On $\eta_\omega$-Groups and Fields

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Abstract. Let $\kappa := \aleph_\omega$. The following are known: two $\eta_\omega$-sets of power $\kappa$ are isomorphic. Let $\alpha > 0$. Two ordered divisible Abelian groups that are $\eta_\alpha$-sets of power $\kappa$ are isomorphic, two real closed fields that are $\eta_\alpha$-sets of power $\kappa$ are isomorphic. The following is shown: (1) there exist $2^\kappa$ nonisomorphic ordered Abelian groups (respectively ordered fields) that are $\eta_\alpha$-sets of power $\kappa$; (2) there exist $2^\kappa$ nonisomorphic ordered divisible Abelian groups (respectively real closed fields) of power $\kappa$ all having the same order type; (3) there exist $2$ nonisomorphic ordered divisible Abelian groups (respectively real closed fields) that are $\eta_\alpha$-sets having the same order type.

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1. Introduction

A partially ordered set $(S,<)$ is an ordered set if $\forall x, y \in S, x \leq y$ or $x \geq y$. Assume throughout the rest of the paper that all groups are Abelian.

Let $A_1$ and $A_2$ be subsets of an ordered set $A$. Let $A_1 < A_2$ if $a_1 \in A_1$ and $a_2 \in A_2 \Rightarrow a_1 < a_2$. If $A_1 < A_2$ and if the cardinality $\text{card}(A_1 \cup A_2) < \aleph_\omega$ implies the existence of $a \in A$ such that $A_1 < \{a\} < A_2$, Hausdorff [9, pp. 180, 181] called $A$ an $\eta_\alpha$-set. Thus $A$ is a dense ordered set without end points $\iff A$ is an $\eta_0$-set. Hausdorff also proved that the order type of an $\eta_\alpha$-set of power $\aleph_\alpha$ is unique up to isomorphism. Let $\alpha > 0$. In 1955 Erdős, Gillman, and Henriksen [7] proved that a real closed field that is an $\eta_\alpha$-set of power $\aleph_\alpha$ is determined by its order type. In 1960 the first author [1] proved the same result for ordered divisible groups.

Erdős, Gillman, and Henriksen asked the following question [7, Section 5.1]: "Is a non-denumerable real-closed field, in particular, if it is non-archimedean characterized by its type of order as an ordered set"? One might go on to look for a complete
set of invariants for a real closed field. Dense ordered sets without end points, ordered divisible groups, and real closed fields all have the following important model theoretic property: they are all o-minimal. (A structure $M$ is o-minimal if every subset of it that is definable with parameters in $M$ is a finite union of intervals of $M$. See [12] for definitions and properties.) Let $\kappa := \aleph_\alpha$ with $\alpha > 0$. o-minimal structures have many interesting properties: e.g., the following.

THEOREM 1.1. $M$ is $\kappa$-saturated if and only if $M$ has a $\kappa$-saturated order type.

The order type of a dense ordered set without end points, an ordered divisible group, or a real closed field, is $\kappa$-saturated if and only if it is an $\eta_\alpha$-set. Thus from the uniqueness of saturated models and Theorem 1.1, one can prove that the theorems cited of Hausdorff; Erdős, Gillman, and Henriksen; and the first author, follow immediately. In this language the question engendered by that of Erdős, Gillman, and Henriksen translates as follows: what invariants characterize an o-minimal structure?

The search for a natural solution of this classification problem for o-minimal theories is suggested by such important model theoretic properties as the existence and uniqueness of prime models, and Vaught's conjecture (see [12] and [11] for details). However, order type often does not characterize ordered structures. Having made standard assumptions about $\kappa$ in Section 3, we prove that there exist $2^\kappa$ pairwise nonisomorphic ordered groups (respectively ordered fields) that are $\eta_\alpha$-sets of power $\kappa$, necessarily all having the same order type. These examples are not divisible (respectively not real closed) and thus not o-minimal (see [12]). An analogous result is obtained in Section 4, even under the additional hypothesis of o-minimality, but without the condition that the examples are $\eta_\alpha$-sets. In Section 5, having dropped the condition that the examples are of power $\kappa$, it is shown that there exist two ordered divisible groups that are $\eta_\alpha$-sets which are isomorphic as ordered sets, but not as ordered groups, and that there exist two real closed fields that are $\eta_\alpha$-sets which are isomorphic as ordered additive groups, but not as ordered fields. However, the second author has proved [10] that the order type does determine a group up to isomorphism for the class of Archimedean-complete, ordered divisible groups of reverse ordinal rank.

