REAL ALGEBRAIC GEOMETRY LECTURE NOTES (15: 08/12/09)

SALMA KUHLMANN

Contents

1. Algebraic sets and constructible sets

Definition 1.1. Let K be a field. Let $f_1, \ldots, f_k \in K[\underline{\mathrm{x}}] = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. A set of the form

$$Z(f_1, \ldots, f_k) := \{\underline{x} \in K^n : f_1(\underline{x}) = \cdots = f_k(\underline{x}) = 0\}$$

is called an **algebraic set**.

Definition 1.2. A subset $C \subseteq K^n$ is **constructible** if it is a finite Boolean combination of algebraic sets.

Remark 1.3.

(1) A constructible subset of K is either finite or cofinite.

(2) Let $K = \mathbb{R}$ and consider the algebraic set

$$Z = \{(x, y) \in K^2 : x^2 - y = 0\}.$$

Its image under the projection $\pi(x, y) = y$ is $\pi(Z) = [0, \infty[$ which is neither finite nor cofinite.

This shows that in general a Boolean conbination of algebraic sets is not closed under projections.

Definition 1.4. A function $F\colon K^n \to K^m$ is a **polynomial map** if there are polynomials $F_1, \ldots, F_m \in K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ such that for every $\underline{x} \in K^n$,

$$F(\underline{x}) = (F_1(\underline{x}), \ldots, F_m(\underline{x})) \in K^m.$$

Example 1.5. The projection map

$$\prod_n : \quad K^{n+m} \longrightarrow K^n$$

$$(x_1, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n)$$

is a polynomial map, where for every i, $1 \leqslant i \leqslant n$,

$$P_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = x_i$$

and $\prod_n = (P_1, \ldots, P_n)$.

By Chevalley's Theorem (Quantifier elimination for algebraically closed fields), if K is an algebraically closed field, then the image of a constructible set over K under a polynomial map is constructible (in particular under projections).

Let R be now a real closed field.

Remark 1.6.

(1) A semialgebraic subset of R^n is the projection of an algebraic subset of R^{n+m} for some $m \in \mathbb{N}$, e.g. the semialgebraic set

$$\{\underline{x} \in R^n : f_1(\underline{x}) = \cdots = f_l(\underline{x}) = 0, g_1(\underline{x}) > 0, \ldots, g_m(\underline{x}) > 0\}$$

is the projection of the algebraic set

$$\{(\underline{x}, \underline{y}) \in R^{n+m} : f_1(\underline{x}) = \cdots = f_l(\underline{x}) = 0,\ y_1^2 g_1(\underline{x}) = 1, \ldots,\ y_m^2 g_m(\underline{x}) = 1\}.$$

(2) Every semialgebraic subset of R^n is in fact the projection of an algebraic subset of R^{n+1} (Motzkin, The real solution set of a system of algebraic inequalities is the projection of a hypersurface in one more dimension, 1970 Inequalities, II Proc. Second Sympos., U.S. Air Force Acad., Colo., 1967 pp. 251–254 Academic Press, New York).

2. Topology

For $\underline{x} = (x_1, \ldots, x_n) \in R^n$, we have the norm $||\,\underline{x}\,|| := \sqrt{x_1^2 + \cdots + x_n^2}$.
Let $r \in R$, $r > 0$.

$$B_n(\underline{x}, r) = \{\underline{y} \in R^n : ||\,\underline{y} - \underline{x}\,|| < r\} \qquad \text{is an open ball.}$$

$$\bar{B}_n(\underline{x}, r) = \{\underline{y} \in R^n : ||\,\underline{y} - \underline{x}\,|| \leqslant r\} \qquad \text{is a closed ball.}$$

$$S^{n-1}(\underline{x}, r) = \{\underline{y} \in R^n : ||\,\underline{y} - \underline{x}\,|| = r\} \qquad \text{is a } n-1\text{-sphere.}$$

$$S^{n-1} = S^{n-1}(\underline{0}, 1) \qquad \underline{0} \in R^n.$$

Exercise 2.1.

– $B_n(\underline{x}, r)$, $\bar{B}_n(\underline{x}, r)$, $S^{n-1}(\underline{x}, r)$ are semialgebraic.

