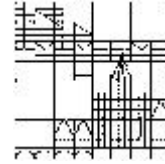


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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 4

These exercises will be collected Tuesday 17 November in the mailbox number 15 of the Mathematics department.

Definition 0.1 (a) Let K be a field and let $T \subset K$ such that

(i) $T + T \subset T$ (where $T + T := \{t_1 + t_2 \mid t_1, t_2 \in T\}$);

(ii) $T \cdot T \subset T$ (where $T \cdot T := \{t_1 t_2 \mid t_1, t_2 \in T\}$);

(iii) $a^2 \in T$ for every $a \in K$.

Then T is said to be a **preordering (Präordnung)** or **cone** of K .

(b) A preordering T of a field K is said to be **proper (echte Präordnung)** if $-1 \notin T$.

(c) A proper preordering T of a field K is said to be a **positive cone (Positivkegel)** if $-T \cup T = K$, where $-T := \{-t \mid t \in T\}$.

Given a field K , recall that there is a bijection between the set of **orderings** \leq on K and the set of positive cones of K . Therefore by abuse of terminology, we say that “ T is an ordering of K ” whenever T is a positive cone of K .

1. Show that any ordering T on a field K extends to an order on the field of rational functions in several variables $K(x_1, \dots, x_n)$, using induction on n and two different methods:

(a) by contradiction using the criterion for extension of orderings in the case of a field extension L/K ;

(b) by construction of an order relation generalising the one defined in the lectures for the field $\mathbb{R}(x)$.

Recall that a field R is said to be **real** if it admits an ordering. From now on, R will denote a **real closed field**, i.e. a real field which has no proper real algebraic extension (*keine echte algebraische reelle Erweiterung*).

2. Given a polynomial $f(x) = dx^m + d_{m-1}x^{m-1} + \dots + d_0 \in R[x]$ with $d \neq 0$, we set

$$D := 1 + \sum_{i=m-1}^0 \left| \frac{d_i}{d} \right| \in R.$$

Show that any real root a of f is such that $|a| < D$.

3. Let $f(x) = x^m + d_{m-1}x^{m-1} + \dots + d_0$ be a monic polynomial in $R[x]$, whose roots a_1, \dots, a_m are all reals. Show that:

$$a_i \geq 0 \text{ for all } i = 1, \dots, m \Leftrightarrow (-1)^{m-i} d_i \geq 0 \text{ for all } i = 0, \dots, m-1.$$

4. Let $f(x) = dx^m + d_{m-1}x^{m-1} + \dots + d_0$ be a polynomial in $R[x]$ with $d \neq 0$. Show that the following assertions are equivalent:

(a) $f \geq 0$ on R (i.e. $f(x) \geq 0$ for any $x \in R$);

(b) $d > 0$ and all the real roots of f have **even multiplicity (gerade Vielfachheit)**;

(c) $f = g^2 + h^2$ for some $g, h \in R[x]$.