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On η_α-Groups and Fields

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Abstract. Let $\kappa := \aleph_\alpha$. The following are known: two η_α-sets of power κ are isomorphic. Let $\alpha > 0$. Two ordered divisible Abelian groups that are η_α-sets of power κ are isomorphic, two real closed fields that are η_α-sets of power κ are isomorphic. The following is shown: (1) there exist 2^κ nonisomorphic ordered Abelian groups (respectively ordered fields) that are η_α-sets of power κ; (2) there exist 2^κ nonisomorphic ordered divisible Abelian groups (respectively real closed fields) of power κ all having the same order type; (3) there exist 2 nonisomorphic ordered divisible Abelian groups (respectively real closed fields) that are η_α-sets having the same order type.

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1. Introduction

A partially ordered set $(S, <)$ is an *ordered set* if $\forall x, y \in S$, $x \leqslant y$ or $x \geqslant y$. Assume throughout the rest of the paper that all groups are Abelian.

Let A_1 and A_2 be subsets of an ordered set A. Let $A_1 < A_2$ if $a_1 \in A_1$ and $a_2 \in A_2 \Rightarrow a_1 < a_2$. If $A_1 < A_2$ and if the cardinality card$(A_1 \cup A_2) < \aleph_\alpha$ implies the existence of $a \in A$ such that $A_1 < \{a\} < A_2$, Hausdorff [9, pp. 180, 181] called A an η_α-*set*. Thus A is a dense ordered set without end points $\Longleftrightarrow$ A is an η_0-set. Hausdorff also proved that the order type of an η_α-set of power $\aleph_\alpha$ is unique up to isomorphism. Let $\alpha > 0$. In 1955 Erdős, Gillman, and Henriksen [7] proved that a real closed field that is an η_α-set of power $\aleph_\alpha$ is determined by its order type. In 1960 the first author [1] proved the same result for ordered divisible groups.

Erdős, Gillman, and Henriksen asked the following question [7, Section 5.1]: "Is a non-denumerable real-closed field, in particular, if it is non-archimedean characterized by its type of order as an ordered set"? One might go on to look for a complete set of invariants for a real closed field. Dense ordered sets without end points, ordered divisible groups, and real closed fields all have the following important model theoretic property: they are all o-minimal. (A structure M is *o-minimal* if every subset of it that is definable with parameters in M is a finite union of intervals of M. See [12] for definitions and properties.) Let $\kappa := \aleph_\alpha$ with $\alpha > 0$. o-minimal structures have many interesting properties: e.g., the following.

THEOREM 1.1. *M is κ-saturated if and only if M has a κ-saturated order type.*

The order type of a dense ordered set without end points, an ordered divisible group, or a real closed field, is κ-saturated if and only if it is an η_α-set. Thus from the uniqueness of saturated models and Theorem 1.1, one can prove that the theorems cited of Hausdorff; Erdős, Gillman, and Henriksen; and the first author, follow immediately. In this language the question engendered by that of Erdős, Gillman, and Henriksen translates as follows: what invariants characterize an o-minimal structure?

The search for a natural solution of this classification problem for o-minimal theories is suggested by such important model theoretic properties as the existence and uniqueness of prime models, and Vaught's conjecture (see [12] and [11] for details). However, order type often does *not* characterize ordered structures. Having made standard assumptions about κ in Section 3, we prove that there exist 2^κ pairwise nonisomorphic ordered groups (respectively ordered fields) that are η_α-sets of power κ, necessarily all having the same order type. These examples are not divisible (respectively not real closed) and thus not o-minimal (see [12]). An analogous result is obtained in Section 4, even under the additional hypothesis of o-minimality, but without the condition that the examples are η_α-sets. In Section 5, having dropped the condition that the examples are of power κ, it is shown that there exist two ordered divisible groups that are η_α-sets which are isomorphic as ordered sets, but not as ordered groups, and that there exist two real closed fields that are η_α-sets which are isomorphic as ordered additive groups, but not as ordered fields. However, the second author has proved [10] that the order type does determine a group up to isomorphism for the class of Archimedean-complete, ordered divisible groups of reverse ordinal rank.

