

POSITIVE POLYNOMIALS LECTURE NOTES

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Contents

1. Generalities about polynomials	1
2. PSD- and SOS- polynomials	2
3. Convex sets, cones and extremality	3

1. GENERALITIES ABOUT POLYNOMIALS

Definition 1.1. For a **polynomial** $p \in \mathbb{R}[X_1, \dots, X_n]$, we write

$$p(\underline{X}) = \sum_{\underline{i} \in \mathbb{Z}_+^n} c_i \underline{X}^{\underline{i}}; \quad c_i \in \mathbb{R},$$

where $\underline{X}^{\underline{i}} = X_1^{i_1} \dots X_n^{i_n}$ is a monomial of degree $= |\underline{i}| = \sum_{k=1}^n i_k$ and $c_i \underline{X}^{\underline{i}}$ is a term.

Definition 1.2. A polynomial $p(\underline{X}) \in \mathbb{R}[\underline{X}]$ is called **homogeneous** or **form** if all terms in p have the same degree.

Notation 1.3. $\mathcal{F}_{n,m} := \{F \in \mathbb{R}[X_1, \dots, X_n] \mid F \text{ is a form and } \deg(F) = m\}$, the set of all forms in n variables of degree m (also called set of n -ary m -ics forms), for $n, m \in \mathbb{N}$.

Convention: $0 \in \mathcal{F}_{n,m}$.

Definition 1.4. Let $p \in \mathbb{R}[X_1, \dots, X_n]$ of degree m . The **homogenization** of p w.r.t X_{n+1} is defined as

$$p_h(x_1, \dots, x_n, x_{n+1}) := x_{n+1}^m p\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)$$

Note that p_h is a homogeneous polynomial of degree m and in $n + 1$ variables i.e. $p_h \in \mathcal{F}_{n+1,m}$.

Proposition 1.5. (1) Let $p(\underline{X}) \in \mathbb{R}[\underline{X}]$, $\deg(p) = m$, then

number of monomials of $p \leq \binom{m+n}{n}$

(2) Let $F(\underline{X}) \in \mathcal{F}_{n,m}$, then

number of monomials of $F \leq N := \binom{m+n-1}{n-1}$ □

Remark 1.6. $\mathcal{F}_{n,m}$ is a finite dimensional real vector space with $\mathcal{F}_{n,m} \simeq \mathbb{R}^N$.

2. PSD- AND SOS- POLYNOMIALS

Definition 2.1. (1) $p(\underline{x}) \in \mathbb{R}[\underline{X}]$ is **positive semidefinite (psd)** if

$$p(\underline{x}) \geq 0 \quad \forall \underline{x} \in \mathbb{R}^n.$$

(2) $p(\underline{x}) \in \mathbb{R}[\underline{X}]$ is **sum of squares (SOS)** if $\exists p_i \in \mathbb{R}[\underline{X}]$ s.t.

$$p(\underline{x}) = \sum_i p_i(\underline{x})^2.$$

Notation 2.2. $\mathcal{P}_{n,m}$:= set of all forms $F \in \mathcal{F}_{n,m}$ which are psd, and

$\Sigma_{n,m}$:= set of all forms $F \in \mathcal{F}_{n,m}$ which are sos.

Lemma 2.3. If a polynomial p is psd then p has even degree. □

Remark 2.4. From now on (using lemma 2.3) we will often write $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$.

Lemma 2.5. Let p be a homogeneous polynomial of degree $2d$, and p sos. Then every sos representation of p consists of homogeneous polynomials only, i.e.

$$p(\underline{x}) = \sum_i p_i(\underline{x})^2 \Rightarrow p_i(\underline{x}) \text{ homogenous of degree } d, \text{ i.e. } p_i \in \mathcal{F}_{n,d}. \quad \square$$

Remark 2.6. The properties of psd-ness and sos-ness are preserved under homogenization (see the following lemma).

Lemma 2.7. Let $p(\underline{x})$ be a polynomial of degree m . Then

(1) p is psd iff p_h is psd,

(2) p is sos iff p_h is sos. □

So we can focus our investigation of psdness of polynomials versus sosness of polynomials to those of forms, i.e. study and compare $\Sigma_{n,m} \subseteq \mathcal{P}_{n,m}$.

Theorem 2.8. (Hilbert) $\Sigma_{n,m} = \mathcal{P}_{n,m}$ iff

- (i) $n = 2$ [i.e. binary forms] or
- (ii) $m = 2$ [i.e. quadratic forms] or
- (iii) $(n, m) = (3, 4)$ [i.e. ternary quartics].

For the ternary quartics case ($\mathcal{F}_{3,4}$), we shall study the **convex cones** $\mathcal{P}_{n,m}$ and $\Sigma_{n,m}$.

3. CONVEX SETS, CONES AND EXTREMALITY

Definition 3.1. A subset C of \mathbb{R}^n is **convex set** if $\underline{a}, \underline{b} \in C \Rightarrow \lambda \underline{a} + (1 - \lambda) \underline{b} \in C$, for all $0 < \lambda < 1$.

