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MODEL THEORY – EXERCISE 7

To be submitted on Wednesday 25.05.2011 by 14:00 in the mailbox.

Definition.

(1) Let A be a set, and $F \subseteq \mathcal{P}(A)$ is called a *filter* on A if the following holds:
 (a) $\emptyset \notin F$.
 (b) $F \neq \emptyset$.
 (c) If $X, Y \in F$ then $X \cap Y \in F$.
 (d) If $X \in F$ and $X \subseteq Y \subseteq A$ then $Y \in F$.

(2) F is said to be a *principle filter* if there exists some $X \subseteq A$ such that $F = \{Y \subseteq A \,|\, Y \supseteq X\}$.

(3) F is said to be an *ultrafilter* if for every $X \subseteq A$, $X \in F$ or $A \backslash X \in F$.

(4) Let A be an infinite set. A family $C \subseteq \mathcal{P}(A)$ is called *independent* if for every $n, m \in \mathbb{N}$ and $X_1, \ldots, X_n \in C$, $Y_1, \ldots, Y_m \in C$ such that no two are equal, $\bigcap_i X_i \cap \bigcap (A \backslash Y_j) \neq \emptyset$.

(5) A structure M is said to be *countably saturated* if given a set $\{\varphi_n(x) \in L(M) \,|\, n < \omega\}$ such that each finite subset has a solution a_n, then $\{\varphi_n(x) \,|\, n < \omega\}$ has a common solution a. (Recall that $L(M)$ is the signature L where we also allow parameters from M).

Question 1.

(1) Suppose F is a filter on a set A and there exists some finite $X \subseteq A$ such that $X \in F$. Show that F is principle.
Solution: Let $X_0 \subseteq X$ be the smallest set such that $X_0 \in F$. Then F is generated by X_0: if $Y \in F$, then $Y \cap X_0 \in F$ but it cannot be that $Y \cap X_0 \subsetneq X_0$.

(2) Show that if F is a principle ultrafilter on a set A then there exists $a \in A$ such that $F = \{X \in A \,|\, X \supseteq \{a\}\}$.
Solution: suppose F is generated by X. Let $a \in X$ (note that $X \neq \emptyset$). It cannot be that $A \backslash \{a\} \in F$.

(3) Suppose F is a principle ultrafilter on a set A, and that M_i for $i \in A$ is an L-structure. Show that the ultra-product $\prod_{i \in A} M_i / F$ is isomorphic to M_{i_0} for some $i_0 \in A$.
Solution: by (2) there is some i_0 that generates F. Map $\langle x_i \,|\, i/A \rangle / D$ to x_{i_0}.

(4) Suppose A is an infinite set. Let I be the set of all finite subsets of A. For each $s \in I$, let $X_s = \{t \in I \,|\, t \supseteq s\}$. Show that $\{X \subseteq A \,|\, \exists s \in I \,(X \supseteq X_s)\}$ is a filter on A.
Solution: $X_s \cap X_t = X_{s \cup t}$.

Question 2.
Let A be an infinite set. Compute the cardinality of the set $U = \{D \,|\, D \text{ is an ultrafilter on } A\}$. Use the following steps:

(1) Show that $|U| \leq 2^{2^{|A|}}$.
Solution: $U \subseteq \mathcal{P}(\mathcal{P}(A))$.

(2) Show that there exists a family $C \subseteq \mathcal{P}(A)$, such that $|C| = 2^{|A|}$ and if $X_1 \neq X_2 \in C$ then $X_1 \subsetneq X_2$, $X_2 \subsetneq X_1$.
Hint: look at graphs of functions, use the fact that $|A|^2 = |A|$.
Solution: let $f : A^2 \to A$ be an isomorphism. For each function $r : A \to A$, $f[r] \subseteq A$. Let $C = \{f[r] \,|r : A \to A\}$. If $r_1 \neq r_2$, then there is some $a \in A$ such that $r_1(a) \neq r_2(a)$, $(a, r_1(a)) \notin r_2$ while $(a, r_2(a)) \notin r_1$.

