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Übungen zur Vorlesung Reelle algebraische Geometrie

Blatt 9 - Solution

Definition 0.1 *A **first order formula** in the language of real closed fields is obtained as follows recursively:*

1. *if $f(\underline{x}) = \mathbb{Q}[x_1,\dots,x_n]$, $n > 1$, then $f(\underline{x}) \geq 0$, $f(\underline{x}) > 0$, $f(\underline{x}) = 0$, $f(\underline{x}) \neq 0$ are first order formulas (with free variables $\underline{x} = (x_1,...,x_n)$);*
2. *if Φ and Ψ are first order formulas, then $\Phi \vee \Psi$, $\Phi \wedge \Psi$ and $\neg\Phi$ are also first order formulas (with free variables given by the union of the free variables of Φ and the free variables of Ψ);*
3. *if Φ is a first order formula then*
$$\exists x\ \Phi \ \text{ and } \ \forall x\ \Phi$$
are first order formulas (with same free variables as Φ minus $\{x\}$).

*The formulas obtained using just 1. and 2. are called **quantifier free**.*

*Let R be a real closed field, $n \geq 1$. A subset $A \subset R^n$ is said to be **definable** (with parameters from R) in R if there is a first order formula $\Phi(\underline{t},\underline{x})$ with parameters $\underline{t} \in R^m$ and free variables $\underline{x} = (x_1,...,x_n)$, such that*
$$A = \{\underline{x} \in R^n : \Phi(\underline{t},\underline{x}) \text{ is true in } R\}.$$

Proposition 0.2 *For any real closed field R, the class of definable sets (with parameters) in R coincides with the class of semialgebraic sets.*

Theorem 0.3 (Tarski's quantifier elimination theorem for real closed fields) *Every first order formula in the language of real closed fields is equivalent to a quantifier free formula.*

Theorem 0.4 (Tarski-Seidenberg, geometric version) *Consider the projection map*
$$\begin{array}{cccc} \Pi: & R^{m+n} & \rightarrow & R^m \\ & (\underline{x},\underline{y}) & \mapsto & \underline{x}. \end{array}$$
Then, for any semi-algebraic set $A \subset R^{m+n}$, $\Pi(A)$ is a semi-algebraic subset of R^m.

1. We suppose R endowed with the interval topology, which coincides with the euclidean topology (see Ü.B. 5).

   (a) Let A be a semi-algebraic set, $A \subset R^n$ for some $n \in \mathbb{N}$.
   (i) We have

$$\begin{aligned}
\mathrm{Cl}(A) &= \left\{\underline{x} \in R^n \mid \forall t \in R,\ \exists \underline{y} \in A,\ \left(\|\underline{x}-\underline{y}\|^2 = \sum_{i=1}^{n}(x_i-y_i)^2 < t^2 \text{ or } t=0\right)\right\};\\
\mathrm{Int}(A) &= \left\{\underline{x} \in R^n \mid \exists t \in R,\ \forall \underline{y} \in R^n,\ \left(\|\underline{x}-\underline{y}\|^2 < t^2 \Rightarrow \underline{y} \in A \text{ and } t \neq 0\right)\right\};\\
\partial A &= \Big\{\underline{x} \in R^n \mid \forall t \in R,\ \exists \underline{y} \in A \text{ and } \exists \underline{z} \notin A,\\
&\qquad \left[\left(\|\underline{x}-\underline{y}\|^2 < t^2 \text{ and } \|\underline{x}-\underline{z}\|^2 < t^2\right) \text{ or } t=0\right]\Big\};
\end{aligned}$$

