Primes and Irreducibles in Truncation Integer Parts of Real Closed Fields. *

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Abstract

Berarducci (2000) studied irreducible elements of the ring $k((G^{<0}))\oplus\mathbb{Z}$, which is an integer part of the power series field $k((G))$ where G is an ordered divisible abelian group and k is an ordered field. Pitteloud (2001) proved that some of the irreducible elements constructed by Berarducci are actually prime. Both authors mainly concentrated on the case of archimedean G. In this paper, we study *truncation integer parts* of any (non-archimedean) real closed field and generalize results of Berarducci and Pitteloud. To this end, we study the canonical integer part $\mathrm{Neg}\,(F)\oplus\mathbb{Z}$ of *any truncation closed subfield* F of $k((G))$, where $\mathrm{Neg}\,(F) := F\cap k((G^{<0}))$, and work out in detail how the general case can be reduced to the case of archimedean G. In particular, we prove that $k((G^{<0}))\oplus\mathbb{Z}$ has (cofinally many) prime elements for any ordered divisible abelian group G. Addressing a question in the paper of Berarducci, we show that every truncation integer part of a non-archimedean exponential field has a cofinal set of irreducible elements. Finally, we apply our results to two important classes of exponential fields: exponential algebraic power series and exponential-logarithmic power series.

0 Introduction

An **integer part** (IP for short) Z of an ordered field K is a discretely ordered subring, with 1 as the least positive element, and such that for every $x\in K$, there is a $z\in Z$ such that $z\le x<z+1$. It follows that the ring of integers $\mathbb{Z}$ is a convex subring of Z. If K is archimedean, then $\mathbb{Z}$ is the only IP of K, so we will be interested in the case of non-archimedean K.

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Shepherdson [S] showed that IP's of real closed fields are precisely the models of a fragment of Peano Arithmetic called Open Induction (OI for short). OI is the first-order theory, in the language $L := \{+, \cdot, <, 0, 1\}$, of discretely ordered commutative rings with 1 whose set of non-negative elements satisfies, for each quantifier-free formula $\Phi(x, y)$, the associated induction axiom $I(\Phi)$:

$$\forall y\, [\Phi(0, y) \text{ and } \forall x\, [\Phi(x, y) \to \Phi(x + 1, y)] \to \forall x \Phi(x, y)]\,.$$

This correspondence led Shepherdson to investigate the arithmetic properties of IP's of real closed fields. Given a field k and an ordered abelian group G, let us denote by $k((G))$ the field of **generalized power series** with exponents in G and coefficients in k (see Section 1). We write $k(G) := k(t^g \mid g \in G)$ for the subfield generated by k and by the monomials t^g. This subfield is the quotient field of the **group ring** $k[G]$ (which is a domain by the same argument as for the ring of polynomials). If k is an ordered field, then $k((G))$ can be ordered lexicographically. Shepherdson considered the countable recursive "algebraic Puiseux series field" $\mathbb{Q}(t^g \mid g \in \mathbb{Q})^r$, which can be viewed as a subfield of the field $\mathbb{Q}^r((\mathbb{Q}))$ of power series with coefficients in the field of real algebraic numbers $\mathbb{Q}^r$ and exponents in the additive group of rational numbers $\mathbb{Q}$. (The superscript r denotes real closure.) He constructed an IP of this field in which the only irreducible elements are those in $\mathbb{Z}$. In particular, the set of primes is not cofinal in this model of OI. Thus the "cofinality of primes" is not provable from Open Induction. On the other hand, subsequent to the work of Shepherdson, several authors (e.g. [M], [B–O], [Bi]) constructed various models of OI with (a cofinal set of) infinite primes.

In [M–R], Mourgues and Ressayre establish the existence of an IP for any real closed field K as follows. Let V be the natural valuation on K. Denote by k the residue field and by G the value group of K (see Section 1). [M–R] show that there is an order preserving embedding φ of K into the field of generalized power series $k((G))$ such that $\varphi(K)$ is a truncation closed subfield (see Section 2 for details). They observe that for the field $k((G))$, an integer part is given by $k((G^{<0})) \oplus \mathbb{Z}$, where $k((G^{<0}))$ is the (non-unital) ring of power series with negative support. It follows easily (see Proposition 2.1) that for any truncation closed subfield F of $k((G))$, an integer part is given by $Z_F = \mathrm{Neg}\,(F) \oplus \mathbb{Z}$, where $\mathrm{Neg}\,(F) := k((G^{<0})) \cap F$. We shall call Z_F the **canonical integer part** of F. Finally $\varphi^{-1}(Z_F)$ is an integer part of K if we take $F = \varphi(K)$. An integer part Z of K obtained in this way from a truncation closed embedding shall be called a **truncation integer part** of K.

In [R], a proof for an exponential analogue of the main result of [M–R] is sketched: every exponential field (see Section 5) has an *exponential* integer part (EIP for short). An EIP is an IP that satisfies some closure conditions under the exponential function (see Section 5).

In [Bo2], Boughattas considers the following extension of OI in the language $L \cup \{2^x\}$ containing a symbol for the exponential function. He defines $\mathrm{OI}(2^x)$ to be the first-order theory of discretely ordered commutative rings with 1 whose set of non-negative elements satisfies, for each quantifier-free formula $\Phi(x, y)$ of $L \cup \{2^x\}$, the associated induction axiom $I(\Phi)$, and also the following basic axioms for 2^x:

$$2^0 = 1, \quad 2^1 = 2, \quad 2^{x+y} = 2^x 2^y, \quad x < 2^x.$$

Analogously to Shepherdson's result, Boughattas constructs a model of $\mathrm{OI}(2^x)$ in which the only irreducible elements are those in $\mathbb{Z}$, and thus the set of primes is not cofinal in this model of $\mathrm{OI}(2^x)$. We note that, unlike the case of OI, an algebraic description of models of $\mathrm{OI}(2^x)$ is not known. In particular, the relationship between models of $\mathrm{OI}(2^x)$ on the one hand and EIP's of exponential fields on the other hand remains unclear. In [B; Concluding Remarks], Berarducci asks (and attributes the question to Ressayre) for an explicit axiomatisation of the class of EIP's of exponential fields. We do not consider this question in this paper.

Using a new kind of valuation whose values are ordinal numbers, Berarducci [B] studies irreducible elements in the ring $k((G^{<0})) \oplus \mathbb{Z}$, that is, in the canonical IP of the power series field $k((G))$, focusing mainly on the case when G is *archimedean.* He gives a test for irreducibility based only on the order type of the support of a series. It is not known if every irreducible element of $k((G^{\leq 0}))$ is prime (that is, generates a prime ideal). Refining the methods of [B], Pitteloud [P] shows that some of the irreducible series constructed in [B] are actually prime, in the case when G is *archimedean* or *contains a maximal proper convex subgroup.* In particular, [P; Theorem 4.1] says that, in the case of archimedean divisible group G, if $s \in k((G^{<0}))$ is such that support s has order type ω and least upper bound 0, then $s + 1$ is prime in $k((G^{<0})) \oplus \mathbb{Z}$.