(3) Show that there exists an independent family $C \subseteq \mathcal{P}(A)$ of subsets of A of cardinality $2^{|A|}$.
Hint: let C_0 be as above. Let $M = \omega \times A^{<\omega}$ (where $A^{<\omega}$ is the set of finite sequences from A). For all $X \in C_0$, let $\bar{X}$ be the set of pairs $(n, \bar{a})$ from M such that: the length of $\bar{a}$ is divisible by n, so $\bar{a} = \bar{a}_0 \frown \ldots \frown \bar{a}_{k-1}$ where $\lg(\bar{a}_i) = n$, and there is $i < k$ such that all elements from $\bar{a}_i$ are in X. Show that $\left\{\bar{X} \,|X \in C\right\}$ is an independent set in M and conclude.
Solution: By a theorem from class $|M| = |A|$. If $X_0, \ldots, X_{n-1}, Y_0 \ldots, Y_{m-1} \in C_0$ as in the definition, for each pair $i < n, j < m$, let $a_{i,j} \in X_i \backslash Y_j$. For $i < n$, let $\bar{a}_i = \langle a_{i,0}, a_{i,1}, \ldots, a_{i,m-1}\rangle$. Finally, $(n, \bar{a}_0 \frown \ldots \frown \bar{a}_{n-1}) \in \bar{X}_i$ for $i < n$ but not in Y_j for $j < m$.

(4) Conclude that $|U| = 2^{2^{|A|}}$.
Solution: For every choice s of a subset of C, we can find an ultrafilter that contains $\{X \,|X \in s\} \cup \{A \backslash X \,|X \notin s\}$. It satisfies the finite intersection property, hence is contained in an ultrafilter.

Question 3.

(1) Suppose A is an infinite set, and D is a filter on an infinite set I. Show that $|A| \leq \left|A^I/D\right| \leq |A|^{|I|}$.
Solution: Let $\varphi : A \to A^I$ be the diagonal embedding. Now if $a \neq b$, then $\varphi(a)/D \neq \varphi(b)/D$ (the set of indices where they are equal is $\emptyset$).

(2) Show that for every A and I there exists an ultrafilter on I such that $\left|A^I/D\right| = |A|$.
Solution: take a principle ultrafilter.

(3) Let D be a non-principle filter on ω. Show that $|\mathbb{N}^\omega/D| = 2^{\aleph_0}$.
Hint: use the fact that every natural number has a unique presentation in base 2.
Solution: By (1), we have $\leq \aleph_0^{\aleph_0} \leq \left(2^{\aleph_0}\right)^{\aleph_0} = 2^{\aleph_0}$. For the other direction, for every function $f : \omega \to 2$, let $n_i^f = \sum_{j<i} 2^j f(j)$. If $f_1 \neq f_2$, then $n_i^{f_1} \neq n_i^{f_2}$ for big enough i, and this set of indices is cofinite, so it cannot be that its complement is in D.

Question 4.

Prove the following:
Let $\{A_i \,|i \in \mathbb{N}\}$ be structures in a signature L. Let D be an ultrafilter containing the filter of co-finite sets, and let A be the ultraproduct. Then A is countably saturated.
Follow these steps:

(1) We are given a set $\{\varphi_n(x) \in L(M) \,|\, n < \omega\}$ as in the definition, and we want to find a common solution. Show that it's enough to show it assuming $\varphi_n \in L$ (i.e. no parameters).
Solution: Add the parameters appearing in this set to L, interpret them in A_i in the obvious way: $c^{A_i} = c^A(i)$.

(2) For $n < \omega$, pick $a_n \in A_n$ such that $A_n \models \varphi_1(a_n), \ldots, \varphi_k(a_n)$ for $k \leq k(n)$, where $k(n) \leq n$ is as large as possible (can be 0). Show that $\langle a_n \,|\, n \in \omega \rangle / D$ is a common solution.
Solution: we must show that $\{n \,|\, A_n \models \varphi_n(a_n)\}$ is in D.