(ii) Consider the sets

$$\begin{aligned}
B_1 &:= \left\{(\underline{x},\underline{y},t) \in R^{2n+1} \mid \underline{y} \in A \text{ and } \left(\|\underline{x}-\underline{y}\|^2 < t^2 \text{ or } t=0\right)\right\};\\
B_2 &:= \left\{(\underline{x},\underline{y},t) \in R^{2n+1} \mid \underline{y} \in R^n \backslash A \text{ and } \left(\|\underline{x}-\underline{y}\|^2 < t^2 \text{ or } t=0\right)\right\};
\end{aligned}$$

and the following projections

$$\begin{aligned}
\Pi_1 : \quad R^{2n+1} &\to R^{n+1}\\
(\underline{x},\underline{y},t) &\mapsto (\underline{x},t);
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2 : \quad R^{n+1} &\to R^{n}\\
(\underline{x},t) &\mapsto \underline{x}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{Cl}(A) &= R^n \backslash \Pi_2\left[R^{n+1} \backslash \Pi_1(B_1)\right];\\
\mathrm{Int}(A) &= \Pi_2\left[R^{n+1} \backslash \Pi_1(B_2)\right];\\
\partial A &= R^n \backslash \Pi_2\left[R^{n+1} \backslash \Pi_1(B_1)\right] \cap \Pi_2\left[R^{n+1} \backslash \Pi_1(B_2)\right].
\end{aligned}$$

(b) Consider the following semi-algebraic set

$$A := \{(x,y) \in R^2 \mid x^3 - x^2 - y^2 > 0\}.$$

Remark that, for any $(x,y) \in R^2$, we have $x^3 - x^2 - y^2 > 0 \Leftrightarrow x^2(x-1) > y^2$. So, the closure $\mathrm{Cl}(A)$ of A is given by

$$\mathrm{Cl}(A)\{(x,y) \in R^2 \mid x^3 - x^2 - y^2 \geq 0 \text{ and } x \geq 1\}.$$

which is equal to $B \backslash \{(0,0)\}$ where

$$B := \{(x,y) \in R^2 \mid x^3 - x^2 - y^2 \geq 0\}.$$

Note that the closure of a semi-algebraic set is not obtained just by relaxing the strict inequalities.

2. Let $A \subset R^n$, $B \subset R^m$ be semi-algebraic sets, $m,n \in \mathbb{N}^*$.

(a) Take a polynomial map

$$\begin{aligned}
f : \quad A &\to B\\
\underline{x} &\mapsto (f_1(\underline{x}), \ldots, f_m(\underline{x})).
\end{aligned}$$

where $f_i \in R[\underline{X}]$, $i = 1, \ldots, m$. The graph of f is

$$\Gamma(f) := \{(\underline{x},\underline{y}) \in A \times B \mid \forall i = 1, \ldots, m,\ f_i(\underline{x}) = y_i\}$$

which is indeed a semi-algebraic subset of R^{n+m} (and even an algebraic one);

(b) Take a regular rational map

$$\begin{array}{rccc} f: & A & \to & B \\ & \underline{x} & \mapsto & \left(\dfrac{f_1(\underline{x})}{g_1(\underline{x})}, \ldots, \dfrac{f_m(\underline{x})}{g_m(\underline{x})}\right). \end{array}$$

where $f_i, g_i \in R[\underline{X}]$, $i = 1, \ldots, m$. The graph of f is

$$\Gamma(f) := \{(\underline{x}, \underline{y}) \in A \times B \mid \forall i = 1, \ldots, m,\ f_i(\underline{x}) = y_i g_i(\underline{x})\}$$

which is indeed a semi-algebraic subset of R^{n+m} (and even an algebraic one);

(c) Consider semi-algebraic maps $f : A \to R$ and $g : A \to R$ with corresponding graphs

$$\begin{array}{rcl} \Gamma(f) & := & \{(\underline{x}, y) \in A \times R \mid y = f(\underline{x})\}; \\ \Gamma(g) & := & \{(\underline{x}, y) \in A \times R \mid y = g(\underline{x})\}. \end{array}$$

