# REAL ALGEBRAIC GEOMETRY LECTURE NOTES (19: 22/12/09)

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Contents



Let $R$ be a real closed field.

## 1. Motivation

**Theorem 1.1.** *(**Curve-selection Lemma**: Kurvenauswahllemma) Let $A$ be a semialgebraic subset of $R^n$, $x \in R^n$, $x \in \bar{A} = \operatorname{clos}(A)$. Then there exists a continuous semialgebraic map $f\colon [0,1] \to R^n$ such that $f(0) = x$ and $f(]0,1]) \subset A$.*

This has important consequences such as

(1) The image of a closed and bounded semialgebraic set under a continuous semialgebraic map is a closed and bounded semialgebraic set.

(2) A semialgebraic set is semialgebraic connected if and only if it is semialgebraic **path connected** (*wegzusammenhängend*).

## 2. Closed and bounded semialgebraic sets

**Definition 2.1.** A subset $A \subseteq R^n$ is **bounded** if $\exists\, r \in R$ such that $||a|| < r$ $\forall\, a \in A$.

We have seen that for $R \neq \mathbb{R}$ we have to replace the notion of "connected" by "semialgebraic connected".

Similarly the notion of compactness is problematic for $R \neq \mathbb{R}$. In fact, closed and bounded subsets of $R$ need not be compact.

**Example 2.2.** Let $R = \mathbb{R}_{alg} = \{\text{real algebraic numbers}\} =$ the real closure of $\mathbb{Q}$ in $\mathbb{R}$. The interval $[0,1] \subseteq R$ is not compact. For example the set

$$\mathcal{U} = \{\, [0,r[\, \subset R \ : r < \pi/4 \,\} \cup \{\, ]s,1] \subset R \ : s > \pi/4 \,\}$$

is an open cover of $[0,1]$ by semialgebraic subsets of $R$ and it is not possible to extract from it a finite subcover!

This example shows that, unlike the notion of semialgebraic connectness, a notion of of semialgebraic compactness given just with semialgebraic open coverings is not appropriate. Instead, we shall suffice ourselves with studying "closed and bounded" semialgebraic sets and bounded semialgebraic functions.

**Definition 2.3.** A function $f\colon A \to R$ is **bounded** if $\forall\, a \in A\ \exists\, r \in R$ with $||f(a)|| < r$.

**Proposition 2.4.** *Let $r \in R$, $r > 0$ and $\varphi\colon ]0, r] \to R$ a continuous bounded semialgebraic function. Then $\varphi$ extends to a continuous function on $[0, r]$.*

For the proof we need the following lemma:

**Lemma 2.5.** *Let $A \subseteq R$ be a semialgebraic set and $\varphi\colon A \to R$ a semialgebraic function. Then there exists a non-zero polynomial $f \in R[\mathrm{x}, \mathrm{y}]$ such that $f$ vanishes on $\Gamma(\varphi)$, i.e.*

$$\forall\, x \in A \quad f(x, \varphi(x)) = 0.$$

(For its proof see Lemma 1.1 of Lecture 21)

*Proof of Proposition 2.4.* Assuming Lemma 2.5, let $f \in R[\mathrm{x}, \mathrm{y}]$ be a non-zero polynomial such that $f$ vanishes on $\Gamma(\varphi)$. We shall proceed by induction on $d = \deg f$ in y.

Suppost first $d = 1$. We write

$$f = Q_1(\mathrm{x})\mathrm{y} + Q_0(\mathrm{x}), \quad Q_0, Q_1 \in R[\mathrm{x}], \quad Q_1 \not\equiv 0.$$

We have that

$$f(x, \varphi(x)) = 0 \;\Rightarrow\; Q_1(x)\varphi(x) + Q_0(x) = 0 \qquad \forall x \in\, ]0, r].$$

We may assume that $Q_1(\mathrm{x})$, $Q_0(\mathrm{x}) \in R[\mathrm{x}]$ are relatively prime (otherwise we divide by the common factor). So we get that

$$\varphi(x) = \frac{-Q_0(x)}{Q_1(x)}$$

(we may assume that $Q_1(x) \neq 0$ for all $x \in ]0, r]$, otherwise we take an opportune subinterval $]0, r'] \subset ]0, r]$).

Note that $Q_1(x)$ does not have a zero at $x = 0$ (i.e. x does not divide $Q_1(\mathrm{x})$), otherwise by continuity

$$\lim_{x \to 0^+} \varphi(x) = \pm\infty$$

which contradicts our assumptions that $\exists\, M \in R$ such that $|\varphi(x)| < M$ for all $x \in ]0, r]$. So we can set

$$\varphi(0) := \frac{-Q_0(0)}{Q_1(0)}$$

and with this new definition the map

$$\varphi\colon [0, r] \;\longrightarrow\; R$$

is continuous.

