POSITIVE POLYNOMIALS LECTURE NOTES (24: 08/07/10)

SALMA KUHLMANN

Contents

1. TOPOLOGICAL $\mathbb{R}-$VECTOR SPACE

1.1. Fix E $\mathbb{R}$ -vector space (no assumptions in the dimension)
Notation: $\bar{0} \in E$, $0 \in \mathbb{R}$ (to distinguish them).
Let τ be a topology on E making it a topological $\mathbb{R}$-vector space, i.e. the maps

$$\begin{aligned} E \times E &\to E \\ (x, y) &\mapsto x + y, \text{ and} \end{aligned}$$

$$\begin{aligned} \mathbb{R} \times E &\to E \\ (\lambda, x) &\mapsto \lambda x \quad \text{are continuous,} \end{aligned}$$

where $\mathbb{R}$ has Euclidean topology τ_E,
$E \times E$ has the product topology $\tau \times \tau$, and
$\mathbb{R} \times E$ has the product topology $\tau_E \times \tau$.
Recall that $\{A_1 \times A_2 \mid A_1 \in \tau_1, A_2 \in \tau_2\}$ is a base for the product topology $\tau_1 \times \tau_2$.
Let $\mathcal{U}_\tau = \{U \in \tau \mid \bar{0} \in U\} = \{\tau -$ neighbourhood of $\bar{0}\}$.
Since $\forall x \in E$ the map

$$E \to E, a \mapsto a + x$$

is a τ-homeomorphism,
$\forall a \in E$, $a + \mathcal{U}_\tau = \{a + U \mid U \in \mathcal{U}_\tau\} = \{\tau -$ neighbourhood of $a \in E\}$.
Namely $\mathcal{U}_\tau$ determines all the topology τ.

We want to prove the following theorem:

Theorem 1.2. There is a finest locally convex topology τ_{max} on E. Moreover τ_{max} is Hausdorff.

Definition 1.3. Let $(p, \leq)$ be a partial order.

1. $F \subseteq P$ is a **filter** if
   - $\forall\, x, y \in F, \exists\, z \in F$ such that $z \leq x$ and $z \leq y$;
   - $\forall\, x, \in F, \forall\, y \in P$: $x \leq y \Rightarrow y \in F$
2. Let $F \subseteq P$ is a filter. Then $B \subseteq F$ is a **base** for the filter if $\forall\, x \in F\ \exists\, y \in B$ such that $y \leq x$.

Example 1.4. Let (X, τ) be a topological space and $x \in X$. Then

$$F_x = \{A \in \tau \mid x \in A\} = \{\tau -\text{ neighbourhoods of } x \in X\}$$

is a filter of the partial order $(\tau, \subseteq)$:

- $A_1, A_2 \in F_x \Rightarrow A_1 \cap A_2 \in F_x$ and $A_1 \cap A_2 \subseteq A_1$, $A_1 \cap A_2 \subseteq A_2$.
- For $A \in F_x, U \in \tau$: $A \subseteq U \Rightarrow U \in F_x$.

In particular $\mathcal{U}_\tau = \{\tau -\text{ neighbourhoods of } \bar{0}\}$ is a filter of $(\tau, \subseteq)$

Let $\mathcal{B} \subseteq \mathcal{U}_\tau$ be a base of the filter $\mathcal{U}_\tau$ (in sense of the above definition).

Definition 1.5. A topological space (X, τ) is said to be **locally convex** if $\forall x \in X$ and $\forall\, U_x \in \tau$ containing x, $\exists\, V \in \tau$ convex such that $x \in V \subset U_x$.