2. Background

For $x, y, z$ in an ordered group $(G, +, 0, <)$, let

$$|x| := \max\{x, -x\}.$$ 

$x \in G \mapsto |x| \in G$ has all the expected properties.

Let $x \sim y$ if $\exists n \in \mathbb{N}$ such that $|x| \leq n|y|$ and $|y| \leq n|x|$. $\sim$ is an equivalence relation on $G$. Let $[G]$ be the set of equivalence classes of $G$ mod $\sim$. Let $[x] \in [G]$ such that $x \in [x]$. $[\cdot] : x \in G \mapsto [x] \in [G]$ is surjective.
Let $x \ll y$ if $\forall n \in \mathbb{N}, n|x| < |y|$. 

$x \in G \Rightarrow x \ll x$. $\forall x, y, z \in G$, $x \ll y$, and $y \ll z \Rightarrow x \ll z$.

Let $[x] < [y] \iff x \ll y$.


Let $m$ be an order-preserving map of $[G]$ onto an ordered set $\Gamma$; then $v := m(\cdot)$ is a natural valuation of $G$, and $\Gamma$ is a value set of $G$. Let $\Gamma^* := m([G]^*)$.

For all $x, y \in G$, $v(x + y) \leq \max \{v(x), v(y)\}$.

Call $C \subseteq G$ convex if $c \in C$, if $g \in G$, and if $|g| \leq |c|$ imply $g \in C$.

For all $\gamma \in \Gamma^*$ let $G^\gamma := \{x \in G: v(x) \leq \gamma\}$ and let $G_\gamma := \{x \in G: v(x) < \gamma\}$.

Let $G_{-\infty} := 0$ and let $G_{-\infty} := 0$.

$G^\gamma$ and $G_\gamma$ are convex subgroups of $G$. $B(G, \gamma) := G^\gamma/G_\gamma$ has a unique order making it an Archimedean ordered group such that the canonical homomorphism of $G^\gamma$ onto $B(G, \gamma)$ preserves $\leq$. $B(G, -\infty) = 0$, and $\gamma \in \Gamma^* \Rightarrow B(G, \gamma) \neq 0$. $(B(G, \gamma))_{\gamma \in \Gamma^*}$ is called a skeleton $S(G)$ of $G$. Let $f$ be an isomorphism of $G$ onto an ordered group $G'$.

Since $x \sim y \iff f(x) \sim f(y)$ and $x \ll y \iff f(x) \ll f(y)$, $f$ induces the following order-preserving surjection of value sets:

$f_v$: $v(x) \in \Gamma_G \longmapsto v'(f(x)) \in \Gamma_{G'}$.

For each $\gamma \in \Gamma_G$, $f$ induces an isomorphism $f_\gamma$ of ordered groups as follows:

$x + G_\gamma \in B(G, \gamma) \longmapsto f(x) + G'_{f_\gamma(\gamma)} \in B(G', f_\gamma(\gamma))$. (1)

Hence $S(G)$ is an invariant of $G$. Let $G^* := G \setminus \{0\}$.

A family $(B(a))_{a \in A^*}$ of nonzero, Archimedean, ordered groups indexed by an ordered set $A^*$ will be called an ordered system of Archimedean ordered groups. Note: $S(G)$ is such a family. Let $(B(a))_{a \in A^*}$ be an ordered system of Archimedean ordered groups, and let $\Pi$ be its Cartesian product. For $p \in \Pi$ let

$supp(p) := \{a \in A^*: p(a) \neq 0\}$,

and call it the support of $p$. 

Let $H := \{ p \in \Pi : \text{supp}(p) \text{ is an anti-well-ordered subset of } A^* \}$.

$H$ is a subgroup of $\Pi$. For $h \in H^*$, let $v(h)$ be the greatest element of supp$(h)$. Let $v(0) := -\infty$. $v$ maps $H$ onto $A := A^* \cup \{-\infty\}$. Let the order on $A^*$ be extended to $A$ by stipulating that $a > -\infty, \forall a \in A^*$. Let $H$ be given the lexicographic order. $H$ is an ordered group called the Hahn product of $(B(a))_{a \in A^*}$.

Let $H_\kappa := \{ h \in H : \text{card}(\text{supp}(h)) < \kappa \}$.

$H_\kappa$ is a subgroup of $H$. Let $v_\kappa := v|H_\kappa$. $v$ (respectively, $v_\kappa$) is a natural valuation of $H$ (respectively, $H_\kappa$), having essential value set $A^*$, and skeleton $(B(a))_{a \in A^*}$.

