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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 4

These exercises will be collected Tuesday 17 November in the mailbox number 15 of the Mathematics department.

Definition 0.1 *(a) Let K be a field and let $T \subset K$ such that*
(i) $T + T \subset T$ (where $T + T := \{t_1 + t_2 \mid t_1, t_2 \in T\}$);
(ii) $T.T \subset T$ (where $T.T := \{t_1 t_2 \mid t_1, t_2 \in T\}$);
(iii) $a^2 \in T$ for every $a \in K$.
Then T is said to be a ***preordering (Präordnung)*** *or* ***cone*** *of K.*

(b) A preordering T of a field K is said to be ***proper (echte Präordnung)*** *if $-1 \notin T$.*

(c) A proper preordering T of a field K is said to be a ***positive cone (Positivkegel)*** *if $-T \cup T = K$, where $-T := \{-t : t \in T\}$.*

Given a field K, recall that there is a bijection between the set of **orderings** $\leq$ on K and the set of positive cones of K. Therefore by abuse of terminology, we say that "T is an ordering of K" whenever T is a positive cone of K.

1. Show that any ordering T on a field K extends to an order on the field of rational functions in several variables $K(x_1,...,x_n)$, using induction on n and two different methods:

   (a) by contradiction using the criterion for extension of orderings in the case of a field extension L/K;

   (b) by construction of an order relation generalising the one defined in the lectures for the field $\mathbb{R}(x)$.

Recall that a field R is said to be **real** if it admits an ordering. Frow now on, R will denote a **real closed field**, *i.e. a real field which has no proper real algebraic extension (keine echte algebraische reelle Erweiterung).*

2. Given a polynomial $f(x) = dx^m + d_{m-1}x^{m-1} + \cdots + d_0 \in R[x]$ with $d \neq 0$, we set
$$D := 1 + \sum_{i=m-1}^{0} \left| \frac{d_i}{d} \right| \in R.$$
Show that any real root a of f is such that $|a| < D$.

3. Let $f(x) = x^m + d_{m-1}x^{m-1} + \cdots + d_0$ be a monic polynomial in $R[x]$, whose roots $a_1, \ldots, a_m$ are all reals. Show that:
$$a_i \geq 0 \text{ for all } i = 1, \ldots, m \iff (-1)^{m-i} d_i \geq 0 \text{ for all } i = 0, \ldots, m-1.$$

4. Let $f(x) = dx^m + d_{m-1}x^{m-1} + \cdots + d_0$ be a polynomial in $R[x]$ with $d \neq 0$. Show that the following assertions are equivalent:

(a) $f \geq 0$ on R (i.e. $f(x) \geq 0$ for any $x \in R$);

(b) $d > 0$ and all the real roots of f have **even multiplicity** (**gerade Vielfachheit**);

(c) $f = g^2 + h^2$ for some $g, h \in R[x]$.