2. Background

For x, y, z in an ordered group $(G, +, 0, <)$, let

$$|x| := \max\{x, -x\}.$$

$x \in G \mapsto |x| \in G$ has all the expected properties.

Let $x \sim y$ if $\exists n \in \mathbb{N}$ such that $|x| \leqslant n|y|$ and $|y| \leqslant n|x|$. $\sim$ is an equivalence relation on G. Let $[G]$ be the set of equivalence classes of G mod $\sim$. Let $[x] \in [G]$ such that $x \in [x]$. $[\cdot]$: $x \in G \mapsto [x] \in [G]$ is surjective.

Let $x \ll y$ if $\forall n \in \mathbb{N}$, $n|x| < |y|$.

$x \in G \Rightarrow x \not\ll x$. $\forall x, y, z \in G$, $x \ll y$, and $y \ll z \Rightarrow x \ll z$.

Let $[x] < [y] \iff x \ll y$.

$[G]$ is an ordered set, $-\infty := [0]$ being its least element. Call $[\cdot]$ *the natural valuation* of G, call $[G]$ its *value set*, call $[G]^* := [G] \setminus \{-\infty\}$ its *essential value set*, and call the order type of $[G]^*$ the *rank* of G.

Let m be an order-preserving map of $[G]$ onto an ordered set Γ; then $v := m([\cdot])$ is a *natural valuation* of G, and Γ is *a value set* of G. Let $\Gamma^* := m([G]^*)$.

For all $x, y \in G$, $v(x+y) \leqslant \max\{v(x), v(y)\}$.

Call $C \subseteq G$ *convex* if $c \in C$, if $g \in G$, and if $|g| \leqslant |c|$ imply $g \in C$.

For all $\gamma \in \Gamma^*$ let $G^\gamma := \{x \in G\colon\ v(x) \leqslant \gamma\}$ and let $G_\gamma := \{x \in G\colon\ v(x) < \gamma\}$.

Let $G^{-\infty} := 0$ and let $G_{-\infty} := 0$.

G^γ and G_γ are convex subgroups of G. $B(G,\gamma) := G^\gamma/G_\gamma$ has a unique order making it an Archimedean ordered group such that the canonical homomorphism of G^γ onto $B(G,\gamma)$ preserves $\leqslant$. $B(G, -\infty) = 0$, and $\gamma \in \Gamma^* \Rightarrow B(G,\gamma) \neq 0$. $(B(G,\gamma))_{\gamma\in\Gamma^*}$ is called a *skeleton* $S(G)$ of G. Let f be an isomorphism of G onto an ordered group G'. Since

$$x \sim y \iff f(x) \sim f(y) \quad \text{and} \quad x \ll y \iff f(x) \ll f(y),$$

f induces the following order-preserving surjection of value sets:

$$f_v\colon\ v(x) \in \Gamma_G \longmapsto v'\big(f(x)\big) \in \Gamma_{G'}.$$

For each $\gamma \in \Gamma_G$, f induces an isomorphism f_γ of ordered groups as follows:

$$x + G_\gamma \in B(G,\gamma) \longmapsto f(x) + G'_{f_v(\gamma)} \in B\big(G', f_v(\gamma)\big). \tag{1}$$

Hence $S(G)$ is an invariant of G. Let $G^* := G \setminus \{0\}$.

A family $(B(a))_{a\in A^*}$ of nonzero, Archimedean, ordered groups indexed by an ordered set A^* will be called an *ordered system of Archimedean ordered groups*. Note: $S(G)$ is such a family. Let $(B(a))_{a\in A^*}$ be an ordered system of Archimedean ordered groups, and let Π be its Cartesian product. For $p \in \Pi$ let

$$\operatorname{supp}(p) := \{a \in A^*\colon\ p(a) \neq 0\},$$

and call it the *support* of p.

Let $H := \{p \in \Pi\colon \operatorname{supp}(p)$ is an anti-well-ordered subset of $A^*\}$.