Let $(F, +, \cdot, 0, 1, <)$ be an ordered field. $(F, +, 0, <)$ is an ordered group, $[\cdot ]$ is its natural valuation, and $[F]$ is its value set. Let $m$ be an order-preserving mapping of $[F]$ onto an ordered set $\Gamma$. $v := m([\cdot])$ is a natural valuation of $(F, +, 0, <)$, which has $\Gamma$ as its value set. Let $G := \Gamma^*$. $G$ has a unique addition such that $v|F^*$ is a homomorphism of $(F^*, \cdot, 1, <)$ onto the ordered group $(G, +, <)$.

Let $O := \{ x \in F : v(x) \leq 0 \}$, and let $M := \{ x \in F : v(x) < 0 \}$.

$v$ is a valuation of $F$, $G$ is its value group, $O$ is its valuation ring, and $M$ is the maximal ideal of $O$. Let $\rho$ be a homomorphism of $O$ having kernel $M$. $\rho$ is a place of $F$ associated with $v$. Let $\rho(O) := K$ be called a residue class field of $v$. $K$ has a unique ordering such that $\rho$ preserves $\leq$. $K$ is an Archimedean ordered field. $K$ and $G$ are invariants of $F$. (See, e.g., [13] for details.)

Let $K$ be an Archimedean field, and let $G$ be an ordered group; then $(K, +, 0, <)^G$ is an ordered system of Archimedean ordered groups. Its Hahn product $H$ is an ordered group. $H_\kappa$ is a subgroup of $H$, which may also be denoted by $H(G, K)_\kappa$.

For all $x, y \in H$, and for all $g \in G$, let $(x \cdot y)(g) := \sum_{a+b=g} x(a)y(b)$.

Hahn [8] proved that $(H, +, \cdot, 0, 1, <)$ is an ordered field. $v$ is a valuation of $H$ having $K$ as its residue class field and $G$ as its value group. $H_\kappa$ is a subfield of $H$. $v_\kappa := v|H_\kappa$ is a valuation on $H_\kappa$ having $K$ as its residue class field and $G$ as its value group.

3. Examples, Part 1

ASSUMPTION 3.1. $\alpha > 0, \kappa := 2^{\aleph_\alpha}$ is regular, and $\sum_{\beta < \alpha} 2^{\aleph_\beta} \leq \kappa$.

It is well-known (see e.g., [14]) that Assumption 3.1 is equivalent to:

$\alpha > 0$, and there exists an $\eta_\alpha$-set $E$ of power $\kappa$. 

Let $\mathbb{N}$ denote the set of all positive integers, let $\mathbb{Z}$ denote the ring of integers, let $\mathbb{Q}$ denote the field of rational numbers, and let $\mathbb{R}$ denote the field of real numbers. $\mathbb{R}^E$ is an ordered system of Archimedean ordered divisible groups. Let $H$ be its Hahn product, and let $v$ be the natural valuation of $H$ having value set $\Gamma := E \cup \{-\infty\}$. (See Section 2 for definitions.)

For all $T \in \wp(E)$, let $G(T) := \{h \in H_{\kappa}: h(T) \subseteq \mathbb{Z}\}$.

**Lemma 3.1.** (a) $G(T)$ is a subgroup of $H_{\kappa}$ that is an $\eta_\alpha$-set of power $\kappa$. (b) $G(\emptyset) = H_{\kappa}$, and thus is divisible. (c) $T \neq \emptyset \Rightarrow G(T)$ is not divisible.

**Proof.** [2, 3] $\Rightarrow$ (a). (b) and (c) are obvious. $\square$

**Lemma 3.2.** (a) There exists a family $\{E_\lambda: \lambda \in 2^\kappa\}$ of nonempty, pair-wise nonisomorphic, ordered subsets of $E$. (b) $2^\kappa$ is the maximal power of a family of nonempty, pair-wise nonisomorphic, ordered sets, each of power $\leq \kappa$.

**Proof.** A proof of the existence of such a family, each element of which has power at most $\kappa$, may be found in [6, pp. 156–157]. On applying Hausdorff's Theorem, [9, p. 181], we see that each $E_\lambda$ may be embedded in $E$, proving (a). Since $\text{card}(\wp(E)) = 2^\kappa$, (b) holds. $\square$

**Theorem 3.1.** (a) $G(E_\lambda)$ is a nondivisible, ordered group that is an $\eta_\alpha$-set of power $\kappa$. (b) $\forall \lambda, \lambda' \in 2^\kappa$, $G(E_\lambda)$ and $G(E_{\lambda'})$ are isomorphic as ordered sets. (c) $\forall \lambda \neq \lambda' \in 2^\kappa$, $G(E_\lambda)$ and $G(E_{\lambda'})$ are not isomorphic as ordered groups.

