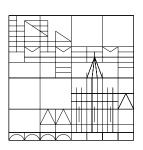
Universität Konstanz

Fachbereich Mathematik und Statistik

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Real Algebraic Geometry II Exercise Sheet 10

Definition: An ordered field K is *root closed for positive elements*, if for every $n \in \mathbb{N}$ and every $x \in K^{>0}$ there exists $y \in K$ such that $y^n = x$.

Definition: Let K be an ordered field and v the natural valuation on K. $S \subseteq K^{>0}$ is called *value group section*, if there exists an order preserving embedding $t : v(K^{\times}) \hookrightarrow K^{>0}$ with v(t(g)) = g for all $g \in v(K^{\times})$, such that $S = t(v(K^{\times}))$.

Definition: Let K be an ordered field, v the natural valuation on K and k the residue field of (K, v). $T \subseteq K$ is called *residue field section*, if there exists an order preserving embedding $\iota : k \hookrightarrow K$ with $\overline{\iota(c)} = c$ for all $c \in K$, such that $T = \iota(k)$.

Exercise 1

Let K be an ordered field. Let K be root closed for positive elements. Let v be the natural valuation on K.

- (a) Show that (K, v) admits a value group section.
- (b) Assume in addition that K is real closed. Show that K admits a residue field section.

Definition: Let K and L be fields. We extend addition and multiplication from L to $L \cup \{\infty\}$ as follows. For $a \in L$ and $b \in L \setminus \{0\}$

$$\infty + a := a + \infty := \infty$$
$$\infty \cdot b := b \cdot \infty := \infty$$
$$\infty \cdot \infty := \infty$$

We will not define $\infty + \infty$, $\infty \cdot 0$ and $0 \cdot \infty$.

A map

$$\varphi: K \longrightarrow L \cup \{\infty\}$$

is called *place* (German *Stelle*) on K, if for all $x, y \in K$ the following is true whenever the right side is defined:

- (i) $\varphi(x+y) = \varphi(x) + \varphi(y)$
- (ii) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$
- (iii) $\varphi(1) = 1$.

Exercise 2

Let K be a field.

(a) Assume L is a field and $\varphi: K \longrightarrow L \cup \{\infty\}$ is a place on K. Show that

$$\mathcal{O} := \varphi^{-1} \left(L \right)$$

is a valuation ring on K with maximal ideal

$$\mathcal{M} = \varphi^{-1}\left(\{0\}\right)$$

and residue field

$$\mathcal{O}/\mathcal{M} \cong \varphi(\mathcal{O})$$
.

(b) Let ${\mathcal O}$ be a valuation ring on K with maximal ideal ${\mathcal M}.$ Let

$$\widetilde{arphi}:\mathcal{O}\twoheadrightarrow\mathcal{O}/\mathcal{M}$$

be the residue map of (K, \mathcal{O}) . Define

$$\varphi: K \longrightarrow \mathcal{O}/\mathcal{M} \cup \{\infty\}$$

by

$$\varphi\left(x\right) := \begin{cases} \widetilde{\varphi}\left(x\right), & x \in \mathcal{O} \\ \infty & , & x \notin \mathcal{O} \end{cases}$$

for $x \in K$.

Show that φ is a place on K.

Definition: Let (K, \mathcal{O}) be a valued field. The place as in Exercise 2 (b) is called the canonical place of \mathcal{O} .

Exercise 3

Let (K, \leq) be an ordered field. Let v be a valuation on K with valuation ring \mathcal{O} , maximal ideal \mathcal{M} and residue field \overline{K} . Let $\varphi : \mathcal{O} \to \overline{K}$ be the canonical place of \mathcal{O} .

Show that the following are equivalent

- (i) If $a, b \in K$ with 0 < a < b, then $v(a) \ge v(b)$.
- (ii) \mathcal{M} is convex in (K, \leq) .
- (iii) \mathcal{O} is convex in (K, \leq) .
- (iv) $\overline{P} := \{ p + \mathcal{M} \mid p \in K^{\geq 0} \cap \mathcal{O} \}$ is an ordering on \overline{K} .
- (v) If $x, y \in \mathcal{O}$ with $x \leq y$, then $\varphi(y) \varphi(x) \in \overline{P}$.
- (vi) $1 + \mathcal{M} \subseteq K^{>0}$.
- (vii) If $x \in \mathcal{M}$, then |x| < q for all $q \in \mathbb{Q}$ (where |.| is the absolute value induced by \leq).

The exercise will be collected **Thursday**, 25/06/2015 until 10.00 at box 13 near F 441.

http://www.math.uni-konstanz.de/~ dupont/rag.htm