H is a subgroup of Π. For $h \in H^*$, let $v(h)$ be the greatest element of $\operatorname{supp}(h)$. Let $v(0) := -\infty$. v maps H onto $A := A^* \cup \{-\infty\}$. Let the order on A^* be extended to A by stipulating that $a > -\infty$, $\forall a \in A^*$. Let H be given the lexicographic order. H is an ordered group called the *Hahn product* of $(B(a))_{a\in A^*}$.

Let $H_\kappa := \{h \in H\colon \operatorname{card}(\operatorname{supp}(h)) < \kappa\}$.

H_κ is a subgroup of H. Let $v_\kappa := v|H_\kappa$. v (respectively, v_κ) is a natural valuation of H (respectively, H_κ), having essential value set A^*, and skeleton $(B(a))_{a\in A^*}$.

Let $(F, +, \cdot, 0, 1, <)$ be an ordered field. $(F, +, 0, <)$ is an ordered group, $[\cdot]$ is its natural valuation, and $[F]$ is its value set. Let m be an order-preserving mapping of $[F]$ onto an ordered set Γ. $v := m([\cdot])$ is a natural valuation of $(F, +, 0, <)$, which has Γ as its value set. Let $G := \Gamma^*$. G has a unique addition such that $v|F^*$ is a homomorphism of $(F^*, \cdot, 1, <)$ onto the ordered group $(G, +, <)$.

Let $O := \{x \in F\colon v(x) \leqslant 0\}$, and let $M := \{x \in F\colon v(x) < 0\}$.

v is a valuation of F, G is its value group, O is its valuation ring, and M is the maximal ideal of O. Let ρ be a homomorphism of O having kernel M. ρ is a *place* of F associated with v. Let $\rho(O) := K$ be called *a residue class field* of v. K has a unique ordering such that ρ preserves $\leqslant$. K is an Archimedean ordered field. K and G are invariants of F. (See, e.g., [13] for details.)

Let K be an Archimedean field, and let G be an ordered group; then $(K, +, 0, <)^G$ is an ordered system of Archimedean ordered groups. Its Hahn product H is an ordered group. H_κ is a subgroup of H, which may also be denoted by $H(G, K)_\kappa$.

For all $x, y \in H$, and for all $g \in G$, let $(x \cdot y)(g) := \sum\limits_{a+b=g} x(a)y(b)$.

Hahn [8] proved that $(H, +, \cdot, 0, 1, <)$ is an ordered field. v is a valuation of H having K as its residue class field and G as its value group. H_κ is a subfield of H. $v_\kappa := v|H_\kappa$ is a valuation on H_κ having K as its residue class field and G as its value group.

3. Examples, Part 1

ASSUMPTION 3.1. *$\alpha > 0$, $\kappa := \aleph_\alpha$ is regular, and $\sum_{\beta<\alpha} 2^{\aleph_\beta} \leqslant \kappa$.*

It is well-known (see e.g., [14]) that Assumption 3.1 is equivalent to: $\alpha > 0$, and there exists an η_α-set E of power κ.

Let $\mathbb{N}$ denote the set of all positive integers, let $\mathbb{Z}$ denote the ring of integers, let $\mathbb{Q}$ denote the field of rational numbers, and let $\mathbb{R}$ denote the field of real numbers. $\mathbb{R}^E$ is an ordered system of Archimedean ordered divisible groups. Let H be its Hahn product, and let v be the natural valuation of H having value set $\Gamma := E \cup \{-\infty\}$. (See Section 2 for definitions.)

For all $T \in \wp(E)$, let $G(T) := \{h \in H_\kappa\colon\ h(T) \subseteq \mathbb{Z}\}$.