Proposition 3.2. The intersection of an arbitrary collection of convex sets is convex.

Notation 3.3. $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$.

Definition 3.4. Let $\underline{c}_1, \dots, \underline{c}_k \in \mathbb{R}^n$. A **convex combination** of $\underline{c}_1, \dots, \underline{c}_k$ is any vector sum

$$\alpha_1 \underline{c}_1 + \dots + \alpha_k \underline{c}_k, \text{ with } \alpha_1, \dots, \alpha_k \in \mathbb{R}_+ \text{ and } \sum_{i=1}^k \alpha_i = 1.$$

Theorem 3.5. A subset $C \subseteq \mathbb{R}^n$ is convex if and only if it contains all the convex combinations of its elements.

Proof. (\Leftarrow) clear

(\Rightarrow) Let $C \subseteq \mathbb{R}^n$ be a convex set. By definition C is closed under taking convex combinations with two summands. We show that it is also closed under finitely many summands.

Let $k > 2$. By Induction on k , assuming it true for fewer than k .

Given a convex combination $\underline{c} = \alpha_1 \underline{c}_1 + \dots + \alpha_k \underline{c}_k$, with $\underline{c}_1, \dots, \underline{c}_k \in C$

Note that we may assume $0 < \alpha_i < 1$ for $i = 1, \dots, k$; otherwise we have fewer than k summands and we are done.

Consider $\underline{d} = \frac{\alpha_2}{1 - \alpha_1} \underline{c}_2 + \dots + \frac{\alpha_k}{1 - \alpha_1} \underline{c}_k$

we have $\frac{\alpha_2}{1 - \alpha_1}, \dots, \frac{\alpha_k}{1 - \alpha_1} > 0$ and $\frac{\alpha_2}{1 - \alpha_1} + \dots + \frac{\alpha_k}{1 - \alpha_1} = 1$

Thus \underline{d} is a convex combination of $k - 1$ elements of C and $\underline{d} \in C$ by induction.

Since $\underline{c} = \alpha_1 \underline{c}_1 + (1 - \alpha_1) \underline{d}$, it follows that $\underline{c} \in C$. □

Definition 3.6. The intersection of all convex sets containing a given subset $S \subseteq \mathbb{R}^n$ is called the **convex hull** of S and is denoted by $\text{cvx}(S)$.

Remark 3.7. The convex hull of $S \subseteq \mathbb{R}^n$ is a convex set and is the uniquely defined smallest convex set containing S .

Theorem 3.8. For any $S \subseteq \mathbb{R}^n$,
 $\text{cvx}(S)$ = the set of all convex combinations of the elements of S .

Proof. (\supseteq) The elements of S belong to $\text{cvx}(S)$, so all their convex combinations belong to $\text{cvx}(S)$ by Theorem 3.5.

(\subseteq) On the other hand we observe that the set of convex combinations of elements of S is itself a convex set:

let $\underline{c} = \alpha_1 \underline{c}_1 + \dots + \alpha_k \underline{c}_k$ and $\underline{d} = \beta_1 \underline{d}_1 + \dots + \beta_l \underline{d}_l$, where $\underline{c}_i, \underline{d}_i \in S$, then

$\lambda \underline{c} + (1 - \lambda) \underline{d} = \lambda \alpha_1 \underline{c}_1 + \dots + \lambda \alpha_k \underline{c}_k + (1 - \lambda) \beta_1 \underline{d}_1 + \dots + (1 - \lambda) \beta_l \underline{d}_l$, $0 \leq \lambda \leq 1$ is just another convex combination of elements of S .

So by minimality property of $\text{cvx}(S)$, it follows that $\text{cvx}(S) \subseteq$ the set of all convex combinations of the elements of S . \square

Corollary 3.9. The convex hull of a finite subset $\{\underline{s}_1, \dots, \underline{s}_k\} \subseteq \mathbb{R}^n$ consists of all the vectors of the form $\alpha_1 \underline{s}_1 + \dots + \alpha_k \underline{s}_k$ with $\alpha_1, \dots, \alpha_k \geq 0$ and $\sum_i \alpha_i = 1$. \square

Definitions 3.10. (1) A set which is the convex hull of a finite subset of \mathbb{R}^n is called a **convex polytope**, i.e. $C \subseteq \mathbb{R}^n$ is a convex polytope if $C = \text{cvx}(S)$ for some finite $S \subseteq \mathbb{R}^n$.

(2) A point in a polytope is called a **vertex** if it is not on the line segment joining any other two distinct points of the polytope.

Remark 3.11. (1) Convex polytope is necessarily closed and bounded, i.e. compact.

(2) A convex polytope is always the convex hull of its vertices.

More general version for compact sets is the Krein Milman theorem:

Theorem 3.12. (Krein-Milman) Let $C \subseteq \mathbb{R}^n$ be a compact and convex set. Then C is the convex hull of its extreme points. \square

Definitions 3.13. $\underline{x} \in C$ is **extreme** if $C \setminus \{\underline{x}\}$ is convex.