POSITIVE POLYNOMIALS LECTURE NOTES (22: 01/07/10)

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Contents

1. CLOSED FINITELY GENERATED PREORDERINGS(continue)

Theorem 1.1. (Theorem 2.2 (ii) of last lecture) Let K be a basic closed semialgebraic set. Assume $C \subseteq K$ is a non empty open cone. Let $S = \{g_1, \dots, g_s\}$ such that $K = K_S$. Then T_S is closed.

Proof. It is enough to prove the following lemma, which is a generalization of lemma 2.4 of last lecture. □

Lemma 1.2. Let $S = \{g_1, \dots, g_s\}$ such that K_S contains a non-empty open cone. Let $f \in A_d \cap M_S := M_d$; $f = b_0 + b_1 g_1 + \dots + b_s g_s$ where $b_i \in \sum \mathbb{R}[\underline{X}]^2$, then

1. $\deg f = \max\{\deg b_0, \deg(b_1 g_1), \dots \deg(b_s g_s)\}$
2. If $f = \sum_{j=1}^{m_0} (h_{0j})^2 + \sum_{j=1}^{m_1} (h_{1j})^2 g_1 + \dots + \sum_{j=1}^{m_s} (h_{sj})^2 g_s$ then $\deg h_{0j} \leq \frac{d}{2}$ and $\deg(h_{ij}) \leq \frac{d - \deg g_i}{2}$; $i = 1, \dots, s$.

   So w.l.o.g. $f \in M_d$ has the form
$$f = \sum_{j=1}^{m_0} (h_{0j})^2 + \sum_{j=1}^{m_1} (h_{1j})^2 g_1 + \dots + \sum_{j=1}^{m_s} (h_{sj})^2 g_s \text{ with } \deg(h_{ij}) \leq \frac{d}{2}.$$

To prove 1.) of this lemma we need the following two propositions:

Proposition 1.3. Let $C \in \mathbb{R}^n$ be a cone, $h \in \mathbb{R}[\underline{X}]$ and $h = h_0 + \ldots + h_\nu$ be the decomposition of h into homogeneous components, i.e. $\deg h_i = i$ and $\deg h = \deg h_\nu = \nu$. Write $LT(h) = h_\nu$.

If $h \geq 0$ in C then $LT(h) \geq 0$ on C.

Proof. Let $c \in C$. We show that $h + \nu(c) \geq 0$. Wlog $h_\nu(c) \neq 0$. Consider the following variable in one real variable λ: $P_c(\lambda) := h(\lambda c) = h_0 + h_1(c)\lambda + h_2(c)\lambda^2 + \ldots + h_\nu(c)\lambda^\nu$. For all $\lambda > 0$, $\lambda c \in C$ so $P_c(\lambda) = h(\lambda c) \geq 0$. So $P_c(\lambda) \geq 0$ on $[0, \infty) \subseteq \mathbb{R}$. So it must have positive leading coefficient i.e. $h_\nu(c) > 0$ as required. □

Proposition 1.4. Let $p_0, \ldots, p_s \in \mathbb{R}[\underline{X}]$ and assume that there is a nonempty open cone C such that $p_i \geq 0$ on $C, \forall i = 1, \ldots s$ then $\deg(p_0 + \ldots + p_s) = \max(\deg p_0, \ldots, \deg p_s)$.

Proof. Let $m = \max(\deg p_0, \ldots, \deg p_s)$. Let us gather those leading terms of degree m say $LT(p_0), \ldots, LT(p_l)), l \leq s$. We want to show that $LT(p_0) + \ldots + LT(p_l) \not\equiv 0$ (once this is shown we are done because this sum, if nonzero, is the $LT(p_0 + \ldots + p_s)$ and is of degree m so this will establish that $\deg(p_0 + \ldots + p_s) = m$ indeed). Now $LT(p_1) \neq 0$ so there is $c \in C$ such that $LT(p_1)$ does not vanish at c (a nonzero polynomial does not vanish on a nonempty open set). By proposition 1.3 we must have $LT(p_1)$ evaluated at c is > 0. Since $LT(p_i)$ evaluated at c for $i = 1, \ldots, l$ are all ≥ 0 (again proposition 1.3), we se that there are no cancellations and $LT(p_0) + \ldots + LT(p_l)$ evaluated at c is > 0. So $LT(p_0) + \ldots + LT(p_l) \not\equiv 0$ □

2. APPLICATIONS TO THE *K*-MOMENT PROBLEM

Corollary 2.1. $K \subseteq \mathbb{R}^n, n \geq 3$ bcsas. K contains a non empty open cone $\Rightarrow$ KMP is not finitely solvable.

Proof.

1. $\mathrm{Dim}(K) \geq 3$; $K = K_S, S$ − finite $\Rightarrow T_S$ is not saturated.
2. But T_S is closed so S solves KMP iff T_S is saturated.
3. So S does not solve KMP.

□

Corollary 2.2. $K \subseteq \mathbb{R}^n, n \geq 2$. If K contains cone of dimension 2 then KMP is not finitely solvable. Note that we do not claim that T is closed.

Corollary 2.3. If K is non compact b.c.s.a. set $K = K_S$, S any finite description. Then T_S is closed.

Proof. K contains an open infinite half line $\Rightarrow$ K contains open cone. □

3. THE FINEST LOCALLY CONVEX TOPOLOGY ON A $\mathbb{R}$-VECTOR SPACE

Recall:

1. Hausdorff: If $x_1 \neq x_2, \exists u_1, u_2$ open such that $u_1 \cap u_2 = \phi$ and $x_i \in u_i$.
2. Topological vector space: Topology continuous with + and scalar multiplication.
3. A topology is locally convex if V is a topological vector space and has a basis of convex open sets.

Theorem 3.1. Tychonoff theorem On a finite dimensional vector space there is a unique topology making it into a Hausdorff topological vector space namely the ET. (much stronger statement then the fact that all—-topologies are equivalent!)

Theorem 3.2. If V is a (Hausdorff) topological vector space and W is a subspace then W is a (Hausdorff) topological vector space with the induced topology.

We first claim the following general fact:
Let X be a topological space and $Y \subseteq X$. Then the product topology of the induced topologies on X on $Y \times Y$ is induced topology of the product topology of $X \times X$ on $Y \times Y$.

- Fact 1: Any vector space admits the finest topology (greatest number of open sets) making it into a locally convex topological vector space.
- Fact 2: This finest locally convex topology is Haudorff.

Theorem 3.3. Let V be a countable dimensional real vector space. Then the finest locally convex topology (from Fact 1) is the finite topology.

Proof. Let $u \subset V$ be open in the finest locally convex topology then we want to show that u is open in the finite topology. Let $W \subset V$ be finite dimensional subspace. We show that $W \cap u$ is open in W in ET. Now W inherits the finite locally convex topology and $W \cap u$ is open in the inherited f.l.c. topology by definition of relative topology. But the induced f.l.c. topology on W makes it into a Hausdorff topological vectorspace by theorem 3.2 and therefore is the ET by theorem 3.1. So $W \cap u$ is open in W for the ET.

Conversely, let u be an open set in the finite topology on V. it must be open in the finest locally convex topology because finite topology on a countable dimensional vector space is a locally convex topolgy. Therefore u is open in the finest locally convex topology. □

Remark 3.4. Let V be a real vector space of arbitrary dimension and define a topology on V as follows: $u \subset V$ is open iff $u \cap W$ is open for every finite dimensional subspace W of V. Then V need not to be a topological vector space as addition as a binary map is not necessarily continuous. Furthermore the topology need not be locally convex.