**Proof.** Lemma 3.2, and parts (a) and (c) of Lemma 3.1 $\Rightarrow$ (a). (a) and Hausdorff's Theorem II [9, p. 181] $\Rightarrow$ (b). Let $\lambda \neq \lambda' \in 2^\kappa$. Assume for a moment that there exists an isomorphism $f$ of $G(E_\lambda)$ onto $G(E_{\lambda'})$. As we saw in Section 2, $f$ induces an order-preserving mapping $f_0$ of $E$ onto itself. There we also saw that for each $e \in E$ the isomorphism $f$ induces an isomorphism $f_e$ of $B(G(E_\lambda), e)$ onto $B(G(E_{\lambda'}), f_0(e))$. By definition and by (1) we have the following:

$$B(G(E_\lambda), e) \cong \mathbb{Z} \iff e \in E_\lambda.$$  
$$B(G(E_{\lambda'}), f_0(e)) \cong \mathbb{Z} \iff f_0(e) \in E_{\lambda'}.$$  

Thus $f_0|E_\lambda$ is an order-preserving map onto $E_{\lambda'}$; but this violates part (a) of Lemma 3.2, and thus proves (c). $\square$

For all $T \in \wp(E)$, let $F(T) := H(G(T), \mathbb{R})$.

**Corollary 3.1.** (a) $F(E_\lambda)$ is a non-real closed field that is an $\eta_\alpha$-set of power $\kappa$. (b) $\forall \lambda, \lambda' \in 2^\kappa$, $F(E_\lambda)$ and $F(E_{\lambda'})$ are isomorphic as ordered additive groups. (c) $\forall \lambda \neq \lambda' \in 2^\kappa$, $F(E_\lambda)$ and $F(E_{\lambda'})$ are not isomorphic as ordered fields.

**Proof.** Parts (a) and (c) of Lemma 3.1, Lemma 3.2, and [2, 3] imply (a). (a) and [1] $\Rightarrow$ (b). Let $\lambda \neq \lambda' \in 2^\kappa$. Assume, for a moment that there exists an isomorphism $f$ of $F(E_\lambda)$ onto $F(E_{\lambda'})$; then $f$ induces an isomorphism of $G(E_\lambda)$ onto $G(E_{\lambda'})$, which violates part (c) of Theorem 3.1, proving (c). $\square$
4. Examples, Part 2

Continuing with the notation and assumptions of Section 3, let $\lambda \in 2^\kappa$.

For all $e \in E_\lambda$, let $C^\lambda(e) := \mathbb{Q}$, and for all $e \in E \setminus E_\lambda$, let $C^\lambda(e) := \mathbb{Q}(\sqrt{2})$.

$(C^\lambda(e))_{e \in E}$ is an ordered system of Archimedean ordered divisible groups. Let $H(\lambda)$ be its Hahn product.

For all $\lambda \in 2^\kappa$, let $C(\lambda) := H(\lambda)_\kappa$.

**THEOREM 4.1.** (a) $C(\lambda)$ is an ordered divisible group of power $\kappa$. (b) $\forall \lambda, \lambda' \in 2^\kappa$, $C(\lambda)$ and $C(\lambda')$ are isomorphic as ordered sets. (c) $\forall \lambda \neq \lambda' \in 2^\kappa$, $C(\lambda)$ and $C(\lambda')$ are not isomorphic as ordered groups.

**Proof.** [2, 3] implies (a). Cantor has shown [5, pp. 504–506] there exists an order-preserving mapping $f$ of $\mathbb{Q}$ onto $\mathbb{Q}(\sqrt{2})$. Let $\lambda \neq \lambda' \in 2^\kappa$, and let $c \in C(\lambda)$. Recall that $c \in \mathbb{Q}(\sqrt{2})$. Let a mapping $F$ be defined as follows.

$$
eq F(c)(e) = c(e).$$

$$e \in (E_\lambda \setminus E_{\lambda'}) \cup ((E \setminus E_\lambda) \cap (E \setminus E_{\lambda'})) \implies F(c)(e) = f(c(e)).$$

$F$ is an order-preserving mapping of $C(\lambda)$ onto $C(\lambda')$, proving (b). Note that $(\mathbb{Q}, +, 0)$ and $(\mathbb{Q}(\sqrt{2}), +, 0)$ are vector spaces over $\mathbb{Q}$ of dimensions 1 and 2 respectively; thus they are not isomorphic as groups. Using this we may modify the proof of Theorem 3.1 to establish (c).