Remark 1.6. Let (E, τ) be a topological $\mathbb{R}$-vector space. In order to prove that (E, τ) is locally convex, it is enough to prove that the filter $\mathcal{U}_\tau$ of τ-neighbourhoods of $\bar{0}$ has a base B (in the sense of base of a filter) made of convex set:

Let $\mathcal{B}$ be a base for the filter $\mathcal{U}_\tau$ such that each $U \in \mathcal{B}$ is convex. Let $x \in X$, $U_x \in \tau$ containing x. Then (see page 1) $U_x = x + U$ for some $U \in \mathcal{U}_\tau$. Let $C \in \mathcal{B}$ such that $C \subseteq U$ ($\exists$ such C because $\mathcal{B}$ is a base), then $x + C \subset U_x$ is convex and contains x.

1.7. Fact 1: $U \in \mathcal{U}_\tau \Rightarrow U$ is absorbing (i.e. $\forall x \in E, \exists\, \mu > 0$ such that $|\lambda| \geq \mu \Rightarrow x \in \lambda U$).

Proof. Fix $U \in \mathcal{U}_\tau$ and $x \in E$. The map
$$f_x : \mathbb{R} \to E; \lambda \mapsto \lambda x$$
is continuous everywhere, in particular in $0 \in \mathbb{R}$.
So $f_x^{-1}(U) \subseteq \mathbb{R}$ is open and contains $0 \in \mathbb{R}$.
So $\exists\, \epsilon > 0$ such that $f_x(-\epsilon, \epsilon) \subseteq U$, (we can assume $\epsilon < 1$). In other words, $c < \epsilon \Rightarrow cx \in U \Leftrightarrow x \in c^{-1}U$. So we can take for instance $\mu = \epsilon^{-1} + 1$ □

1.8. Fact 2: $U \in \mathcal{U}_\tau \Rightarrow \exists\, V \in \mathcal{U}_\tau$ such that $V + V \subseteq U$.

Proof. The map
$$+ : E \times E \to E; (x, y) \mapsto x + y$$
is continuous in $(\bar{0}, \bar{0})$. So $+^{-1}(U)$ is open in $E \times E$. So there are $V_1, V_2 \in \mathcal{U}_\tau$ such that $V_1 + V_2 \subseteq U$ and we can take $V = V_1 \cap V_2$. □

1.9. Fact 3: Let $U \in \mathcal{U}_\tau$. Set $b(U) := \bigcap_{|\mu| \geq 1} \mu U$. Then $b(U) \subseteq U, b(U) \in \mathcal{U}_\tau$, and $b(U)$ is **balanced** (i.e. $\lambda b(U) \subseteq b(U)\ \forall\, \lambda \in \mathbb{R}, |\lambda| \leq 1$).

Proof. The map
$$\mathbb{R} \times E \to E, (\lambda, x) \mapsto \lambda x$$
is continuous at $(0, \bar{0})$. So $\exists\, \epsilon > 0, \exists\, V \in \mathcal{U}_\tau$ such that $\lambda V \subseteq U\ \forall\, \lambda \in \mathbb{R}, |\lambda| \leq \epsilon$.
Claim: $\epsilon V \subseteq b(U)$.
Let $|\mu| \geq 1$, we want $\epsilon V \subseteq \mu U$. We can take $\lambda := \dfrac{|\epsilon|}{|\mu|} < \epsilon$ and $\lambda V \subseteq U \Rightarrow \epsilon V \subseteq \mu U$. □

Proposition 1.10. If (E, τ) is locally convex then $\exists\, \mathcal{B} \subseteq \mathcal{U}_\tau$ base for the filter $\mathcal{U}_\tau$ with the following properties:

1. Every $U \in \mathcal{B}$ is absorbing and absolutely convex (i.e. convex and balanced).
2. If $U \in \mathcal{B}$ and $\lambda \neq 0$, then $\lambda U \in \mathcal{B}$.

Conversely, given a base $\mathcal{B}$ for a filter on E with above properties (1.) and (2.) above, there is a unique topology on E such that E is a (locally convex) topological vector space with $\mathcal{B}$ as a base for the filter of neighbourhoods of $\bar{0} \in E$.

