POSITIVE POLYNOMIALS LECTURE NOTES (25: 13/07/10)

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Contents

1. TOPOLOGICAL $\mathbb{R}-$VECTOR SPACE (continued)

Theorem 1.1. There is unique Hausdorff topology τ on a finite dimensional $\mathbb{R}$-vector space making it a topological $\mathbb{R}$-vector space.

Remark 1.2. Lets see why the discrete topology τ_D is not good. Let V be an $\mathbb{R}$-vector space. When we ask that the map
$\cdot : \mathbb{R} \times V \to V,$
$(\lambda, v) \longmapsto \lambda v$ is continuous,
we assume that $\mathbb{R}$ is endowed with euclidean topology τ_E and $\mathbb{R} \times V$ with the product topology.
So, for instance, $\{\bar{0}\} \in \tau_D = \mathcal{P}(V)$,
and $\cdot^{-1}(\{\bar{0}\}) = (\mathbb{R} \times \{\bar{0}\}) \cup (\{0\} \times V$, which is not open in the product topology $\tau_E \times \tau_D$.

Remark 1.3. If we do not assume Hausdorffness, there are other topologies as $\tau_I = \{\phi, V\}$ (the indiscrete topology).

1.4. Let V be an $\mathbb{R}$-vector space, $\dim(V) = n \in \mathbb{N}$.
Claim: We may assume $V = \mathbb{R}^n$
Proof of claim: Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a base of V (as a $\mathbb{R}$-vector space).
Let $\Phi_{\mathcal{B}} : V \to \mathbb{R}^n$

$$\sum_{i=1}^{n} a_i v_i \mapsto (a_1, \ldots, a_n)$$

$\Phi_{\mathcal{B}}$ is an isomorphism of $\mathbb{R}$-vector space.
We define:

$A \subset V$ open $\Leftrightarrow \Phi_{\mathcal{B}}(A) \in \tau_E$ (the Euclidean topology on $\mathbb{R}^n$).
This defines a topology τ on V that does not depend on $\mathcal{B}$ and such that (v, τ) is homeomorphic to $(\mathbb{R}^n, \tau_E)$.
Since $(\mathbb{R}^n, \tau_E)$ is a topological $\mathbb{R}$-vector space, also (V, τ) is a topological $\mathbb{R}$-vector space, and so Theorem 1.1 is equivalent to:

Theorem 1.5. The Euclidean topology τ_E on $\mathbb{R}^n$ is the unique Hausdorff topology on $\mathbb{R}^n$ such that the following maps are continuous:
$\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$; $(\lambda, x) \mapsto \lambda x$, and
$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$; $(x, y) \mapsto x + y$.

Proposition 1.6. Let $(P, \leq)$ be a partial order. Let F_1, F_2 be a filter of P , and $B_1 \subseteq F_1$, $B_2 \subseteq F_2$ base. Suppose that

(i) $\forall\ x \in B_1\ \exists\ y \in B_2$ s.t. $y \leq x$.

(ii) $\forall\ x \in B_2\ \exists\ y \in B_1$ s.t. $y \leq x$.

Then we conclude that $F_1 = F_2$.

Proof. "$F_1 \subseteq F_2$": Let $z \in F_2$. B_2 base for $F_2 \Rightarrow\ \exists\ x \in B_2$ s.t. $x \leq z$.
(ii) $\Rightarrow\ \exists\ y \in B_1$ s.t. $y \leq x \leq z$.
F_1 filter, $B_1 \subseteq F_1 \Rightarrow z \in F_1$.
"$F_2 \subseteq F_1$" is symmetric using (i) instead of (ii). □

1.7. Proof of Theorem 1.5:
Let τ be a topology on $\mathbb{R}^n$ s.t. τ is Hausdorff and $(\mathbb{R}^n, \tau)$ is a topological $\mathbb{R}$-vector space.
We want to show that: $\tau = \tau_E$ $\quad\ldots(\star)$
Since the topology is determined from what happens around $\overline{0} \in \mathbb{R}^n$, so

$$(\star) \Leftrightarrow \mathcal{U}_\tau = \mathcal{U}_{\tau_E}.$$

Consider $F_\tau = \{X \subset \mathbb{R}^n \mid \overline{0} \in U \subset X, \text{for some } U \in \tau\}$. Then F_τ is a filter.
We will show that $F_\tau = F_{\tau_E}$, by applying Proposition 1.6, where $(P, \leq) = (\mathcal{P}(\mathbb{R}^n), \subseteq\ )$, $F_1 = F_\tau, F_2 = F_{\tau_E}$, and B_1 and B_2 two bases for F_1 and F_2 with properties (i) and (ii). We will find next a good base for F_τ.

Definition 1.8. Let (E, τ) be a topological $\mathbb{R}$-vector space. $X \subset E$ is said to be **circled** if $\alpha \in \mathbb{R}, |\alpha| < 1, x \in X \Rightarrow \alpha x \in X$.

Proposition 1.9. Any topological $\mathbb{R}$-vector space (E, τ) has a base of circled neighbourhoods of $\overline{0} \in E$.