In this paper, we extend the results of [B] and [P] to the canonical integer part $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ of any *truncation closed subfield* F,

$$k(G) \subset F \subset k((G)) \,,$$

for an *arbitrary* divisible ordered abelian group $G \neq 0$. We shall denote by $k[G^{<0}]$ (respectively, by $k[G^{\leq 0}]$) the **semigroup ring** consisting of power series with negative (respectively, non-positive) and *finite* support. Note that $k[G^{<0}] \subset \mathrm{Neg}\,(F)$ since $k(G) \subset F$. [B; Theorem 11.2] says that, in the case of archimedean G, all irreducible elements of $k[G^{<0}] \oplus \mathbb{Z}$ remain irreducible in $k((G^{<0})) \oplus \mathbb{Z}$. We study, for an arbitrary G, the behaviour of primes and irreducibles under the ring extensions

$$k[G^{<0}] \oplus \mathbb{Z} \subset \mathrm{Neg}\,(F) \oplus \mathbb{Z} \subset k((G^{<0})) \oplus \mathbb{Z}. \tag{1}$$

In [B; Concluding Remarks], the author asks whether every EIP of an exponential field contains a cofinal set of irreducible elements. We give a partial answer to this question: applying our results to truncation EIP's of non-archimedean exponential fields, we show that these EIP's indeed contain a cofinal set of irreducible elements. Note that in this case, the rank of G (see Section 2) is a dense linearly ordered set without endpoints [K ; Corollary 1.23], in particular G cannot be archimedean, nor can it contain a maximal proper convex subgroup.

The structure of this paper is the following. In Section 1, we fix the notation and review the necessary background concerning ordered groups and fields. In Section 2, we establish some straightforward facts used in the subsequent sections, in particular, to study the extensions (1). The main results of Section 3 are Proposition 3.8 and Corollary 3.10 which provides (under mild conditions) cofinal sets of irreducibles (=primes) in $k[G^{<0}] \oplus \mathbb{Z}$.

In Section 4, we generalize [B; Theorem 11.2] (see Corollaries 4.4 and 4.5) using the reduction to the case of an archimedean group G given in our Theorem 4.2 and Corollary 4.3 (some special cases of Corollary 4.3 already appeared in [B] and [P]). As a consequence, we extend Corollary 3.10 to the ring $k((G^{<0})) \oplus \mathbb{Z}$ obtaining cofinal sets of *irreducibles* with finite support in $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ (see Corollary 4.10). In the special case when F is the **field of algebraic power series**, that is, $F = k(G)^r$, we can improve the result to obtain cofinal sets of *primes* with finite support in $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ (see Corollary 4.11). Using our generalization of [P; Theorem 4.2] (see Theorem 4.12), we show that $k((G^{<0})) \oplus \mathbb{Z}$ *has a cofinal set of primes* with infinite support (see Corollary 4.16).

In Section 5, we apply the results of Section 4 to study EIP's of exponential fields. The main application is given in Theorem 5.3: we establish that *every exponential field has an EIP with a cofinal set of irreducible elements.* We work out two examples in detail. In Example 5.4, we consider the countable "Algebraic Power Series Fields with Exponentiation" described in [K; Example 1.45]. These fields are of the form $E(t^g \mid g \in G)^r$, where E is a countable exponentially closed subfield of the reals, and G is the lexicographic sum, taken over the rationals, of copies of the additive group of E. These fields may be viewed as truncation closed subfields of $E((G))$. We start by studying the canonical IP of such a field and show that this canonical IP is an EIP. We establish that this canonical EIP has a cofinal set of *prime elements with finite support.* In Example 5.5 we study the canonical EIP of the "Exponential-Logarithmic Power Series Fields" introduced in [K; Chapter 5 p. 79]. We show that this EIP has cofinally many *primes with infinite support.*

1 Preliminaries.

Let G be an ordered abelian group. Set $|g| := \max\{g, -g\}$ for $g \in G$. For non-zero $g_1, g_2 \in G$ we say that g_1 is **archimedean equivalent to** g_2 if there exists $r \in \mathbb{N}$ such that

$$r|g_1| \geq |g_2| \quad \text{and} \quad r|g_2| \geq |g_1|. \tag{2}$$

We write $g_1 << g_2$ if $r|g_1| < |g_2|$ for all $r \in \mathbb{N}$. Denote by $[g]$ the equivalence class of $g \neq 0$, and by v the **natural valuation** on G, that is, $v(g) := [g]$ for $g \neq 0$, and $v(0) := \infty$. The **rank** of G is defined to be $\Gamma := v(G \setminus \{0\})$. The relation $<<$ on G induces a linear order on the set Γ by setting $v(g_1) > v(g_2)$ if $g_1 << g_2$ (note the reversed order). For $g_1,\ g_2 \in G$, we have $v(g_1 - g_2) \geq \min\{v(g_1), v(g_2)\}$. For each $\gamma \in \Gamma$, fix $g_\gamma \in G$ such that $v(g_\gamma) = \gamma$. Let C_γ and D_γ denote, respectively, the smallest convex subgroup containing g_γ and the largest convex subgroup not containing g_γ. Note that C_γ and D_γ are independent from the choice of the representative g_γ. In fact

$$C_\gamma = \{g \in G \,|\, v(g) \geq \gamma\} \text{ and } D_\gamma = \{g \in G \,|\, v(g) > \gamma\}.$$

Note also that D_γ is a maximal proper convex subgroup of C_γ and thus the quotient

$$A_\gamma := C_\gamma / D_\gamma$$

is archimedean. We call A_γ an **archimedean component** of G.

Let Q be an archimedean field. We now recall some general definitions and facts about ordered Q-vector spaces. Clearly, if G is an ordered Q-vector space, then for $g_1, g_2 \in G$, g_1 is archimedean equivalent to g_2 if and only if there exists $r \in Q$ such that (2) holds. A subset $B \subset G \setminus \{0\}$ is **strongly independent** if B consists of pairwise archimedean inequivalent elements. We say that $B = \{b_i \mid i \in I\}$ is **Q-valuation independent** if for all $r_i \in Q$ such that $r_i = 0$ for all but finitely many $i \in I$,

$$v\left(\sum_{i \in I} r_i b_i\right) = \min_{\{i \in I | r_i \neq 0\}} \{v(b_i)\}\ .$$

A **Q-valuation basis** is a Q-basis which is Q-valuation independent.

If G is a Q-vector space, then C_γ and D_γ are Q-subspaces, so A_γ is a Q-vector space, isomorphic to any maximal archimedean *subspace* of G containing g_γ, and to any Q-vector space complement to D_γ in C_γ.

Remark 1.1 For $Q = \mathbb{Q}$, A_γ is isomorphic to any maximal archimedean subgroup of G containing g_γ. This is because if A is a subgroup of G, then A is archimedean if and only if the divisible hull $\langle A \rangle_{\mathbb{Q}}$ of A is archimedean, which implies that a maximal archimedean subgroup of G is necessarily a $\mathbb{Q}$-subspace.

We let π_γ denote the natural homomorphism $C_\gamma \to A_\gamma = C_\gamma / D_\gamma$. The following characterization of valuation independence is useful and easy to prove (see [K]).