LEMMA 3.1. (a) *$G(T)$ is a subgroup of H_κ that is an η_α-set of power κ.* (b) *$G(\varnothing) = H_\kappa$, and thus is divisible.* (c) *$T \neq \varnothing \Rightarrow G(T)$ is not divisible.*

Proof. [2, 3] $\Rightarrow$ (a). (b) and (c) are obvious. □

LEMMA 3.2. (a) *There exists a family $\{E_\lambda\colon\ \lambda \in 2^\kappa\}$ of nonempty, pair-wise nonisomorphic, ordered subsets of E.* (b) *2^κ is the maximal power of a family of nonempty, pair-wise nonisomorphic, ordered sets, each of power $\leqslant \kappa$.*

Proof. A proof of the existence of such a family, each element of which has power at most κ, may be found in [6, pp. 156–157]. On applying Hausdorff's Theorem, [9, p. 181], we see that each E_λ may be embedded in E, proving (a). Since $\mathrm{card}(\wp(E)) = 2^\kappa$, (b) holds. □

THEOREM 3.1. (a) *$G(E_\lambda)$ is a nondivisible, ordered group that is an η_α-set of power κ.* (b) *$\forall\lambda, \lambda' \in 2^\kappa$, $G(E_\lambda)$ and $G(E_{\lambda'})$ are isomorphic as ordered sets.* (c) *$\forall\lambda \neq \lambda' \in 2^\kappa$, $G(E_\lambda)$ and $G(E_{\lambda'})$ are not isomorphic as ordered groups.*

Proof. Lemma 3.2, and parts (a) and (c) of Lemma 3.1 $\Rightarrow$ (a). (a) and Hausdorff's Theorem II [9, p. 181] $\Rightarrow$ (b). Let $\lambda \neq \lambda' \in 2^\kappa$. Assume for a moment that there exists an isomorphism f of $G(E_\lambda)$ onto $G(E_{\lambda'})$. As we saw in Section 2, f induces an order-preserving mapping f_v of E onto itself. There we also saw that for each $e \in E$ the isomorphism f induces an isomorphism f_v of $B(G(E_\lambda), e)$ onto $B(G(E_{\lambda'}), f_v(e))$. By definition and by (1) we have the following:

$$B(G(E_\lambda), e) \simeq \mathbb{Z} \iff e \in E_\lambda.$$

$$B(G(E_{\lambda'}), f_v(e)) \simeq \mathbb{Z} \iff f_v(e) \in E_{\lambda'}.$$

Thus $f_v|E_\lambda$ is an order-preserving map onto $E_{\lambda'}$; but this violates part (a) of Lemma 3.2, and thus proves (c). □

For all $T \in \wp(E)$, let $F(T) := H(G(T), \mathbb{R})_\kappa$.

COROLLARY 3.1. (a) *$F(E_\lambda)$ is a non-real closed field that is an η_α-set of power κ.* (b) *$\forall\lambda, \lambda' \in 2^\kappa$, $F(E_\lambda)$ and $F(E_{\lambda'})$ are isomorphic as ordered additive groups.* (c) *$\forall\lambda \neq \lambda' \in 2^\kappa$, $F(E_\lambda)$ and $F(E_{\lambda'})$ are not isomorphic as ordered fields.*

Proof. Parts (a) and (c) of Lemma 3.1, Lemma 3.2, and [2, 3] imply (a). (a) and [1] $\Rightarrow$ (b). Let $\lambda \neq \lambda' \in 2^\kappa$. Assume, for a moment that there exists an isomorphism f of $F(E_\lambda)$ onto $F(E_{\lambda'})$; then f induces an isomorphism of $G(E_\lambda)$ onto $G(E_{\lambda'})$, which violates part (c) of Theorem 3.1, proving (c). □

4. Examples, Part 2

Continuing with the notation and assumptions of Section 3, let $\lambda \in 2^{\kappa}$.

For all $e \in E_\lambda$, let $C^\lambda(e) := \mathbb{Q}$, and for all $e \in E \setminus E_\lambda$, let $C^\lambda(e) := \mathbb{Q}(\sqrt{2})$.

$(C^\lambda(e))_{e \in E}$ is an ordered system of Archimedean ordered divisible groups. Let $H(\lambda)$ be its Hahn product.

For all $\lambda \in 2^\kappa$, let $C(\lambda) := H(\lambda)_\kappa$.

