want to show that $\sum_d$ is closed in A_d in the Euclidean topology (i.e. convergence of coefficients).
Let $f \in A_d$; $f_j \in \sum_d$ so that $f_j \to f$ coefficientwise in A_d $\quad (\star)$

To show: $f \in \sum_d$

Write without loss of generality: $f_j = \sum\limits_{i=1}^{N} h_{ij}^2$, $\deg h_{ij} \leq \dfrac{d}{2}$ $\forall j$; $N = \binom{n+d/2}{d/2}$.
$(\star) \Rightarrow f_j(\underline{a}) \to f(\underline{a})\ \forall\ \underline{a} \in \Delta$ as $j \to \infty$
i.e. $\sum\limits_{i}^{N} (h_{ij}(\underline{a}))^2 \to f(\underline{a})$ in $\mathbb{R}\ \forall\ \underline{a} \in \Delta$.
So $\exists\, \delta > 0$ s.t.

$$h_{ij}^2(\underline{a}) \leq f_j(\underline{a}) \leq \delta\ \forall\ \underline{a} \in \Delta, \forall j \in \mathbb{N}, \forall\ i = 1, \ldots, N$$

So for each fixed $\underline{a} \in \Delta$ and each fixed $i \in \{1, \ldots, N\}$, $\{h_{ij}(\underline{a})\}_{j\in\mathbb{N}}$ is a bounded sequence of reals so has a convergent subsequence.
Also since Δ is finite there is therefore a subsequence $\{h_{ij_k}\}_{k\in\mathbb{N}}$ of $\{h_{ij}\}$ for each fixed $i \in \{1, \ldots, N\}$ such that $\{h_{ij_k}(\underline{a})\}_{k\in\mathbb{N}}$ is convergent for each $\underline{a} \in \Delta$. So by Corollary 2.6 above:
for each $i \in \{1, \ldots, N\}$: $\{h_{ij_k}\}_{k\in\mathbb{N}}$ is convergent in $A_{d/2}$ say to h_i.

So $\sum\limits_{i=1}^{N} h_i^2 = \lim\limits_{k\to\infty} \sum\limits_{i=1}^{N} {h_{ij_k}}^2 = \lim\limits_{k\to\infty} f_{j_k} = f$.

So $f \in \sum_d$ as required. □(proof of theorem 2.2 (i))