Proof. $\mathcal{B}_\tau = \{ \cup_{|\alpha|\le 1} \alpha V \mid V \in \mathcal{U}_\tau\}$ is a base for F_τ.
(We will actually show that $\mathcal{B}_\tau$ is a base for $\mathcal{U}_\tau$, since it is equivalent)
Fix $V \in \mathcal{U}_\tau$. By continuity in $(\overline{0}, 0)$ of the product $\exists\, \epsilon > 0,\ \exists\, W \in \mathcal{U}_\tau$ s.t.
$|\lambda| \le \epsilon$ and $x \in W \Rightarrow \lambda x \in V$.
Set $U := \epsilon W$. Then $\alpha V \subset U\ \forall\, \alpha, |\alpha| \le 1$.
So, $\cup_{|\alpha|\le 1}\alpha V \subseteq U$. □

1.10. Topological fact: Let (X, τ) be a topological space, $K \subseteq X$. Then
$x \in \overline{K} \Leftrightarrow\ \forall\, V_x\ \ \tau-$ open containing x, $V_x \cap K \neq \phi$.

Proof. "⇒" Suppose, for a contradiction $V_x\ \ \tau-$ open containing x, with $V_x \cap K = \phi$. Then $x \notin K$, and $A = (X \setminus \overline{K}) \cup V_x$ is open, so $A \cap K = \phi$ in contradiction with the fact that $X \setminus \overline{K}$ is the biggest open set disjoint from K (because $\overline{K}$ is the smallest closed set containing K).
"⇐" Suppose $x \notin \overline{K}$, so $x \in X \setminus \overline{K}$ which is open. Then $\exists\, V_x$ open containing x s.t. $V_x \subset V \setminus \overline{K}$, contradiction. □

Lemma 1.11. Let (X, τ) be a Hausdorff topological space. If $K \subseteq X$ is τ-compact, then K is τ-closed.

Proof. Let $x \in \overline{K}$. We want $x \in K$. Suppose on contrary $x \notin K$.
$x \in \overline{K} \Leftrightarrow \forall\, V_x\ \tau-$ open containing x, $V_x \cap K \neq \phi$.
X Hausdorff $\Rightarrow \forall\, a \in K:\ \exists\, \tau-$ open $V_a \ni a, V_a^x \ni x$ such that $V_a \cap V_a^x = \phi$.
$\{V_a \mid a \in K\}$ is an open covering of K.
K compact $\rightarrow\ \exists$ finite subcovering $\{V_{a_1}, \dots, V_{a_n}\}$. Set $V_x := V_{a_1}^x \cap \dots \cap V_{a_n}^x$.
Then V_x is $\tau-$open (since finite intersection of open sets is open) containing x and $V_x \cap K = \phi$, a contradiction
(otherwise if $e \in V_x \cap K$, then $\exists\, i = 1, \dots, n$ s.t. $e \in V_x \cap V_{a_i}^x = \phi$). □

1.12. Proof of Theorem 1.5 continued:
<u>To prove:</u> $\tau = \tau_E$

"$\tau \subseteq \tau_E$ " : Let U be circled $\tau-$neighbourhood of $\overline{0}$, and let V be a circled $\tau-$neighbourhood of $\overline{0}$ s.t. $\underbrace{V + \dots + V}_{n-\text{ times}} \subseteq U$.
V absorbing (see Fact 1 of last lecture) $\Rightarrow\ \exists\, k > 0$ s.t. $ke_i \in V\ \forall\, i = 1, \dots, n$.
$\Rightarrow k \sum_{i=1}^{n} \alpha_i e_i \in U$ if $\sum_i |\alpha_i|^2 \le 1$.
Therefore $B_k := \{x \in \mathbb{R}^n \mid \|x\|_2 < k\} \subset U$.

"$\tau_E \subseteq \tau$ " : Let $B = \{x \in \mathbb{R}^n \mid \|x\|_2 < 1\}$ and $S := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$.
S τ_E-compact, $\tau \subseteq \tau_E \Rightarrow S$ is $\tau-$compact.
By Lemma 1.11, S is $\tau-$ closed.

$\overline{0} \notin S \Rightarrow \exists$ a circled $\tau-$neighbourhood V of $\overline{0}$ s.t. $V \cap S = \phi$.
We want $V \subset B$. Suppose not: $\exists\, x \in V$ s.t. $||x||_2 \geq 1$ ($\Leftrightarrow x \notin B$),
then $\dfrac{x}{||x||_2} \in V \cap S = \phi$, a contradiction.
Thus B is a $\tau-$ neighbourhood of $\overline{0}$. Multipying by scalars we have a $\tau-$ neighbourhood base at $\overline{0}$, so $\tau_E \subseteq \tau$.

Remark 1.13. The hypothesis that $\dim V = n \in \mathbb{N}$ cannot be avoided. Consider for instance $V = \mathbb{R}^{\mathbb{N}}$:
We saw that τ_{fin} is a topology on $\mathbb{R}^{\mathbb{N}}$ making it a topological $\mathbb{R}-$vector space. τ_{fin} is Hausdorff.
It is not the only use !
Consider for instance the product topology τ on $\mathbb{R}^{\mathbb{N}}$. τ is Hausdorff and makes $\mathbb{R}^{\mathbb{N}}$ a topological $\mathbb{R}-$ vectore space.
$\tau \subseteq \tau_{\text{fin}}$, but $\tau \neq \tau_{\text{fin}}$. For instance: $(0,1)^{\mathbb{N}} \in \tau_{\text{fin}} \setminus \tau$.