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VALUED FIELDS – EXERCISE 4

To be submitted on Wednesday 24.11.2010 by 14:00 in the mailbox.

Question 1.
Prove the following:

(1) A domain R is a valuation ring in its field of fractions K iff the set of ideals of R is linearly ordered by inclusion.
(2) A field K posses only trivial places iff K is an algebraic extension of $\mathbb{F}_p$ for some prime p.

Question 2.
In the next 2 questions we shall prove the following theorem:

Theorem. *(weak form of Hilbert's Nullstellensatz) Suppose I is a proper ideal in the ring $k[X_1, \ldots, X_n]$ with k an algebraically closed field.*
Then $V(I) := \{(a_1, \ldots, a_n) \mid \forall f \in I\,(f(a_1, \ldots, a_n) = 0)\} \neq \emptyset$.

(1) Show that the theorem is equivalent to:
  $\star$ Suppose k is a field, and $B \supseteq k$ is a field which is finitely generated as a ring over k. Then B is a finite algebraic extension.

  Hint: Note that if k is algebraically closed, B is an algebraic extension iff $B = k$.
(2) Show that $\star$ follows from the following Proposition:

  Proposition. *Let $A \subseteq B$ be integral domains, B finitely generated as a ring over A, and let Ω be an algebraically closed field. Let $0 \neq v \in B$ then there exists $0 \neq u \in A$ with the following property:*
  Any homomorphism $f : A \to \Omega$ such that $f(u) \neq 0$ can be extended to a homomorphism $g : B \to \Omega$ such that $g(v) \neq 0$.

  Hint: let $A = k$, $\Omega = \bar{k}$ (the algebraic closure of k), $v = 1$.
(3) Bonus: Show that the Proposition above is not true without the assumption that B is finitely generated as a ring over A.

Question 3.
Prove the proposition in Question 2 using the following steps:

(1) Show that it is enough to prove the Proposition for the case where $B = A[x]$ (i.e. B is generated by just one element over A).
(2) First case: x is transcendental over A. Then $v = v(x) = \sum_{i=0}^{n} a_i x^i$ is a polynomial over A. Choose $u = a_n$.
  Hint: note that if f is a homomorphism as in the proposition, and $f(u) \neq 0$, then there exists some $\varepsilon \in \Omega$ such that $\sum_{i=0}^{n} f(a_i)\,\varepsilon^i \neq 0$.
(3) Second case: x is algebraic over A. So v^{-1} is also algebraic over A – explain why.
  Suppose $\sum_{i=0}^{m} a_i v^{-i} = 0$ and $\sum_{i=0}^{n} b_i x^i = 0$ (and $a_i, b_i \in A$). Choose $u = a_m b_n$. Let $K = \mathrm{quot}(A)$. Suppose $f(u) \neq 0$.

(4) Extend f to $A\left[u^{-1}\right]$ by taking u^{-1} to $f(u)^{-1}$ – make sure this defines a well defined homomorphism.
(5) Next extend f to a place $P : K_P \to \Omega$ – explain why this is possible.
(6) Show that by choice of u, both x and v^{-1} are integral over $A\left[u^{-1}\right]$.
(7) Conclude that $B \subseteq K_P$ and $v^{-1} \in K_P$ and finish the proof.