POSITIVE POLYNOMIALS LECTURE NOTES (23: 06/07/10)

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Contents

1. THE FINEST LOCALLY CONVEX TOPOLOGY ON A VECTOR SPACE (continued)

Let E be a vectorspace. There is a finest topology making E into a locally convex topological vector space. This topology is Hausdorff. It is called the **finest locally convex topology**.
Let E be a topological vector space.

Remark 1.1. Since translation for $u \in E$, $T_u : X \mapsto X + u$ is a homomorphism of E. If B is a base for neighbourhoods of zero then $u + B$ is a base for all neighbourhoods of u. Therefore the whle topological structure of E determined by all neighbourhoods of the origin.

Definition 1.2. A function $p : E \to [0, \infty)$ is called a seminorm if it has the following properties:

1. Homogeneity: $p(\lambda X) = |\lambda| p(X), \lambda \in \mathbb{R}; X \in E$
2. Subadditivity: $p(X + Y) \leq p(X) + p(Y) \forall X, Y \in E$.

If $p^{-1}(\{0\}) = \{0\}$, then p norm.

Strategy for proof of the theorem

- **Fact 2.** A family of seminorms induces a local convex topology on E making it into a topological vector space.
- **Fact 1.** Conversely (1.4+1.6) the topology of an arbitrary local convex topological vector space is always induced by a family of seminorms.

- **Fact 3.** Take all seminorms. It induces a local convex topology making into a topological vector space by Fact 2. It is the finest by Fact 1.

Definition 1.3. Let $A \subset E$. Then A

1. is **absorbing** if $\forall X \in E$ there exists $M > 0$ such that $X \in \lambda A \forall \lambda \in \mathbb{R}; |\lambda| \geq M$.
2. is **balanced** if $\lambda A \subseteq A$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$.
3. is **absolutely convex** if it is convex and balanced.

Filter of neighbourhoods of zero just means the collection of all neighbourhoods of zero.

Proposition 1.4. Let E be a topological vector space and $\mathcal{U} = \{$ all neighbourhoods of zero$\}$. Then

1. $u \in \mathcal{U}$ is absorbing.
2. for every $u \in \mathcal{U}$ there exists $u' \in \mathcal{U}$ with $u + u' \subseteq \mathcal{U}$.
3. for every $u \in \mathcal{U}$;

$$b(u) := \cap_{|\mu| \geq 1} \mu u \text{ is a balanced neighbourbood of zero contained in } u.$$

It follows that every topological vector space has a base of balanced neighbourhoods of zero.

Proof. For every $X \in E$, the map:

$$\begin{aligned} \mathbb{R} &\to E \\ \lambda &\mapsto \lambda X \end{aligned}$$

is continuous at $\lambda = 0$. This implies (1).
Similarly the continuity at $(0, 0)$ of the map

$$\begin{aligned} E \times E &\to E \\ (X, Y) &\mapsto X + Y \end{aligned}$$

implies (2).
By continuity of

$$\begin{aligned} \mathbb{R} \times E &\to E \\ (\lambda, X) &\mapsto \lambda X \end{aligned}$$

So given $u \in \mathcal{U}$ there exists $\epsilon > 0$ and $v \in \mathcal{U}$ suc that $\lambda v \subseteq u$ for $|\lambda| \leq \epsilon$. Therefore $\epsilon v \subseteq b(u) \subseteq u$. So u contains a balanced set $b(u)$ which is a neighbourhood of zero because ϵv is a neighbourhood of zero; $X \mapsto \epsilon X$ being a homomorphism of E. □

Proposition 1.5. Let E be a locally convex topological vector space then the filter collection of neighbourhoods of zero has a base $\mathcal{B}$ with the following properties:

1. Every $u \in \mathcal{B}$ is absorbing and absolutely convex.
2. If $u \in \mathcal{B}$ and $0 \neq \lambda \in \mathbb{R}$ then $\lambda u \in \mathcal{B}$.

Proof. If u is a neighbourhood of zero then $b(u)$ is absolutely convex (by proposition 1.2). So if $\mathcal{B}_0$ is a base of convex neighbourhoods of zero then the family $\mathcal{B} := \{\lambda b(u) | u \in \mathcal{B}_0; \lambda \neq 0\}$ is a base satisfying (1) and (2). □

Converse of the above proposition: Let E have a base for a filter on E with properties (1) and (2) there is a uniqe topology on E such that E is a locally convex topological vector space with $\mathcal{B}$ as a base of neighbourhoods of zero.

1.1. CONNECTION TO SEMINORMS

Remark 1.6. If p is a seminorm and $\alpha > 0$ then the set $\{X \in E | p(X < \alpha)\}$ is convex and absorbing.

Proof. Exercise □

Let E be a vector space. Associating a seminorm candidate to a subset of E: For $A \neq \phi, A \subseteq E$ sefine a mapping:

$$p_A : E \to [0\infty]$$
$$p_A(X) := \inf\{\lambda > 0 | X \in \lambda A\}$$

(where $p_A(X) = \infty$ if the set $p_A(X)$ is empty).
When p_A is seminorm?

Lemma 1.7. If $A \neq \phi, A \subseteq E$ is

1. absorbing; then $p_A(X) < \infty$ for all $X \in E$.
2. convex, then p_A is subadditive.
3. balanced then p_A is homogeneous and $\{X \in E | p_A(X) < 1\} \subseteq A \subseteq \{A \in E | p_A(X) \leq 1\}$.

If A satisfies (1)-(3) then p_A is called the seminorm determined by A.

Proposition 1.8. Let E be a vector space and $(P_i)_{i\in I}$ a family of seminorms. There exists a coarsest topology on E with the properties that E is a topological vector space and each P_i is continuous under this topology E is locally convex and the familie of sets $\{X \in E | p_{i_1} < \epsilon, \ldots, p_{i_n} < \epsilon\}$ for all $\{i_1, \ldots, i_n\} \in I$ and $n \in \mathbb{N}, \epsilon > 0, \epsilon \in \mathbb{R}$ is a base for the (filter of) neighbourhoods of zero.

Proof. Later □

Proposition 1.9. The topology of an arbitrary locally convex tpological vector space E is always induced by a family of seminorms.

Proof. By proposition 1.4 let $\mathcal{B}$ be the base for neighbourhoods of zero with properties ((i) absorbing and absolutely convex and (ii) $u \in \mathcal{B}, \lambda \neq 0 \Rightarrow \lambda u \in \mathcal{B}$).

Now consider the family $\{p_u | u \in \mathcal{B}\}$. By lemma 1.6 this is a family of seminorms (Moreover since u is open we actually have $u = \{X \in E | p_u(X) < 1\}$). Verify that the topology induced by this family of seminorms (as described in Fact 1) Coincides with the given topology E. □