Proposition 1.2 *Let $B \subset G \setminus \{0\}$. Then B is Q-valuation independent if and only if the following holds: for all $n \in \mathbb{N}$ and distinct $b_1, \ldots, b_n \in B$ with $v(b_1) = \ldots = v(b_n) = \gamma$, the elements $\pi_\gamma(b_1), \ldots, \pi_\gamma(b_n)$ in A_γ are Q-linearly independent.*

It follows that a Q-valuation basis is a maximal Q-valuation independent set. (But in general, a maximal Q-valuation independent set need *not* be a basis.) We also have

Corollary 1.3 *Let $B \subset G$ be Q-linearly independent and consist of archimedean equivalent elements. Then $\langle B \rangle_Q$ is archimedean if and only if B is Q-valuation independent.*

If all archimedean components of G have dimension 1 over Q, then $B \subset G$ is Q-valuation independent if and only if B is strongly independent. (In particular, if $Q = \mathbb{R}$, then B is Q-valuation independent if and only if B is strongly independent, because an archimedean $\mathbb{R}$-vector space has necessarily dimension 1 over $\mathbb{R}$.)

In [Br] it is shown that every ordered Q-vector space of countable dimension has a Q-valuation basis. (In particular, every ordered $\mathbb{R}$-vector space of countable dimension has a strongly independent basis.)

We also need to recall some facts about valued fields. (In this paper, we mainly deal with fields of characterisitc 0.) Let K be a field, G an ordered abelian group and ∞ an element greater than every element of G.

A surjective map $w: K \to G \cup \{\infty\}$ is a **valuation** on K if for all $a, b \in K$
(i) $w(a) = \infty$ iff $a = 0$, (ii) $w(ab) = w(a) + w(b)$, (iii) $w(a-b) \geq \min\{w(a), w(b)\}$.

We say that (K, w) is a **valued field**. The **value group** of (K, w) is $w(K) := G$. The **valuation ring** of w is $\mathcal{O}_w := \{a \mid a \in K \text{ and } w(a) \geq 0\}$ and the **valuation ideal** is $\mathcal{M}_w := \{a \mid a \in K \text{ and } w(a) > 0\}$. The field $\mathcal{O}_w/\mathcal{M}_w$, denoted by Kw, is the **residue field**. For $b \in \mathcal{O}_w$, bw is its image under the residue map.

A valued field (K, w) is **henselian** if given a polynomial $p(x) \in \mathcal{O}_w[x]$, and $a \in Kw$ a simple root of the reduced polynomial $p(x)w \in Kw[x]$, we can find a root $b \in K$ of $p(x)$ such that $bw = a$.

Let $K \subset L$ be an extension of valued fields. By abuse of notation, denote by w the valuation on both K and L. Then K is said to be w**-dense** in L if for every $a \in L$ and $\alpha \in w(L)$ there is some $b \in K$ such that $w(a - b) > \alpha$.

If $(K, +, \cdot, 0, 1, <)$ is an ordered field, we denote by V its natural valuation, that is, the natural valuation V on the ordered abelian group $(K, +, 0, <)$. (The set of archimedean classes becomes an ordered abelian group by setting $[x] + [y] := [xy]$.) For the natural valuation, we shall use the notation $\mathcal{O}_K := \{x \in K \mid V(x) \geq 0\}$ and $\mathcal{M}_K := \{x \in K \mid V(x) > 0\}$, respectively, for the valuation ring and valuation ideal. We shall also often write k for the residue field $\mathcal{O}_K/\mathcal{M}_K$ and G for the value group $V(K)$. Note that k is an archimedean ordered field, and that V is **compatible** with the order, that is, has a convex valuation ring. We denote by $\mathcal{U}_K^{>0}$ the multiplicative group of positive units (invertible elements) of $\mathcal{O}_K$. The subgroup $1 + \mathcal{M}_K$ of $\mathcal{U}_K^{>0}$ is called the group of 1-units.

For ordered fields, there is another notion of density that we would like to review briefly. For more details see [K; Chapter 1, Section 6]. Let $K \subset L$ be an extension of ordered fields. K is **order dense** in L if for all $a, c \in L$ with $a < c$ there is $b \in K$ such that $a \leq b \leq c$. Let $K \subset L$ be an extension of ordered fields, and let

w be a compatible valuation on both of them. Assume that L is non-archimedean. Then K is order dense in L if and only if K is w-dense in L.

We need further well-known facts about ordered fields endowed with a compatible valuation. Let K be an ordered field and w a compatible valuation on K. Then K is real closed if and only if (i) (K, w) is henselian (ii) its residue field Kw is real closed field and (iii) its value group $w(K)$ is divisible.

If K is real closed, then the residue field Kw embeds in K and K admits a cross-section, that is, an embedding η of the value group G into $(K^{>0}, \cdot)$ such that $w(\eta(g)) = g$ for all $g \in G$ (see [PC]).

Therefore, whenever needed, we can assume without loss of generality that $k(G) \subset K$. (More precisely, we identify the residue field k with a maximal subfield of $\mathcal{O}_K$ through the residue map, and G with $\eta(G)$.)

Recall from the Introduction that for G an ordered abelian group, k a field, $k((G))$ denotes the field of power series with coefficients in k and exponents in G. Every series $s \in k((G))$ is of the form $\sum_{g \in G} s_g t^g$ with $s_g \in k$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$. Addition is pointwise, multiplication is given by the usual formula for multiplying power series:

$$\left(\sum_{g \in G} r_g t^g\right)\left(\sum_{g \in G} s_g t^g\right) = \sum_{g \in G}\left(\sum_{g' \in G} r_{g'} s_{g-g'}\right) t^g \ .$$

The **canonical valuation** on $k((G))$ is given by $V_{\min}(s) := \min(\operatorname{support} s)$ for any series $s \in k((G))$. If k is an ordered field, we can endow $k((G))$ with the lexicographic order: a series is positive if its least nonzero coefficient is positive. With this order, $k((G))$ is an ordered field, and $V_{\min}$ is compatible with this order. If k is archimedean, then $V_{\min}$ coincides with the natural valuation V. Clearly, the value group of $(k((G)), V_{\min})$ is (isomorphic to) G and the residue field is (isomorphic to) k. The valuation ring $k((G^{\geq 0}))$ consists of the series with non-negative exponents, and the valuation ideal $k((G^{>0}))$ of the series with positive exponents. The **constant term** of a series $\sum_{g \in G} s_g t^g$ is the coefficient s_0. The units of $k((G^{\geq 0}))$ are the series in $k((G^{\geq 0}))$ with a non-zero constant term.

Every series $s \in k((G))$ can be written as $s = s_{<0} + s_0 + s_{>0}$ where s_0 is the constant term of s and $s_{<0}$, resp. $s_{>0}$, denotes the restriction of s to $G^{<0}$, resp. $G^{>0}$. Thus the (non-unital) ring $k((G^{<0}))$ of generalized power series with negative support is a complement in $(k((G)), +)$ to the valuation ring. Note that it is in fact a k-algebra. We shall denote by $k((G^{\leq 0}))$ the ring of generalized power series with non-positive support.

