Universität Konstanz
Fachbereich Mathematik und Statistik
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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 04

These exercises will be collected Tuesday 25 May in the mailbox n.14 of the Mathematics department.

1. Let $C \subset \mathbb{R}^n$ be a line free convex cone.

(*i*) Let $\underline{x} \in C$, $\underline{x} \neq \underline{0}$. Show that $\underline{x}$ belongs to an extremal ray of C if and only if

$$\underline{x} = \underline{x}_1 + \underline{x}_2, \quad \underline{x}_1, \underline{x}_2 \in C \;\Rightarrow\; \underline{x}_i = \lambda_i \underline{x},\, \lambda_i > 0,\, \lambda_1 + \lambda_2 = 1.$$

(*ii*) Show that the set of convex linear combinations of points in extremal rays of C is equal to the set of sum of points in extremal rays of C.

2. Let $F(\mathrm{x}, \mathrm{y}) = \mathrm{x}^6 + \mathrm{x}^4\mathrm{y}^2 + 3\mathrm{x}^2\mathrm{y}^4 + 3\mathrm{y}^6$. Write $F(\mathrm{x}, \mathrm{y})$ as a sum of two squares.

3. Let $F(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t}) = 2\mathrm{x}^2 + 2\mathrm{xy} + 2\mathrm{y}^2 + 3\mathrm{z}^2 + 2\mathrm{zt} + 3\mathrm{t}^2$. Write $F(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{t})$ as a sum of four quares.

4. Let R be a real closed field. We denote by $\mathcal{P}_{n,m}(R)$ the set of psd forms with coefficients in R of degree m in n variables, and with $\Sigma_{n,m}(R)$ the set of forms with coefficients in R of degree m in n variables which are sums of squares. Show that :

(*a*) for every $d \in \mathbb{N}$, $\mathcal{P}_{2,2d}(R) = \Sigma_{2,2d}(R)$.

(*b*) for every $n \in \mathbb{N}$, $\mathcal{P}_{n,2}(R) = \Sigma_{n,2}(R)$.

Hilbert proved that

$$(*) \qquad f \in \mathcal{P}_{3,4}(\mathbb{R}) \;\Rightarrow\; \exists\, f_1, f_2, f_3 \in \mathcal{F}_{3,2}(\mathbb{R}) \text{ such that } f = f_1^2 + f_2^2 + f_3^2.$$

(the proof uses advanced Algebraic Geometry and we will not see it).

Assuming $(*)$, show that:

(c) $\mathcal{P}_{3,4}(R) = \Sigma_{3,4}(R)$.