For all $\lambda \in 2^\kappa$, let $F(\lambda) := H(C(\lambda), \mathbb{R})_\kappa$.

**COROLLARY 4.1.** (a) $F(\lambda)$ is a real closed field of power $\kappa$. (b) $\forall \lambda, \lambda' \in 2^\kappa$, $F(\lambda)$ and $F(\lambda')$ are isomorphic as ordered additive groups. (c) $\forall \lambda \neq \lambda' \in 2^\kappa$, $F(\lambda)$ and $F(\lambda')$ are not isomorphic as ordered fields.

**Proof.** [2, 3] implies (a). Part (b) of Theorem 4.1 implies (b). Part (c) of Theorem 4.1 implies (c).

5. Examples, Part 3

Continue with the assumptions of Sections 2 and 3. For an ordered group $(G, +, 0, <)$, let $G^{>0} := \{x \in G: x > 0\}$.

**LEMMA 5.1.** Let $F$ be an ordered field. $(F, <)$ and $(F^{>0}, <)$ are isomorphic.

**Proof.** Let us define an order-preserving map $\phi$ as follows.
For all $x \in F$ for which $x \geq 0$, let $\phi(x) := x + 1$.

For all $x \in F$ for which $x < 0$, let $\phi(x) := 1/(1 - x)$.

$\phi|[0, \infty)$ is an order-preserving map onto $[1, \infty)$.

For all $w, x \in F$, $w < x < 0 \Rightarrow 0 < -x < -w \Rightarrow 0 < 1/(1 - w) < 1/(1 - x) < 1$;

thus $\phi|(-\infty, 0)$ is an order-preserving map into $(0, 1)$. Let $y \in (0, 1)$. Thus $1/y > 1$, $(1/y) - 1 > 0$, and $x := 1 - (1/y) < 0$. Hence $1/y = 1 - x$ and $y = 1/(1 - x)$.

**THEOREM 5.1.** There exist two ordered divisible groups that are $\eta_\alpha$-sets which are isomorphic as ordered sets, but are not isomorphic as ordered groups.

**Proof.** Recall (Section 3) that $E$ is an $\eta_\alpha$-set of power $\kappa$. Since $\mathbb{R}^E$ is an ordered system of Archimedean ordered divisible groups, its Hahn product $G$ is an ordered divisible group. By [2, 3] $G$ is an $\eta_\alpha$-set. Since $E$ is an $\eta_\alpha$-set, it contains an anti-well-ordered subset of power $\kappa$; thus $\text{card}(G) = 2^\kappa$. Let $\Phi := H(G, \mathbb{R})$. By [2, 3], $(\Phi, +, \cdot, 0, 1, <)$ is a real closed field that is an $\eta_\alpha$-set.

Let $G_1 := (\Phi, +, 0, <)$ and let $G_2 := (\Phi^{>0}, \cdot, 1, <)$.

Since $\Phi$ is an $\eta_\alpha$-set, so are $G_1$ and $G_2$. By Lemma 5.1, $G_1$ and $G_2$ are isomorphic as ordered sets.

Assume, for a moment, that there exists an order-preserving isomorphism $\theta$ of $(G_1, +, 0, <)$ onto $(G_2, \cdot, 1, <)$. $\exists a \in G_1^{>0}$ such that $\theta(a) = 2 \in G_2$. Following [10, p. 78], let $f(x) := \theta(xa), \forall x \in G_1$. $f$ is (i) an order-preserving group isomorphism of $G_1$ onto $G_2$ such that (ii) $1 + 1/3 < f(1) < 3$, showing that $\Phi$ is an exponentially closed field [4]. By [4, Corollary 1.4], $(G^{>0}, <)$ and $(E, <)$ are isomorphic. However, $\text{card}(G^{>0}) = \text{card}(G) = 2^\kappa > \kappa = \text{card}(E)$, which is absurd. \hfill \Box

**COROLLARY 5.1.** There exist two real closed fields that are $\eta_\alpha$-sets which are isomorphic as ordered additive groups, but not as ordered fields.

**Proof.** Let

$$F_i := H(G_i, \mathbb{R}), \text{ for } i = 1, 2.$$  

By [2, 3], $F_1$ and $F_2$ are real closed fields that are $\eta_\alpha$-sets. Since $G_1$ and $G_2$ are isomorphic as ordered sets, $F_1$ and $F_2$ are isomorphic as ordered additive groups. Since $G_1$ and $G_2$ are not isomorphic as ordered groups, $F_1$ and $F_2$ are not isomorphic as ordered fields. \hfill \Box
References