– Polynomials are continuous with respect to the Euclidean topology.

– The open balls form a basis for the Euclidean topology = norm topology = interval topology.

– The closure and the interior of a semialgebraic set are semialgebraic.

Remark 2.2. It is not true that the closure of a semialgebraic set is obtained by relaxing the inequalities! For instance

$$\{x > 0\} \cap \{x < 0\} = \emptyset.$$

3. Semialgebraic functions

Definition 3.1. Let $A \subseteq R^m$, $B \subseteq R^n$ be two semialgebraic sets. A function

$$f \colon A \longrightarrow B$$

is **semialgebraic** if its graph

$$\Gamma_f = \{(\underline{x}, \underline{y}) \in A \times B : \underline{y} = f(\underline{x})\}$$

is a semialgebraic subset of R^{m+n}.

Example 3.2.

(1) Any polynomial mapping $f \colon A \to B$ between semialgebraic sets is semialgebraic.

(2) More generally, any regular rational mapping $f \colon A \to B$ (i.e. all coordinates are rational functions whose denominators do not vanish on A) is semialgebraic.

(3) If A is a semialgebraic set and $f \colon A \to R$, $g \colon A \to R$ are semialgebraic maps, then $|f|$, $\max(f, g)$, $\min(f, g)$ are semialgebraic maps.

(4) If A is a semialgebraic set and $f \colon A \to R$ is a semialgebraic map with $f \geqslant 0$ on A, then $\sqrt{f}$ is a semialgebraic map.

Proposition 3.3.

(1) *The composition $g \circ f$ of semialgebraic maps f and g is semialgebraic.*

(2) *Let $f \colon A \to B$ and $g \colon C \to D$ semialgebraic maps. Then the map*

$$\begin{array}{rccc} f \times g \colon & A \times C & \longrightarrow & B \times D \\ & (\underline{x}, \underline{y}) & \mapsto & (f(\underline{x}), g(\underline{y})) \end{array}$$

is semialgebraic.

(3) *Let* $f\colon A \to B$ *be semialgebraic.*

(*i*) $S \subseteq A$ *semialgebraic* $\Rightarrow$ $f(S)$ *is semialgebraic.*
(*ii*) $T \subseteq B$ *semialgebraic* $\Rightarrow$ $f^{-1}(T)$ *is semialgebraic.*

(4) *Let* A *be a semialgebraic set. Then*

$$\mathcal{S}(A) = \{f\colon A \to R : f \text{ is semialgebraic}\}$$

is a commutative ring under pointwise addition and pointwise multiplication.

Proposition 3.4. *Let* $A \subseteq R^n$ *be a non-empty semialgebraic set.*

(*i*) *For every* $\underline{x} \in R^n$ *the distance between* $\underline{x}$ *and* A*:*

$$\operatorname{dist}(\underline{x}, A) := \inf(\{||\underline{x} - \underline{y}|| : \underline{y} \in A\})$$

is well-defined.

(*ii*) *The function*

$$\begin{array}{rccc} \operatorname{dist}\colon & R^n & \longrightarrow & R \\ & \underline{x} & \mapsto & \operatorname{dist}(\underline{x}, A) \end{array}$$

is continuous semialgebraic vanishing on the closure of A *and positive elsewhere.*

4. Semialgebraic homeomorphisms

We have that every semialgebraic subset of R can be decomposed as the union of finitely many points and open intervals. We shall generalize this to higher dimension:

Definition 4.1. Let A, B be semialgebraic sets and $f\colon A \to B$. We say that f is a **semialgebraic homeomorphism** if

(1) f is a bijection,
(2) f and f^{-1} are continuous and semialgebraic.

Definition 4.2. Let A, B be semialgebraic sets. We say that they are **semialgebraically homeomorphic** if there is a semialgebraic homeomorphism $f\colon A \to B$.

Our aim is to show that every semialgebraic set can be decomposed as the disjoint union of finetely many pieces which are semialgebraically homeomorphic to open hypercubes $(0,1)^d$ (possibly for different $d \in \mathbb{N}$).