Proof. U convex neighborhood of $\bar{0} \in E \Rightarrow b(U)$ is absolutely convex. If $\mathcal{B}_0$ is a base of convex neighbourhoods, then
$$\mathcal{B} := \{\lambda b(U) \mid U \in \mathcal{B}_0, \lambda \neq 0\}$$
has properties (1.) and (2.) above.

Conversely, Let $\mathcal{B}$ be a base for a filter F on E satisfying properties (1.) and (2.). Then $U \in F \Rightarrow \bar{0} \in U$.
The only topology which makes E a topological $\mathbb{R}$-vector space and such that $F = \mathcal{U}_\tau$, has $a + F$ as a filter of $a \in E$ (see again page 1).
Setting $G \subseteq E$ open if $\forall\, a \in G\ \exists\, U \in \mathcal{B}$ such that $a + U \in G$, we define a topology such that $a + F$ is the filter of neighbourhoods of a and E is a topological $\mathbb{R}$-vector space. □

Definition 1.11. $p : E \to [0, \infty[$ is a **seminorm** if

1. $p(\lambda x) = |\lambda| p(x), \forall x \in E, \forall \lambda \in \mathbb{R}$;
2. $p(x + y) \leq p(x) + p(y), \forall x, y \in E$

If $p^{-1}(\{0\}) = \{0\}$ then p is a **norm**.

Proposition 1.12. Let $(p_i)_{i \in I}$ be a family of seminorms on E. Then $\exists$ a coarsest topology τ_C on E such that

(a) E is a topological $\mathbb{R}$-vector space.

(b) p_i is τ_C-continuous $\forall\ i \in I$.

(E, τ_C) is locally convex and the family of sets of the form

$$\{x \in E \mid p_{i_1}(x) < \epsilon, \ldots, p_{i_n}(x) < \epsilon\}; i_1, \ldots, i_n \in I, n \in \mathbb{N}, \epsilon > 0$$

is a base for $\mathcal{U}_{\tau_C}$ (the τ_C-neighbourhood of $\bar{0}$).

Proof. Let $\mathcal{B}$ be the above family of sets. Then $\mathcal{B}$ is a base for a filter on E having properties (1.) and (2.) of Proposition 1.10 and the unique topology asserted in Proposition 1.10 is the coarsest topology on E making E a topological vector space in which each p_i is continuous. □

The topology given by Proposition 1.12 is said to be the topology induced by the family $(p_i)_{i \in I}$ of seminorms.

Lemma 1.13. Let τ_C be the topology induced by the family of seminorms $(p_i)_{i \in I}$ on E. Suppose that $\forall x \in E \setminus \{\bar{0}\}, \exists\ i \in I$ such that $p_i(x) \neq 0$. Then τ_C is Hausdorff.

Proof. Let $x, y \in E, x \neq y$. Then $\exists\ i \in I, \exists\ \epsilon > 0$ such that $p_i(x - y) = 2\epsilon$. So $U_x := \{u \in E \mid p_i(x - u) < \epsilon\}$ and $U_y := \{u \in E \mid p_i(y - u) < \epsilon\}$ are open disjoint neighbourhoods of x and y respectively. □

1.14. Proof of Theorem 1.2:
If we take the topology induced by the family of all seminorms on E, then we obtain the finest locally convex topology on E such that E is a topological $\mathbb{R}$-vector space. We denote it by τ_{max}. We want to see that τ_{max} is Hausdorff.
We need to verify the hypothesis of above lemma, for the family of all seminorms on E. Let $x \in E \setminus \{\bar{0}\}$. Complete $\{x\}$ to a base $\mathcal{B}$ of E as a $\mathbb{R}$-vector space. Define a linear functional

$$\begin{aligned} \chi : E &\to \mathbb{R} \\ x &\mapsto 1 \\ y &\mapsto 0, \forall\ y \in B \setminus \{x\}. \end{aligned}$$

Then $p := |\chi|$ is a semi norm on E and $p(x) \neq 0$. □