This means that $y = f(\underline{x})$ and $y = g(\underline{x})$ are first order formulas. Then so are $f(\underline{x}) - g(\underline{x}) \geq 0$ and its negation $g(\underline{x}) - f(\underline{x}) > 0$. This means that the sets $\{(\underline{x}, y) \in R^{n+1} \mid f(\underline{x}) - g(\underline{x}) \geq 0\}$ and $\{(\underline{x}, y) \in R^{n+1} \mid g(\underline{x}) - f(\underline{x}) > 0\}$ are semi-algebraic. Then we get that the following sets are also semi-algebraic:

$$\begin{array}{rcl} \Gamma(\max(f,g)) & = & \left[\{(\underline{x}, y) \in R^{n+1} \mid f(\underline{x}) - g(\underline{x}) \geq 0\} \cap \Gamma(f)\right] \\ & & \cup \left[\{(\underline{x}, y) \in R^{n+1} \mid g(\underline{x}) - f(\underline{x}) > 0\} \cap \Gamma(g)\right]; \\ \Gamma(\min(f,g)) & = & \left[\{(\underline{x}, y) \in R^{n+1} \mid f(\underline{x}) - g(\underline{x}) \geq 0\} \cap \Gamma(g)\right] \\ & & \cup \left[\{(\underline{x}, y) \in R^{n+1} \mid g(\underline{x}) - f(\underline{x}) > 0\} \cap \Gamma(f)\right]; \\ \Gamma(|f|)) & = & \left[\Gamma(\max(f, \overline{0})) \cap \{(\underline{x}, y) \in R^{n+1} \mid y > 0\}\right] \\ & & \cup \left[\Gamma(\max(-f, \overline{0})) \cap \{(\underline{x}, y) \in R^{n+1} \mid y > 0\}\right] \\ & & \cup \left[\Gamma(f) \cap \{(\underline{x}, 0) \in R^{n+1}\}\right]; \end{array}$$

where $\overline{0}$ denotes the map $A \to R$ with constant value 0.

(d) Consider a semi-algebraic map $f : A \to R$ with $f \geq 0$. Thus

$$\Gamma(f) = \{(\underline{x}, y) \in A \times R \mid y = f(\underline{x}) \text{ and } y \geq 0\}$$

is semi-algebraic, meaning that ($y = f(\underline{x})$ and $y \geq 0$) is a first order formula. So we get that the set

$$S := \{(\underline{x}, y, z) \in A \times R \mid y = z^2 \text{ and } y = f(\underline{x}) \text{ and } y \geq 0\}$$

is also semi-algebraic. Denote

$$\begin{array}{rccc} \Pi: & R^{n+2} & \to & R^{n+1} \\ & (\underline{x}, y, z) & \mapsto & (\underline{x}, z). \end{array}$$

By the geometric version of Tarski-Seidenberg theorem, we get that $\Gamma(\sqrt{f}) = \Pi(S)$ is semi-algebraic.

3. Let R be a real closed field, and $A \subset R^m$, $B \subset R^n$, $C \subset R^p$ and $D \subset R^q$ be some nonempty semi-algebraic sets, $m, n, p, q \in \mathbb{N}^*$.

(a) The following sets

$$\begin{array}{rcl} \Gamma(f) & := & \{(\underline{x}, \underline{y}) \in R^{m+n} \mid \underline{x} \in A \text{ and } \underline{y} = f(\underline{x})\}; \\ \Gamma(g) & := & \{(\underline{y}, \underline{z}) \in R^{n+p} \mid \underline{y} \in B \text{ and } \underline{z} = g(\underline{y})\}; \end{array}$$

are semi-algebraic. So are the following sets

$$\begin{array}{rcl} \Gamma(f)\times R^p & := & \{(\underline{x},\underline{y},\underline{z})\in R^{m+n+p} \mid \underline{x}\in A \text{ and } \underline{y}=f(\underline{x})\};\\ R^m\times\Gamma(g) & := & \{(\underline{x},\underline{y},\underline{z})\in R^{m+n+p} \mid \underline{y}\in B \text{ and } \underline{z}=g(\underline{y})\}. \end{array}$$

(They are defined by first order formulas). So, $(\Gamma(f)\times R^p)\cap(R^m\times\Gamma(g))$ is also semi-algebraic. Denote

$$\begin{array}{rrcl} \Pi: & R^{m+n+p} & \to & R^{m+p}\\ & (\underline{x},\underline{y},\underline{z}) & \mapsto & (\underline{x},\underline{z}). \end{array}$$