Let now $d > 1$ and assume the result to be true for $\deg_{\mathrm{y}} f(\mathrm{x}, \mathrm{y}) < d$. Without loss of generality we may assume that $f(\mathrm{x}, \mathrm{y})$ is not divisible by x. Otherwise, if

$$f(\mathrm{x}, \mathrm{y}) = \mathrm{x} f_1(\mathrm{x}, \mathrm{y}),$$

we have

$$f(x, \varphi(x)) = x f_1(x, \varphi(x)) = 0 \qquad \forall x \in \,]0, r],$$

therefore

$$f_1(x, \varphi(x)) = 0 \qquad \forall x \in \,]0, r]$$

and we can replace $f$ by $f_1$ if necessary.
Let

$$f' = \frac{\partial f}{\partial \mathrm{y}} \not\equiv 0$$

and let

$$(A_i \ ; \{\xi_{ij}\}_{j=1,\ldots,l_i})_{i \in I}$$

be a slicing of $\{f, f'\}$. So $A_i$ is a partition of $R$ in intervals and points. We may assume without loss of generality that $A_1 = \,]0, r]$ and $\varphi = \xi_{1,j_0}$ (for some $r'$ small enough, i.e. replacing $r$ by $r'$ if necessary).

We have to consider two cases:

- If for $x \in A_1$ $\varphi(x)$ is also a root of $f'(x, \mathrm{y})$ (i.e. $f'$ vanishes on $\Gamma(\varphi)$), then we are done by induction hypothesis, since
$$\deg_{\mathrm{y}} f'(\mathrm{x}, \mathrm{y}) < d.$$
- If not, say $\mathrm{sign}(f'(x, \xi_{1j_0}(x))) = \mathrm{sign}(f'(x, \varphi(x))) > 0$ for $x \in \,]0, r]$.

  <u>*Claim*</u>: There are two continuous semialgebraic functions $\rho$ and $\theta$ such that $\rho, \theta \colon [0, r] \to R$ and
  $$\forall\, x \in \,]0, r] \qquad \rho(x) < \varphi(x) < \theta(x)$$
  and $\mathrm{sign}(f'(x, y))$ is positive for all $y \in \,]\,\rho(x), \theta(x)\,[ \qquad (*)$.

  *Proof of Claim.* We can take
  $$\rho := \xi_{1j_0-1} \ \text{ and } \ \theta = \xi_{1j_0+1}.$$
  If $\varphi = \xi_{1j_0} = \xi_{11}$ then we can take $\rho$ to be the constant function $-(M + 1)$, where $M$ is the bound for $\varphi$.
  If $j_0 = l_1$ we can take $\theta$ to be the constant function $M + 1$.

  Note that these functions are roots of the derivative $f'$, and $\deg f' < d$ in y, so by induction hypothesis the continuous semialgebraic maps $\rho$ and $\theta$ can be extended to $[0, r]$ since $f'$ vanishes on $\Gamma(\rho)$ and $\Gamma(\theta)$). □

Now consider $\rho(0)$ and $\theta(0)$: by continuity we have $\rho(0) \leqslant \theta(0)$.

- If $\rho(0) = \theta(0)$, set $\varphi(0) = \rho(0)$. This gives a continuous extension of $\varphi$ to $[0, r]$.

- Otherwise $\rho(0) < \theta(0)$. Consider the function $f'(0, y)$: it is non-negative for every $y \in [\rho(0), \theta(0)]$ (by continuity together with $(*)$ of Claim).

  Now if $f(0, y)$ is constant, it would be identically zero because we have
$$f(0, \rho(0)) \ \leqslant \ 0 \ \leqslant \ f(0, \theta(0))$$
but this is impossible since x is not a factor of $f$.

  So we must have $f'(0, y) > 0$ and the function $f(0, y)$ is strictly increasing and has a unique root $y_0 \in [\rho(0), \theta(0)]$. Set
$$\varphi(0) := y_0.$$

  It remains to show that with this definition $\varphi$ is continuous at 0 (i.e. that $\lim_{x \to 0^+} \varphi(x) = y_0$).

  Case 1. $\rho(0) < y_0 < \theta(0)$.

  Then for $\varepsilon \in R$, $\varepsilon > 0$ small enough, $f(0, y_0 - \varepsilon) < 0$, $f(0, y_0 + \varepsilon) > 0$, $\rho(0) < y_0 - \varepsilon < y_0 < y_0 + \varepsilon < \theta(0)$. Hence there exists $\eta \in R$, $\eta > 0$ such that for every $x \in ]0, \eta[$:
$$\begin{cases} f(x, y_0 - \varepsilon) < 0 \\ f(x, y_0 + \varepsilon) > 0 \\ \rho(x) < y_0 - \varepsilon \\ y_0 + \varepsilon < \theta(x) \end{cases}$$
  Therefore $\varphi(x) \in ]y_0 - \varepsilon, y_0 + \varepsilon[$ for every $x \in ]0, \eta[$.

  Case 2. $\rho(0) = y_0$.

  We have $f(0, y_0 + \varepsilon) > 0$ for every $\varepsilon \in R$, $\varepsilon > 0$ small enough. Then there exists $\eta \in R$, $\eta > 0$ such that for every $x \in ]0, \eta[$:
$$\begin{cases} f(x, y_0 + \varepsilon) > 0 \\ y_0 - \varepsilon < \rho(x) < y_0 - \varepsilon \end{cases}$$
  Again these imply that $\varphi(x) \in ]y_0 - \varepsilon, y_0 + \varepsilon[$ for every $x \in ]0, \eta[$.

  Case 3. $\theta(0) = y_0$. Analogous.

□