Given $s \in k((G))^{>0}$, we can factor out the monomial of smallest exponent $g \in G$ and write $s = t^g u$ with u a unit with a positive constant term. Thus the multiplicative subgroup $\operatorname{Mon} k((G)) := \{t^g \mid g \in G\}$ consisting of the (monic) monomials t^g is a complement in $(k((G))^{>0}, \cdot)$ to the subgroup of positive units.

2 Truncation Integer Parts.

A subfield F of $k((G))$ is **truncation closed** if whenever $s = \sum_{g \in G} s_g t^g \in F$ and $a \in G$, the restriction $s_{<a} = \sum_{g \in G^{<a}} s_g t^g$ of s to the initial segment $G^{<a}$ of G also belongs to F. As mentioned in the Introduction, [M–R] show that given a real closed field K with residue field k and value group G (for the natural valuation V on K), with $k(G) \subset K$, there is a **truncation closed embedding** of K into $k((G))$ over $k(G)$, that is, an embedding φ such that $F := \varphi(K)$ is truncation closed. Note that since the restriction of φ to $k(G)$ is the identity, φ is in particular an embedding of k-vector spaces.

Proposition 2.1 *Let $F \subset k((G))$ be a truncation closed subfield that contains $k(G)$. Then $Z_F := \mathrm{Neg}\,(F) \oplus \mathbb{Z}$ is an integer part of F (that contains $k[G^{<0}] \oplus \mathbb{Z}$).*

Proof: Clearly, Z_F is a discrete subring of F. Let $s \in k((G))$. Let $\lfloor s_0 \rfloor \in \mathbb{Z}$ be the integer part of $s_0 \in k$. Define

$$z_s = \begin{cases} s_{<0} + s_0 - 1 & \text{if } s_0 \in \mathbb{Z} \text{ and } s_{>0} < 0, \\ s_{<0} + \lfloor s_0 \rfloor & \text{otherwise.} \end{cases}$$

Clearly, $z_s \leq s < z_s + 1$. From truncation closedness of F it follows that if $s \in F$, then $z_s \in Z_F$. □

It follows that $\varphi^{-1}(Z_F)$ is an IP of K. Recall that we refer to IP's obtained in this way via a truncation closed embedding as truncation IP's.

We now state some easy facts about IP's of ordered fields in general. We note that all IP's of a given ordered field are *isomorphic as ordered sets* (if Z_1, Z_2 are IP's, an isomorphism is obtained by mapping $s \in Z_1$ to its integer part z_s with respect to Z_2), but in general, they need not be isomorphic as ordered rings, not even elementarily equivalent (see Remark 2.3). If Z is an IP of K, then the fraction field of Z is an order-dense subfield of K. If L is an order-dense subfield of K, then every IP of L is an IP of K. Conversely, if $L \subset K$ and there exists an IP Z of L that remains an IP of K, then L is order-dense in K.

Remark 2.2 In [Bo] examples of ordered fields without IP's are given. We conjectured that every *henselian* ordered field admits a truncation closed embedding into a field of power series and thus, admits an IP. This conjecture is being studied in [F]. We also asked for a direct proof of the existence of IP's for henselian fields (without arguing via truncation closed embeddings in fields of power series). This question is addressed in [KF].

Remark 2.3 Truncation IP's of non-archimedean real closed fields are very peculiar models of OI. It would be interesting to investigate the algebraic and model-theoretic properties of this class.

(i) They admit $\mathbb{Z}$ as a direct summand. This is not the case for an arbitrary model of OI. For example, as observed by D. Marker (unpublished), $\mathbb{Z}$ cannot be a direct summand of a non-standard model of Peano Arithmetic. This remark implies in particular that *not every IP of a real closed field is a truncation IP.*

(ii) They are intimately related to complements of the valuation ring (see [KF]). A truncation IP Z of K decomposes as $Z = A \oplus \mathbb{Z}$ (lexicographic sum), where the summand A is a k-algebra. It is easily verified that A is an additive complement to $\mathcal{O}_K$. Since additive complements to $\mathcal{O}_K$ are unique up to isomorphism of ordered groups, all truncation IP's of K are *isomorphic as ordered groups*. Note that if an additive complement A to $\mathcal{O}_K$ is closed under multiplication, then $Z(A) := A \oplus \mathbb{Z}$ is an IP of K. However, A need not be a k-algebra, so $Z(A)$ need not be a truncation IP. It is not known to us whether every IP of the form $Z(A)$, with A a k-algebra, is a truncation IP.

(iii) They are never normal (that is, never integrally closed in their field of fractions). This is because $\mathrm{Neg}\,(F)$ is a k-vector space, so $\sqrt{2}t^g \in Z_F$ for $g \in G^{<0}$ and hence $\sqrt{2}$ is rational over Z_F: $\sqrt{2} = \frac{\sqrt{2}t^g}{t^g}$. Since the *normal* model obtained in [B-O] is an IP of a real closed field, which also has a *non-normal* IP, we see that the IP's of a given real closed field need not be elementarily equivalent. This motivates the following

Open Question: Does every real closed field have a *normal* integer part?

Let $F = k(G)$ and F^r its real closure. By [M–R], F and F^r are truncation closed subfields of $k((G))$. We already noted that $k[G^{<0}] \subset \mathrm{Neg}\,(F)$. We are interested in understanding when $k[G^{<0}] = \mathrm{Neg}\,(F)$. The following fact was observed by F.-V. Kuhlmann.

Proposition 2.4 *Assume that G is archimedean and divisible, and that k is a real closed field. Then* $\mathrm{Neg}\,(F) = \mathrm{Neg}\,(F^r) = k[G^{<0}]$.

Proof: We first show that $\mathrm{Neg}\,(F) = k[G^{<0}]$. Let $f \in F = k(G)$. So $f = p/q$ with $p = \sum_{i=0}^{l} a_i t^{g_i}$, $q = \sum_{i=0}^{l} a'_i t^{g'_i} \in k[G]$. By factoring out the monomial of least exponent in q, say $a'_0 t^{g'_0}$, we may rewrite

$$f = \sum_{i=0}^{l} c_i t^{h_i} (1+\varepsilon)^{-1} = \sum_{i=0}^{l} c_i t^{h_i} \sum_{n=0}^{\infty} (-1)^n \varepsilon^n$$

where $c_i = a_i/a'_0$, $h_i = g_i - g'_0$, and $\varepsilon \in k[G^{>0}]$. Since G is archimedean, we may choose N large enough so that $NV_{\min}(\varepsilon) \geq -h_i$ for all $i = 0, \ldots, l$. We now rewrite:

$$f = \sum_{i=0}^{l} c_i t^{h_i} \sum_{n=0}^{N-1} (-1)^n \varepsilon^n + \sum_{i=0}^{l} c_i t^{h_i} \sum_{n=N}^{\infty} (-1)^n \varepsilon^n.$$

By our choice of N, we have that $V_{\min}\left(\sum_{i=0}^{l} c_i t^{h_i} \sum_{n=N}^{\infty} (-1)^n \varepsilon^n\right) \geq 0$. It follows that if $f \in \mathrm{Neg}\,(F)$, then $f = \sum_{i=0}^{l} c_i t^{h_i} \sum_{n=0}^{N-1} (-1)^n \varepsilon^n$, so $f \in k[G]$.