By the geometric version of Tarski-Seidenberg theorem, we get that

$$\Gamma(g\circ f)=\Pi\left[(\Gamma(f)\times R^p)\cap(R^m\times\Gamma(g))\right]$$

is also semi-algebraic.;

(b) Considering the formulas

$$\begin{array}{rcl} \Gamma(f) & := & \{(\underline{x},\underline{y})\in R^{m+n} \mid \underline{x}\in A \text{ and } \underline{y}=f(\underline{x})\};\\ \Gamma(g) & := & \{(\underline{z},\underline{t})\in R^{p+q} \mid \underline{z}\in B \text{ and } \underline{t}=g(\underline{z})\}; \end{array}$$

we deduce that

$$\begin{array}{rcl} \Gamma(f\times g) & := & \{(\underline{x},\underline{z},\underline{y},\underline{t})\in R^{m+p+n+q} \mid \underline{x}\in A \text{ and } \underline{y}=f(\underline{x}) \text{ and } \underline{z}\in B \text{ and } \underline{t}=g(\underline{z})\};\\ & \simeq & \Gamma(f)\times\Gamma(g); \end{array}$$

is also semi-algebraic (it is also defined by first order formulas).

(c) Let $f: A\to B$ be a semi-algebraic map.

(i) Consider a semi-algebraic subset $S\subset A$. Then the set $S\times B$ is semi-algebraic, and therefore is $\Gamma(f)\cap(S\times B)$. Denote

$$\begin{array}{rrcl} \Pi_2: & R^{m+n} & \to & R^n\\ & (\underline{x},\underline{y}) & \mapsto & \underline{y}. \end{array}$$

From the geometric version of Tarski-Seidenberg theorem we deduce that $f(S)=\Pi_2\left[\Gamma(f)\cap(S\times B)\right]$ is also semi-algebraic.

(ii) Consider a semi-algebraic subset $T\subset B$. Then the set $A\times T$ is semi-algebraic, and therefore is $\Gamma(f)\cap(A\times T)$. Denote

$$\begin{array}{rrcl} \Pi_1: & R^{m+n} & \to & R^m\\ & (\underline{x},\underline{y}) & \mapsto & \underline{x}. \end{array}$$

From the geometric version of Tarski-Seidenberg theorem we deduce that $f^{-1}(S)=\Pi_1\left[\Gamma(f)\cap(A\times T)\right]$ is also semi-algebraic.

(d) Define $\mathcal{S}(A):=\{f: A\to R^2 \mid f \text{ is a semi-algebraic map}\}$. To prove that $\mathcal{S}(A)$ is a commutative ring, it remains to show that $\mathcal{S}(A)$ is closed under difference and multiplication.

Consider $f : A \to R$ and $g : A \to R$. The following maps

$$\begin{array}{cccc} + : & R^2 & \to & R \\ & (x,y) & \mapsto & x+y. \\ * : & R^2 & \to & R \\ & (x,y) & \mapsto & xy. \\ - & R & \to & R \\ & x & \mapsto & -x. \end{array}$$

are semi-algebraic. Following the questions (a) and (b), so is the map

$$\begin{array}{cccc} f-g : & A & \to & R \\ & \underline{x} & \mapsto & f(\underline{x}) - g(\underline{x}) \end{array}$$

as it is the following composition

$$\begin{array}{ccccccc} A & \to & A \times A & \to & R \times R & \to & R \\ \underline{x} & \mapsto & (\underline{x},\underline{x}) & \mapsto & (f(\underline{x}), -g(\underline{x})) & \mapsto & f(\underline{x}) - g(\underline{x}). \end{array}$$

Similarly, the multiplication is semi-algebraic as it is the following composition

$$\begin{array}{ccccccc} A & \to & A \times A & \to & R \times R & \to & R \\ \underline{x} & \mapsto & (\underline{x},\underline{x}) & \mapsto & (f(\underline{x}),g(\underline{x})) & \mapsto & f(\underline{x})g(\underline{x}). \end{array}$$