THEOREM 4.1. (a) *$C(\lambda)$ is an ordered divisible group of power κ.* (b) *$\forall \lambda, \lambda' \in 2^\kappa$, $C(\lambda)$ and $C(\lambda')$ are isomorphic as ordered sets.* (c) *$\forall \lambda \neq \lambda' \in 2^\kappa$, $C(\lambda)$ and $C(\lambda')$ are not isomorphic as ordered groups.*

Proof. [2, 3] implies (a). Cantor has shown [5, pp. 504–506] there exists an order-preserving mapping f of $\mathbb{Q}$ onto $\mathbb{Q}(\sqrt{2})$. Let $\lambda \neq \lambda' \in 2^\kappa$, and let $c \in C(\lambda)$. Recall that $c \in \mathbb{Q}(\sqrt{2})^E$. Let a mapping F be defined as follows.

$$e \in (E_\lambda \cap E_{\lambda'}) \cup \big((E \setminus E_\lambda) \cap (E \setminus E_{\lambda'})\big) \Longrightarrow F(c)(e) = c(e).$$

$$e \in E_\lambda \cap (E \setminus E_{\lambda'}) \Longrightarrow F(c)(e) = f\big(c(e)\big).$$

$$e \in (E \setminus E_\lambda) \cap E_{\lambda'} \Longrightarrow F(c)(e) = f^{-1}\big(c(e)\big).$$

F is an order-preserving mapping of $C(\lambda)$ onto $C(\lambda')$, proving (b). Note that $(\mathbb{Q}, +, 0)$ and $(\mathbb{Q}(\sqrt{2}), +, 0)$ are vector spaces over $\mathbb{Q}$ of dimensions 1 and 2 respectively; thus they are not isomorphic as groups. Using this we may modify the proof of Theorem 3.1 to establish (c). □

For all $\lambda \in 2^\kappa$, let $F(\lambda) := H\big(C(\lambda), \mathbb{R}\big)_\kappa$.

COROLLARY 4.1. (a) *$F(\lambda)$ is a real closed field of power κ.* (b) *$\forall \lambda, \lambda' \in 2^\kappa$, $F(\lambda)$ and $F(\lambda')$ are isomorphic as ordered additive groups.* (c) *$\forall \lambda \neq \lambda' \in 2^\kappa$, $F(\lambda)$ and $F(\lambda')$ are not isomorphic as ordered fields.*

Proof. [2, 3] implies (a). Part (b) of Theorem 4.1 implies (b). Part (c) of Theorem 4.1 implies (c). □

5. Examples, Part 3

Continue with the assumptions of Sections 2 and 3. For an ordered group $(G, +, 0, <)$, let $G^{>0} := \{x \in G\colon x > 0\}$.

LEMMA 5.1. *Let F be an ordered field. $(F, <)$ and $(F^{>0}, <)$ are isomorphic.*

Proof. Let us define an order-preserving map ϕ as follows.

For all $x \in F$ for which $x \geqslant 0$, let $\phi(x) := x + 1$.

For all $x \in F$ for which $x < 0$, let $\phi(x) := 1/(1 - x)$.

$\phi|[0, \infty)$ is an order-preserving map onto $[1, \infty)$.

For all $w, x \in F$, $w < x < 0 \Rightarrow 0 < -x < -w \Rightarrow 1 < 1 - x < 1 - w \Rightarrow 0 < 1/(1 - w) < 1/(1 - x) < 1$;

thus $\phi|(-\infty, 0)$ is an order-preserving map into $(0, 1)$. Let $y \in (0, 1)$. Thus $1/y > 1$, $(1/y) - 1 > 0$, and $x := 1 - (1/y) < 0$. Hence $1/y = 1 - x$ and $y = 1/(1 - x)$. □

THEOREM 5.1. *There exist two ordered divisible groups that are η_α-sets which are isomorphic as ordered sets, but are not isomorphic as ordered groups.*

Proof. Recall (Section 3) that E is an η_α-set of power κ. Since $\mathbb{R}^E$ is an ordered system of Archimedean ordered divisible groups, its Hahn product G is an ordered divisible group. By [2, 3] G is an η_α-set. Since E is an η_α-set, it contains an anti-well-ordered subset of power κ; thus $\mathrm{card}(G) = 2^\kappa$. Let $\Phi := H(G, \mathbb{R})$. By [2, 3], $(\Phi, +, \cdot, 0, 1, <)$ is a real closed field that is an η_α-set.

