REAL ALGEBRAIC GEOMETRY LECTURE NOTES (18: 17/12/09)

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Contents

Let R be a real closed field.

1. Semialgebraic connectedness

In the last lecture we showed:

Theorem 1.1. *Every semialgebraic subset of R^n is the disjoint union of a finite number of semialgebraic sets, each of them semialgebraically homeomorphic to an open hypercube $]0,1[^d \subset R^d$, for some $d \in \mathbb{N}$ (where $]0,1[^0$ is a point).*

Question 1.2. Are the $]0,1[^d$ connected? (equivalently is R^d connected?)

If $R = \mathbb{R}$ yes, but in general no, because if $R \neq \mathbb{R}$ then R is not Dedekind complete and therefore is disconnected.

So what is a reasonable notion of connectedness for semialgebraic sets?

Definition 1.3. Let $A \subset R^n$ be a semialgebraic set. We say that A is **semialgebraic connected** (*semialgebraisch zusammenhängend*) if the following equivalent conditions hold:

(1) A is not the disjoint union of two non-empty semialgebraic open (relatively to A) subsets of A.

(2) There are no semialgebraic open sets U_1, U_2 of R^n such that

$$U_1 \cap A \neq \emptyset \qquad U_2 \cap A \neq \emptyset$$
$$U_1 \cap U_2 \cap A = \emptyset \quad \text{and} \quad (U_1 \cup U_2) \cap A = A.$$

(3) If A_1, A_2 are disjoint semialgebraic subsets of A with $A = A_1 \cup A_2$ and A_1, A_2 are open in A, then

$$\text{either } A_1 = \emptyset \quad \text{or } A_2 = \emptyset.$$

(4) Whenever $F_1 \subseteq A$, $F_2 \subseteq A$ are semialgebraic and closed in A with $F_1 \dot\cup F_2 = A$, then

$$F_1 = A \quad \text{or} \quad F_2 = A.$$

Remark 1.4.

(i) A subset $A \subseteq R^n$ is connected if it is not the disjoint union of two nonempty open (relatively to A) subsets of A. So for any semialgebraic set A,

$$A \text{ connected } \Rightarrow A \text{ semialgebraic connected.}$$

(ii) Every interval in R is semialgebraic connected, so

$$A \text{ semialgebraic connected } \not\Rightarrow A \text{ connected.}$$

(iii) The property of being semialgebraic connected (as the property of being connected) is preserved under semialgebraic homeomorphisms.

Theorem 1.5.

(a) *Assume A, $B \subset R^n$ semialgebraic connected with $A \cap \bar{B} \neq \emptyset$. Then $A \cup B$ is semialgebraic connected.*

(a') *If A and B are semialgebraic, with $A \subseteq B \subseteq \bar{A}$,*

$$A \text{ semialgebraic connected } \Rightarrow B \text{ semialgebraic connected.}$$

(b) *$A \subseteq R^m$, $B \subseteq R^n$ semialgebraic connected $\Rightarrow$ $A \times B \subseteq R^{n+m}$ semialgebraic connected.*

(c) *If $A \subseteq R^m$ semialgebraic connected and $f \colon A \to R^n$ a continuous semialgebraic map, then $f(A) \subseteq R^n$ is semialgebraic connected.*

Proof.

(a) Let $A \cup B = U \dot\cup V$ with U, V semialgebraic and open in $A \cup B$. Assume for a contradiction U, $V \neq \emptyset$, say without loss of generality $A \cap U \neq \emptyset$. Since A is semialgebraic connected, we must have $A \subseteq U$. Therefore $A \cap V = \emptyset$, $V \subseteq B$ and B semialgebraic connected $\Rightarrow$ $V = B$ and $U = A$. So A, B are open in $A \cup B$ and disjoint. Therefore $A \cap \bar{B} = \emptyset$, contradiction.

(a') Exercise.

(b) Let $A \times B = U \dot{\cup} V$ with U, V semialgebraic and open in $A \times B$. Set

$$A_1 := \{x \in A : \{x\} \times B \subseteq U\}.$$
$$A_2 := \{x \in A : \{x\} \times B \subseteq V\}.$$

Since B is semialgebraic connected, $A = A_1 \dot{\cup} A_2$. Now $A - A_1 = \pi_1(V)$ is open in A. Therefore A_1 is closed in A, A_2 is closed in A. But A_1, A_2 semialgebraic and A semialgebraic connected $\Rightarrow$ $A_1 = \emptyset$ or $A_2 = \emptyset$, so $U = \emptyset$ or $V = \emptyset$.

(c) Exercise.

□

2. Semialgebraic connected components

Proposition 2.1. *Let $A \subseteq R^n$ be non-empty semialgebraic. There are finitely many pairwise disjoint $A_1, \dots, A_r$ semialgebraic connected, semialgebraic subsets of A which are all open (therefore all closed) in A with*

$$A = A_1 \dot{\cup} \cdots \dot{\cup} A_r$$

and this decomposition is unique (up to permutation).

Proof. We know $A = C_1 \dot{\cup} \cdots \dot{\cup} C_m$ with $C_i \approx R^d$ semialgebraic, semialgebraic connected $C_i \neq \emptyset$. We proceed by induction on m.

- $m = 1$. It is clear.
- $m > 1$. If C_1 is open and closed in A, we can use induction on $C_2 \cup \cdots \cup C_m$. Otherwise $\exists\, i \in \{2, \dots, m\}$ such that $\bar{C_1} \cap C_i \neq \emptyset$ or $C_1 \cap \bar{C_i} \neq \emptyset$. In both cases we get $C_1 \cup C_i$ semialgebraic connected (by 1.5(1.4)(a)) and we are done by induction again.

Uniqueness: Suppose $A = A_1 \dot{\cup} \cdots \dot{\cup} A_r = A'_1 \dot{\cup} \cdots \dot{\cup} A'_q$ with each A_i and each A'_j open and closed in A and semialgebraic connected. Then each A_i is contained in exactly one A'_j and viceversa every A'_j is contained in exactly one A_i (Exercise).

□

Definition 2.2. The $A_1, \dots, A_r$ are called the **semialgebraic connected components** of the semialgebraic set $A \subset R^n$.

Remark 2.3. A semialgebraic subset of $\mathbb{R}^n$ is semialgebraic connected if and only if it is connected, so every semialgebraic subset of $\mathbb{R}^n$ has a finite number of connected components which are semialgebraic.