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ÜBUNGEN ZUR VORLESUNG POSITIVE POLYNOME

BLATT 07

These exercises will be collected Tuesday 22th June in the mailbox n.14 of the Mathematics department.

Let $\underline{\mathrm{x}} = (\mathrm{x}_1, \dots, \mathrm{x}_n)$.

1. Let $L\colon \mathbb{R}[\underline{\mathrm{x}}] \to \mathbb{R}$ be a linear functional, $g \in \mathbb{R}[\underline{\mathrm{x}}]$, $\langle\, ,\rangle_g\colon \mathbb{R}[\underline{\mathrm{x}}] \times \mathbb{R}[\underline{\mathrm{x}}] \to \mathbb{R}$ the symmetric bilinear form defined by $\langle h\, , k\rangle_g := L(hkg)$ and S_g the symmetric matrix with $\alpha\beta$ entry $\langle\, \underline{\mathrm{x}}^\alpha, \underline{\mathrm{x}}^\beta\,\rangle_g$ for all $\alpha, \beta \in (\mathbb{Z}_+)^n$.

Show that the following are equivalent:

(*i*) $L(\sigma g) \geqslant 0$ for all $\sigma \in \sum \mathbb{R}[\underline{\mathrm{x}}]^2$.

(*ii*) $L(h^2 g) \geqslant 0$ for all $h \in \mathbb{R}[\underline{\mathrm{x}}]$.

(*iii*) $\langle\, ,\rangle_g$ is PSD.

(*iv*) S_g is PSD.

2.(*a*) Suppose $n = 1$. Let $L\colon \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ be a linear functional such that

$$L(\mathrm{x}^m) = 0 \quad \text{for} \quad m = 2 \quad \text{and} \quad \forall\, m \geqslant 4.$$

Give necessary and sufficient conditions so that there is a Borel measure μ on $\mathbb{R}$ such that

$$L(f) = \int_{\mathbb{R}} f d\mu \qquad \forall\, f \in \mathbb{R}[\mathrm{x}].$$

(*b*) For $x \in \mathbb{R}$, let $L_x\colon \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ be the evaluation on x, i.e. $L_x(f) = f(x)$ for all $f \in \mathbb{R}[\mathrm{x}]$. For which $x \in \mathbb{R}$ and $K \subseteq \mathbb{R}$ closed there is Borel measure μ on K such that

$$L_x(f) = \int_K f d\mu \qquad \forall\, f \in \mathbb{R}[\mathrm{x}]?$$

3. Let $L\colon \mathbb{R}[\mathrm{x}] \to \mathbb{R}$ be a linear functional and set $L(\mathrm{x}^i) := s_i$ for every $i \in \mathbb{Z}_+$. Show that there is a Borel measure μ on $K = [0,1] \subset \mathbb{R}$ such that

$$L(f) = \int_K f d\mu \qquad \forall\, f \in \mathbb{R}[\mathrm{x}]$$

if and only if the following symmetric matrices are PSD:

$$\begin{pmatrix} s_0 & s_1 & s_2 & \dots \\ s_1 & s_2 & \dots\dots\dots & \\ s_2 & \dots\dots\dots\dots & & \\ \vdots & & & \end{pmatrix}$$

$$\begin{pmatrix} s_1 & s_2 & s_3 & \dots \\ s_2 & s_3 & \dots\dots\dots & \\ s_3 & \dots\dots\dots\dots & & \\ \vdots & & & \end{pmatrix}$$

$$\begin{pmatrix} s_0 - s_1 & s_1 - s_2 & s_2 - s_3 & \dots \\ s_1 - s_2 & s_2 - s_3 & \dots\dots\dots\dots & \\ s_2 - s_3 & \dots\dots\dots\dots\dots\dots\dots & & \\ \vdots & & & \end{pmatrix}$$

4. We recall that there is a natural bijection

$$\{L\colon \mathbb{R}[\underline{\mathrm{x}}] \to \mathbb{R} \text{ linear functional}\} \;\leftrightarrow\; \{\tau\colon (\mathbb{Z}_+)^n \to \mathbb{R}\}$$

given by

$$L(\underline{\mathrm{x}}^\alpha) = \tau(\alpha) \qquad \forall\, \alpha \in (\mathbb{Z}_+)^n.$$

Let

$$\Sigma := \sum \mathbb{R}[\underline{\mathrm{x}}]^2,$$

$$\mathcal{P} := \{f \in \mathbb{R}[\underline{\mathrm{x}}] \mid f(\underline{x}) \geqslant 0 \;\forall\, \underline{x} \in \mathbb{R}^n\}.$$

Describe $\Sigma^\vee$ and $\mathcal{P}^\vee$ in terms of conditions on multisequences.