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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 7

These exercises will be collected Tuesday 08 December in the mailbox number 15 of the Mathematics department.

Definition 0.1 *Let R be a real closed field and $n \in \mathbb{N}\backslash\{0\}$. Denote $\underline{X} = (X_1, \dots, X_n)$.*

The class of ***semi-algebraic sets*** *in R^n is defined to be the smallest class of subsets of R^n which contains all sets of the form*

$$\{\underline{x} \in R^n \mid f(\underline{x}) \triangleright 0\}$$

for $f(\underline{X}) \in R[\underline{X}]$ and $\triangleright \in \{\geq\ ,\ >\ ,\ =\ ,\ \neq\}$, and which is closed under finite boolean (union, intersection, complement) combinations.

For any polynomial $f(\underline{X}) \in R[\underline{X}]$, write

$$Z(f) := \{\underline{x} \in R^n \mid f(\underline{x}) = 0\}$$

and

$$U(f) := \{\underline{x} \in R^n \mid f(\underline{x}) > 0\}.$$

For any polynomials $f_1(\underline{X}), \dots, f_p(\underline{X}) \in R[\underline{X}]$, we call ***basic open semi-algebraic set*** *(generated by $f_1, \dots, f_p$) the set*

$$\begin{array}{rcl} U(f_1, \dots, f_p) & := & \{\underline{x} \in R^n \mid f_1(\underline{x}) > 0, \dots, f_p(\underline{x}) > 0\} \\ & = & U(f_1) \cap \cdots \cap U(f_p). \end{array}$$

Recall that:

Proposition 0.2 (Normal Form for Semi-Algebraic Sets) *Any semi-algebraic set in R^n is a finite union of basic semi-algebraic sets of the form*

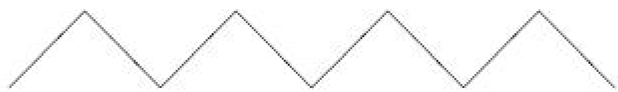
$$Z(f) \cap U(f_1, \dots, f_p).$$

1. (a) Show that for any polynomial $f(\underline{X}) \in R[\underline{X}]$, $U(f)$ is open in the interval topology of R^n (i.e. the product topology of the interval topology of R).

   (b) Show that the basic open semi-algebraic sets form a basis for the interval topology of R^n.

2. Show that the semi-algebraic subsets of R are exactly the finite unions of points and open intervals (bounded or unbounded).

3. (a) Show that the infinite zig zag curve



is not semi-algebraic in $\mathbb{R}^2$.

(b) Show that, for every compact semialgebraic subset K of $\mathbb{R}^2$, the intersection of K with the zigzag is semialgebraic.

Definition 0.3 *Let $A \subset R^m$ and $B \subset R^n$ be two semi-algebraic sets ($m,n \in \mathbb{N}\backslash\{0\}$). A mapping $f : A \to B$ is semi-algebraic if its graph*

$$\Gamma_f := \{(\underline{x},\underline{y}) \in R^{m+n} \mid \underline{y} = f(\underline{x})\}$$

is semi-algebraic in R^{m+n}.

4. The aim of this exercise is to prove that the real exponential function exp is not semi-algebraic.

(a) Consider some polynomials $p_0(X), \ldots, p_n(X) \in \mathbb{R}[X]$, and an infinite subset $U \subset \mathbb{R}$ such that for all $x \in U$

$$p_n(x)(e^x)^n + p_{n-1}(x)(e^x)^{n-1} + \cdots + p_0(x) = 0.$$

Show that $p_0 \equiv \cdots \equiv p_n \equiv 0$:

(i) if U has no bound, by comparison between exp and polynomials .

(ii) if U is bounded, using the following result.

Theorem 0.4 (Identity Theorem of Complex Analysis) *Consider two complex functions $f(z)$ and $g(z)$ holomorphic (i.e. differentiable with respect to their complex variable z) in a domain D of the complex plane $\mathbb{C}$. If the equation*

$$f(z) = g(z)$$

holds for an infinite subset S of D having an accumulation point in D, then it holds in the whole D.

(b) Show by contradiction that exp is not semi-algebraic.
(*Hint: show that $\Gamma_{\exp}$ would have a non empty interior using the normal form for semi-algebraic sets.*)