We now show that $\mathrm{Neg}\,(F) = \mathrm{Neg}\,(F^r)$. By assumptions on k and G, we have $F^r = F^h$:= the henselization of F with respect to $V_{\min}$. Since G is archimedean, F is dense in F^h (see [Ri]). Thus $\mathrm{Neg}\,(F) = \mathrm{Neg}\,(F^h)$ (see [K; Lemma 1.32]). $\square$

Remark 2.5 If G is not archimedean, then $k[G^{<0}] \neq \mathrm{Neg}\,(F)$: choose $g_1, g_2 \in G^{<0}$ with $v(g_1) < v(g_2)$, then $t^{g_1}(1-t^{-g_2})^{-1} \in F$ and $t^{g_1}(1-t^{-g_2})^{-1} = \sum_{n=0}^{\infty} t^{g_1 - ng_2}$ has *infinite* negative support. Also, $\mathrm{Neg}\,(F^r) \neq \mathrm{Neg}\,(F)$: consider $t^{g_1}(1+t^{-g_2})^{1/2} \in F^r$. Since $t^{g_1}(1+t^{-g_2})^{1/2} = t^{g_1} + \frac{1}{2}t^{g_1-g_2} - \frac{1}{8}t^{g_1-2g_2} + \cdots$ has negative support, it belongs to $\mathrm{Neg}\,(F^r)$. On the other hand, $(1+t^{-g_2})^{1/2} \notin F$ so $t^{g_1}(1+t^{-g_2})^{1/2} \notin F$.

We will frequently use the following two lemmas. If R is a ring, we denote by $U(R)$ its group of units.

Lemma 2.6 *Let G be an ordered abelian group, K a field. Let R be a subring of K and* Neg *a (non-unital) K-subalgebra of $K((G^{<0}))$. Let $r \in R$. Then r is prime (resp., irreducible) in R iff r is prime (resp., irreducible) in* $\mathrm{Neg} \oplus R$.

Proof: Clearly, if $a, b \in \mathrm{Neg} \oplus R$ and $ab \in R$, then $a, b \in R$. It follows that $U(\mathrm{Neg} \oplus R) = U(R)$ and that r is irreducible in R iff r is irreducible in $\mathrm{Neg} \oplus R$. Now let $a \in \mathrm{Neg} \oplus R$ and let a_0 be the constant term of a. Clearly, r divides a in $\mathrm{Neg} \oplus R$ iff r divides a_0 in R. It follows that r is prime in R iff r is prime in $\mathrm{Neg} \oplus R$. □

Lemma 2.7 *Let G be an ordered abelian group, K a field. Let R be a subring of K and* Neg *a (non-unital) K-subalgebra of $K((G^{<0}))$. Let f be a non-constant element of* $\mathrm{Neg} \oplus R$ *with constant term 1. Then f is prime (resp., irreducible) in* $\mathrm{Neg} \oplus R$ *iff f is prime (resp., irreducible) in* $\mathrm{Neg} \oplus K$.

Proof: ($\Rightarrow$) Suppose f is irreducible in $\mathrm{Neg} \oplus R$ and assume that $f = ab$ for some non-invertible $a, b \in \mathrm{Neg} \oplus K$. Let a_0, b_0 be the constant terms of a, b. Then $a \neq a_0$, $b \neq b_0$, and $a_0 b_0 = 1$. Let $\tilde{a} = b_0 a$ and $\tilde{b} = a_0 b$. Then $\tilde{a}, \tilde{b}$ are non-invertible elements of $\mathrm{Neg} \oplus R$ and $f = \tilde{a}\tilde{b}$, a contradiction. Now suppose f is prime in $\mathrm{Neg} \oplus R$, $a, b \in \mathrm{Neg} \oplus K$, and f divides ab in $\mathrm{Neg} \oplus K$. Let $\tilde{a} = a/a_0$ if $a_0 \neq 0$ and $\tilde{a} = a$ otherwise. Similarly, let $\tilde{b} = b/b_0$ if $b_0 \neq 0$ and $\tilde{b} = b$ otherwise. Then f divides $\tilde{a}\tilde{b}$ in $\mathrm{Neg} \oplus R$. Hence f divides $\tilde{a}$ or $\tilde{b}$ in $\mathrm{Neg} \oplus R$, which implies that f divides a or b in $\mathrm{Neg} \oplus K$.

($\Leftarrow$) Suppose f is irreducible in $\mathrm{Neg} \oplus K$ and assume that $f = ab$ for some non-invertible $a, b \in \mathrm{Neg} \oplus R$. Then $a_0 b_0 = 1$ and, therefore, $a \neq a_0$ and $b \neq b_0$. But then a, b are not invertible in $\mathrm{Neg} \oplus K$ and $f = ab$, a contradiction. Now suppose f is prime in $\mathrm{Neg} \oplus K$, $a, b \in \mathrm{Neg} \oplus R$, and f divides ab in $\mathrm{Neg} \oplus R$. Then f divides ab in $\mathrm{Neg} \oplus K$. Hence f divides a or b in $\mathrm{Neg} \oplus K$, which implies that f divides a or b in $\mathrm{Neg} \oplus R$ since f has constant term 1. □

We want to study prime and irreducible elements of the rings $k[G^{<0}] \oplus \mathbb{Z}$ and $k((G^{<0})) \oplus \mathbb{Z}$, but it is often more convenient to work with $k[G^{\leq 0}]$ and $k((G^{\leq 0}))$ instead. It is easy to see how this change affects primality and irreducibility.

If f is irreducible in $k[G^{<0}] \oplus \mathbb{Z}$, then either $f \in \mathbb{Z}$ or $f \notin \mathbb{Z}$ and the constant term $f_0 = \pm 1$. Indeed, if $f \notin \mathbb{Z}$ and $f_0 \neq \pm 1$, then we can factor f in $k[G^{<0}] \oplus \mathbb{Z}$ as follows: $f = f_0(f/f_0)$ if $f_0 \neq 0$ and $f = 2(f/2)$ if $f_0 = 0$. Using Lemmas 2.6

and 2.7, we conclude that prime (resp., irreducible) elements of $k[G^{<0}] \oplus \mathbb{Z}$ are of two types: 1) primes in $\mathbb{Z}$ and 2) prime (resp., irreducible) elements of $k[G^{\leq 0}]$ with constant term ± 1. We will show in Section 3 that in fact all irreducible elements of $k[G^{\leq 0}]$ are prime.

By similar considerations, primes (resp., irreducibles) of $k((G^{<0})) \oplus \mathbb{Z}$ are of two types: 1) primes in $\mathbb{Z}$ and 2) primes (resp., irreducibles) of $k((G^{\leq 0}))$ with constant term ± 1. We do not know whether or not all irreducibles of $k((G^{\leq 0}))$ are prime.

The following two lemmas are standard. We omit the proofs.