Let $G_1 := (\Phi, +, 0, <)$ and let $G_2 := (\Phi^{>0}, \cdot, 1, <)$.

Since Φ is an η_α-set, so are G_1 and G_2. By Lemma 5.1, G_1 and G_2 are isomorphic as ordered sets.

Assume, for a moment, that there exists an order-preserving isomorphism θ of $(G_1, +, 0, <)$ onto $(G_2, \cdot, 1, <)$. $\exists a \in G_1^{>0}$ such that $\theta(a) = 2 \in G_2$. Following [10, p. 78], let $f(x) := \theta(xa)$, $\forall x \in G_1$. f is (i) an order-preserving group isomorphism of G_1 onto G_2 such that (ii) $1 + 1/3 < f(1) < 3$, showing that Φ is an exponentially closed field [4]. By [4, Corollary 1.4], $(G^{>0}, <)$ and $(E, <)$ are isomorphic. However, $\mathrm{card}(G^{>0}) = \mathrm{card}(G) = 2^\kappa > \kappa = \mathrm{card}(E)$, which is absurd. □

COROLLARY 5.1. *There exist two real closed fields that are η_α-sets which are isomorphic as ordered additive groups, but not as ordered fields.*

Proof. Let

$$F_i := H(G_i, \mathbb{R}), \quad \text{for } i = 1, 2.$$

By [2, 3], F_1 and F_2 are real closed fields that are η_α-sets. Since G_1 and G_2 are isomorphic as ordered sets, F_1 and F_2 are isomorphic as ordered additive groups. Since G_1 and G_2 are not isomorphic as ordered groups, F_1 and F_2 are not isomorphic as ordered fields. □

References

1. N. L. Alling (1960) On ordered divisible groups, *Trans. Amer. Math. Soc.* **94**, 498–514.
2. N. L. Alling (1961) A characterization of Abelian η_α-groups in terms of their natural valuation, *Proc. Nat. Acad. Sci. (U.S.A.)* **47**, 711–713.
3. N. L. Alling (1962) On the existence of real-closed fields that are η_α-sets of power $\aleph_\alpha$, *Trans. Amer. Math. Soc.* **103**, 341–352.
4. N. L. Alling (1962) On exponentially closed fields, *Proc. Amer. Math. Soc.* **13**, 706–711.
5. G. Cantor (1895) Beiträge zur Begründung der transfiniten Mengenlehre, Part I, *Math. Ann.* **46**, 418–512.
6. C. C. Chang and H. J. Keisler (1973) *Model Theory*, North-Holland, Amsterdam.
7. P. Erdős, L. Gillman and M. Henriksen (1955) An isomorphism theorem for real-closed fields, *Ann. of Math.* **61**, 542–554.
8. H. Hahn (1907) Über die nichtarchimedischen Größensysteme, *S.-B. Akad. Wiss. Wien. math.-naturw. Kl. Abt. IIa*, **116**, 601–655.
9. F. Hausdorff (1914) *Grundzüge der Mengenlehre*, Leipzig, Verlag von Veit.
10. S. Kuhlmann (1991) *Quelques propriétés des espaces vectoriels valués en théorie des modèles*, Doctoral Dissertation, Université Paris 7.
11. L. Mayer (1985) *Vaught's Conjecture for o-Minimal Theories*, Dissertation, Yale University.
12. A. Pillay and C. Steinhorn (1986) Definable sets in ordered structures I, *Trans. Amer. Math. Soc.* **259**, 565–592.
13. S. Prieß-Crampe (1983) *Angeordnete Strukturen: Gruppen, Körper, projective Ebenen*, Springer-Verlag, Berlin.
14. J. G. Rosentein (1982) *Linear Orderings*, Academic Press, New York.