Lemma 2.8 *Let $(I, \leq)$ be a directed set (that is, $\leq$ is a partial order and, for any $i_1, i_2 \in I$, there exists $i_0 \in I$ such that $i_0 > i_1, i_2$). Let R be an integral domain and assume that, for every $i \in I$, we assigned a subring $R_i \subset R$ in such a way that R is the direct limit of R_i (that is, $R_{i_1} \subset R_{i_2}$ for $i_1 \leq i_2$ and $R = \cup_{i \in I} R_i$). Let $r \in R$. If there exists $i_0 \in I$ such that r is prime (resp., irreducible) as an element of R_i for $i \geq i_0$, then r is prime (resp., irreducible) in R.*

Lemma 2.9 *Let R be an integral domain and $T \subset R \setminus \{0\}$ a multiplicative sub-semigroup. Denote by R_T the corresponding ring of fractions. Let $r \in R$.*

1) *If r is prime in R and $r \nmid t$ in R for all $t \in T$, then r is prime in R_T.*

2) *If r is prime in R_T and, for all $t \in T$ and $a \in R$, $r \mid ta$ in $R \Rightarrow r \mid a$ in R, then r is prime in R.*

As an immediate application of Lemma 2.9 (with $R := k[G^{\leq 0}]$ and $T := \{t^g \mid g \in G^{\leq 0}\}$), we obtain

Corollary 2.10 *A non-monomial $f \in k[G^{\leq 0}]$ is prime in $k[G^{\leq 0}]$ if and only if it is prime in $k[G]$.*

3 Primes and Irreducibles in $k[G^{\leq 0}]$.

Let G be a torsion-free abelian group and k a field. Then G is orderable, so the group algebra $k[G]$ is a domain. First, we want to show that all irreducibles of $k[G]$ or $k[G^{\leq 0}]$ are prime.

Proposition 3.1 *Let G be a torsion-free abelian group, k a field. Then every irreducible element of $k[G]$ is prime.*

Proof: Since G is orderable, $R := k[G]$ is a domain and the only invertible elements of R are of the form λt^g where $g \in G$, $0 \neq \lambda \in k$. Let I be the set of finitely generated subgroups of G, partially ordered by inclusion. For each $A \in I$, set $R_A = k[A]$. Then R is the direct limit of R_A. Fix an irreducible element $f \in R$. Suppose that $f \in R_A$ for some $A \in I$. Since $U(R_A) = U(R) \cap R_A$, f is irreducible in R_A. Pick a $\mathbb{Z}$-basis $\{g_1, \ldots, g_n\}$ of A. Then $R_A \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the ring of

Laurent polynomials, where $x_l = t^{g_l}$, $l = 1, \dots, n$, so R_A is a UFD. It follows that f is prime in R_A. Applying Lemma 2.8, we conclude that f is prime in R. □

Corollary 3.2 *Let G be an ordered abelian group, k a field. Then every irreducible element of $k[G^{\leq 0}]$ is prime.*

Proof: Let $f \in k[G^{\leq 0}]$ be irreducible. Then either f has a nonzero constant term or else $f = \lambda t^{g_0}$ where $0 \neq \lambda \in k$ and $g_0 = \max G^{<0}$. In the second case, $k[G^{\leq 0}]/(f) \cong k$, so f is prime. In the first case, using the fact that every element $0 \neq s \in k[G]$ can be written uniquely in the form $\tilde{s}t^g$ where $g \in G$ and $\tilde{s} \in k[G^{\leq 0}]$ with nonzero constant term, one checks that f is irreducible in $k[G]$. Then by Proposition 3.1, f is prime in $k[G]$. Applying Corollary 2.10, we conclude that f is prime in $k[G^{\leq 0}]$. □

Remark 3.3 Note, however, that $k[G^{\leq 0}]$ and $k[G]$ are rarely UFD's. For instance, if $g \in G^{<0}$ is divisible in G (that is, the equation $nx = g$ has a solution for every $n \in \mathbb{N}$), then obviously the (non-invertible) element t^g cannot have a factorization into primes in $k[G^{\leq 0}]$. As to the group ring $k[G]$, if $g \neq 0$ is divisible in G and k is real closed, then the (non-invertible) element $t^g + 1$ does not have a factorization into primes in $k[G]$. Indeed, assume $t^g + 1 = p_1 \cdots p_l$ is such a factorization. Let $H = \langle g \rangle_{\mathbb{Q}}$, the divisible hull of $\langle g \rangle$, and pick a complement H' for H in G. Then $k[G] = R[H']$ where $R = k[H]$. Since $t^g + 1 \in R$, all $p_i \in R$ (otherwise the product $p_1 \cdots p_l$ would contain a term of the form rt^h where $0 \neq r \in R$ and $0 \neq h \in H'$ and thus $p_1 \cdots p_l \notin R$). Since $U(R) = U(k[G]) \cap R$, p_i are irreducible(=prime) in R. But this is a contradiction by Proposition 3.5, below.

The following simple observation will be useful:

Lemma 3.4 *Let $H \subset G$ be a divisible subgroup and $f \in k[G]$ (resp., $f \in k[G^{\leq 0}]$). Assume that* support $f \subset H$. *Then f is prime in $k[G]$ (resp., in $k[G^{\leq 0}]$) iff f is prime in $k[H]$ (resp., in $k[H^{\leq 0}]$).*

Proof: Pick a complement H' for H in G. Let $R = k[H]/(f)$. Then $k[G]/(f) \cong R[H']$. Obviously, $R[H']$ is a domain iff R is a domain. This proves that f is prime in $k[G]$ iff it is prime in $k[H]$. The case of $k[G^{\leq 0}]$ and $k[H^{\leq 0}]$ follows from here by Corollary 2.10. Indeed, since H is divisible, $H^{<0}$ does not have a greatest element. Hence any $f \in k[H^{\leq 0}]$ that is prime in $k[H^{\leq 0}]$ or in $k[G^{\leq 0}]$ must have nonzero constant term. Then by Corollary 2.10, f is prime in $k[H^{\leq 0}]$, resp. $k[G^{\leq 0}]$, iff f is prime in $k[H]$, resp. $k[G]$. □

¿From now on, we assume that G is divisible, that is, a $\mathbb{Q}$-vector space.

Proposition 3.5 *Let G be a divisible ordered abelian group, k a real closed field. If $f \in k[G^{\leq 0}]$ is a non-constant prime, then* support f *does not lie in a 1-dimensional $\mathbb{Q}$-subspace of G.*

Proof: Let support $f = \{g_1, \ldots, g_n, 0\}$. Write $f = a_1 t^{g_1} + \cdots + a_n t^{g_n} + a_0$. Assume that $g_i = \frac{m_i}{n_i} g_1$ for $m_i, n_i \in \mathbb{N}$. Set $N = \operatorname{lcm}\{n_i\}$, $g = \frac{1}{3N} g_1$, and $x = t^g$. Rewriting f in terms of x, we obtain the polynomial $f(x) = a_1 x^{N_1} + \cdots + a_n x^{N_n} + a_0$ (where $N_i = 3m_i M_i$ and $N = M_i n_i$). Since $f(x) \in k[x]$ has degree ≥ 3, it factors in $k[x]$, which provides a factorization of f in $k[G^{\leq 0}]$. □

Remark 3.6 In [B; p.555] it is wrongly asserted that the support must be a $\mathbb{Q}$-linearly independent set. A counterexample is constructed as follows. Let $g_1, g_2 \in G^{<0}$ be $\mathbb{Q}$-linearly independent, then the element $f = t^{2g_1} + t^{g_1+g_2} + t^{g_1} + t^{g_2} + 1$ is prime in $k[G]$ by an argument similar to the proof of Proposition 3.8, below. Indeed, fix any $m \in \mathbb{N}$ and set set $x = t^{\frac{1}{m}g_1}$, $y = t^{\frac{1}{m}g_2}$, then rewriting f in terms of x and y, we obtain an irreducible polynomial $f(x, y) = x^{2m} + (y^m + 1)x^m + (y^m + 1)$. This implies that f is prime in $k[G]$ and, consequently, in $k[G^{\leq 0}]$.

Remark 3.7 Shepherdson's model mentioned at the beginning of the Introduction is the canonical integer part of $\mathbb{Q}(t^g \mid g \in \mathbb{Q})^r$. By Proposition 2.4, this is just $\mathbb{Q}^r[\mathbb{Q}^{<0}] \oplus \mathbb{Z}$. By Proposition 3.5, a non-constant element cannot be prime. This explains why Shepherdson's model has no non-standard primes.

Next, we want to construct prime elements in $k[G^{\leq 0}]$. We first establish the following result:

Proposition 3.8 *Let G be a divisible torsion-free abelian group (resp., divisible ordered abelian group), k a field of characteristic zero. Let f be an element of $k[G]$ (resp., $k[G^{\leq 0}]$). Assume that* support $f = \{g_1, \ldots, g_n, 0\}$ *where $g_1, \ldots, g_n$ are linearly independent over $\mathbb{Q}$ and $n \geq 2$. Then f is prime in $k[G]$ (resp., in $k[G^{\leq 0}]$).*

Proof: Write $f = a_1 t^{g_1} + \cdots + a_n t^{g_n} + a_0$ where $0 \neq a_i \in k$. Let $H = \langle g_1, \ldots, g_n \rangle_{\mathbb{Q}}$. First we show that f is prime in $k[H]$. For $m \in \mathbb{N}$, let $H_m = \langle \frac{1}{m} g_1, \ldots, \frac{1}{m} g_n \rangle_{\mathbb{Z}}$. Then $k[H]$ is the direct limit of $k[H_m]$. By Lemma 2.8, it suffices to show that f is prime in $k[H_m]$ for sufficiently large m.

Clearly, $k[H_m] \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where $x_i = t^{\frac{1}{m} g_i}$. Under this isomorphism f corresponds to the polynomial $f_m(x_1, \ldots, x_n) = a_1 x_1^m + \cdots + a_n x_n^m + a_0$. This polynomial is irreducible in $k[x_1, \ldots, x_n]$. One can show this e.g. by induction on $n \geq 2$ applying Eisenstein's criterion to $f_m(x_1, \ldots, x_n) = a_1 x_1^m + (a_2 x_2^m + \cdots + a_n x_n^m + a_0)$ viewed as an element of $R[x_1]$ where $R = \bar{k}[x_2, \ldots, x_n]$ and $\bar{k}$ is the algebraic closure of k. For $n = 2$, take $p = x_2 - \xi \in R$ where ξ is an m-th root of $-a_0/a_2$, then p satisfies the conditions of Eisenstein's criterion for f_m. For $n > 2$, take $p = a_2 x_2^m + \cdots + a_n x_n^m + a_0$, which is prime in R by induction hypothesis and satisfies the conditions of Eisenstein's criterion for f_m.

Now by Lemma 2.9, 1) (with $R := k$ and T is generated by $x_1, \ldots, x_n$), f_m is prime in $k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, so f is prime in $k[H_m]$ for any m. Thus we have proved that f is prime in $k[H]$. By Lemma 3.4, we conclude that f is prime in $k[G]$. If $f \in k[G^{\leq 0}]$, then by Corollary 2.10, f is also prime in $k[G^{\leq 0}]$. □

Example 3.9 Let G be a divisible ordered abelian group, k a real closed field. Consider $f = a_1 t^{g_1} + a_2 t^{g_2} + 1$ where $g_1,\, g_2 \in G^{<0}$, $0 \neq a_1, a_2 \in k$. Then f is prime in $k[G^{\leq 0}]$ iff g_1 and g_2 are linearly independent over $\mathbb{Q}$.

Proof: Follows from Propositions 3.5 and 3.8. □

Corollary 3.10 *Let G be a divisible ordered abelian group, k an ordered field. If $\dim_{\mathbb{Q}} G \geq 2$, then the ordered ring $k[G^{<0}] \oplus \mathbb{Z}$ has a cofinal set of primes.*

Proof: Let $s \in k[G^{<0}] \oplus \mathbb{Z}$ and let $g = \min(\text{support}\, s) \in G^{\leq 0}$. Then we can find $\mathbb{Q}$-linearly independent $g_1,\, g_2 \in G^{<0}$ such that $g_1 < g$. Let $f = t^{g_1} + t^{g_2} + 1$. Then $f > s$ and f is prime by Proposition 3.8. □

4 Primes and Irreducibles in $k((G^{\leq 0}))$.

Throughout this section, G **is a divisible ordered abelian group, with rank** Γ**,** k **a field of characteristic zero, and** F **is a truncation closed subfield of** $k((G))$ **that contains** $k(G)$. Note that it follows that the restriction of any series in F to any convex subset of G also belongs to F (because the restriction of the series to the convex subset can be written as the difference of two initial segments of the series). We want to investigate when a prime element of $k[G^{\leq 0}]$ remains prime or at least irreducible through the ring extensions

$$k[G^{\leq 0}] \subset \operatorname{Neg}(F) \oplus k \subset k((G^{\leq 0})).$$

First we consider the following example to see how things can go wrong.

Example 4.1 Suppose we have $g_1, g_2 \in G^{<0}$ such that $v(g_1) < v(g_2)$. Then g_1 and g_2 are linearly independent over $\mathbb{Q}$, so by Proposition 3.8, $f = t^{g_1} + t^{g_2} + 1$ is prime in $k[G^{\leq 0}]$. However, f can be factored in $k((G^{\leq 0}))$ as follows: $f = (t^{g_2}+1)(\frac{t^{g_1}}{t^{g_2}+1}+1) = (t^{g_2}+1)(t^{g_1-g_2}(1+t^{-g_2})^{-1}+1) = (t^{g_2}+1)(\sum_{n=1}^{\infty}(-1)^{n-1}t^{g_1-ng_2}+1)$. Thus f does not remain irreducible in $k((G^{\leq 0}))$.

Generalizing this trick, we can show that if f (non-constant) is irreducible in $\operatorname{Neg} F \oplus k$, then all nonzero elements of $\text{support}\, f$ must be archimedean equivalent, that is,

$$\Gamma_f := v(\text{support}\, f \setminus \{0\})$$

must be a singleton (this observation is due to Gonshor [G]). Indeed, let $\gamma = \min \Gamma_f$ and let $f_\gamma = \sum_{g\,:\,v(g)=\gamma} f_g t^g$. Since F is truncation closed, $f_\gamma \in \operatorname{Neg}(F)$. Let $\varepsilon = f - f_\gamma$. If Γ_f is not a singleton, then $\varepsilon \notin k$. Clearly, for all $g \in \text{support}\, \varepsilon$, $v(g) > \gamma$. It follows that $f = \varepsilon(\frac{f_\gamma}{\varepsilon} + 1)$ is a nontrivial factorization of f in $\operatorname{Neg}(F) \oplus k$.

This phenomenon suggests that the problems of irreducibility and primality in $\mathrm{Neg}\,(F) \oplus k$ should essentially reduce to the case when G is archimedean. Let f be an element of $\mathrm{Neg}\,(F) \oplus k$ such that $\Gamma_f = \{\gamma\}$. Fix a complement for D_γ in C_γ so that we can view the archimedean component A_γ as a subgroup of C_γ. Then $C_\gamma = A_\gamma \oplus D_\gamma$ (lexicographic sum). Consequently, $k((C_\gamma)) = k((D_\gamma))((A_\gamma))$ and

$$k((C_\gamma^{<0})) = k((D_\gamma))((A_\gamma^{<0})) \oplus k((D_\gamma^{<0})). \tag{3}$$

Set $k_\gamma = k((D_\gamma)) \cap F$ and $F_\gamma = k((C_\gamma)) \cap F$. Then $f \in F_\gamma$. Also from truncation closedness of F it follows that $F_\gamma = k_\gamma((A_\gamma)) \cap F$, since each coefficient that appears when we represent a series from F_γ as an element of $k((D_\gamma))((A_\gamma))$, is the restriction of the series to a convex subset of C_γ, divided by a monomial, and thus also belongs to F. So F_γ is a truncation closed subfield of $k_\gamma((A_\gamma))$ containing $k_\gamma(A_\gamma)$. Viewing the elements of k_γ as constants, define $\mathrm{Neg}_{k_\gamma}(F_\gamma) = F_\gamma \cap k_\gamma((A_\gamma^{<0}))$. Since we are changing the field of constants, we have to indicate by a subscript over which ground field each Neg is taken. In particular, what was previously denoted by $\mathrm{Neg}\,(F)$ now will be written as $\mathrm{Neg}_k(F)$.

Theorem 4.2 *Let f be an element of $\mathrm{Neg}_k(F) \oplus k$ with constant term $f_0 \in k$. Assume that $\Gamma_f = \{\gamma\}$. Then f is prime (resp., irreducible) in $\mathrm{Neg}_k(F) \oplus k$ iff the following two conditions hold: 1) f is prime (resp., irreducible) in $\mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma$ and 2) either $f_0 \neq 0$ or $\gamma = \max \Gamma$.*

Proof: First we observe that if condition 2) does not hold, that is, $f_0 = 0$ and there exists $g_0 \in G^{<0}$ such that $v(g_0) > \gamma$, then f can be factored in $\mathrm{Neg}_k(F) \oplus k$: $f = t^{g_0}(ft^{-g_0})$. So from now on we assume that condition 2) holds. The proof that f is prime (resp., irreducible) in $\mathrm{Neg}_k(F) \oplus k$ iff f is prime (resp., irreducible) in $\mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma$ will proceed in two steps.

Step 1. We claim that f is prime (resp., irreducible) in $\mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma$ iff f is prime (resp., irreducible) in $\mathrm{Neg}_k(F_\gamma) \oplus k$.
If $\gamma = \max \Gamma$, then $k_\gamma = k$ and the claim is trivial. So assume $f_0 \neq 0$. Replacing f by f/f_0, we can assume that f has constant term 1 as an element of $k((G))$. Since $\Gamma_f = \{\gamma\}$, f still has constant term 1 when viewed as an element of $k_\gamma((A_\gamma))$. By Lemma 2.7 with $G := A_\gamma$, $K := k_\gamma$, $\mathrm{Neg} := \mathrm{Neg}_{k_\gamma}(F_\gamma)$, and $R := \mathrm{Neg}_k(k_\gamma) \oplus k$, f is prime (resp., irreducible) in $\mathrm{Neg} \oplus K = \mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma$ iff f is prime (resp., irreducible) in $\mathrm{Neg} \oplus R = \mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus \mathrm{Neg}_k(k_\gamma) \oplus k$. It remains to observe that decomposition (3) implies that $\mathrm{Neg}_k(F_\gamma) = \mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus \mathrm{Neg}_k(k_\gamma)$.

Step 2. We claim that f is prime (resp., irreducible) in $\mathrm{Neg}_k(F_\gamma) \oplus k$ iff f is prime (resp., irreducible) in $\mathrm{Neg}_k(F) \oplus k$.
Let E_γ be a complement for C_γ in G. Then $G = E_\gamma \oplus C_\gamma$ (lexicographic sum). Consequently, $k((G)) = k((C_\gamma))((E_\gamma))$ and

$$k((G^{<0})) = k((C_\gamma))((E_\gamma^{<0})) \oplus k((C_\gamma^{<0})). \tag{4}$$

Since F is truncation closed, $F \subset F_\gamma((E_\gamma))$. It remains to apply Lemma 2.6 with $G := E_\gamma$, $K := F_\gamma$, $\mathrm{Neg} := \mathrm{Neg}_{F_\gamma}(F)$, and $R := \mathrm{Neg}_k(F_\gamma) \oplus k$ and observe that $\mathrm{Neg} \oplus R = \mathrm{Neg}_k(F) \oplus k$ since $\mathrm{Neg}_k(F) = \mathrm{Neg}_{F_\gamma}(F) \oplus \mathrm{Neg}_k(F_\gamma)$ by (4). $\square$

Taking $F = k((G))$, we obtain the following important

Corollary 4.3 *Let f be an element of $k((G^{\le 0}))$ with constant term f_0. Assume that $\Gamma_f = \{\gamma\}$. Then f is prime (resp., irreducible) in $k((G^{\le 0}))$ iff the following two conditions hold: 1) f is prime (resp., irreducible) in $k_\gamma((A_\gamma^{\le 0}))$ where $k_\gamma = k((D_\gamma))$ and 2) either $f_0 \neq 0$ or $\gamma = \max \Gamma$.*

As an application of Corollary 4.3, we prove the following generalizations of [B; Theorem 11.2]. First we introduce some notation. Let f be an arbitrary element of $\mathrm{Neg}\,(F) \oplus k$ and let $\gamma \in \Gamma_f$. As before, set $f_\gamma = \sum_{g\,:\,v(g)=\gamma} f_g t^g$. Then $f_\gamma \in F_\gamma = k((C_\gamma)) \cap F$. Set

$$S_f(\gamma) := \pi_\gamma(\mathrm{support}\, f_\gamma) \subset A_\gamma.$$

Once a complement for D_γ in C_γ is fixed, we can view F_γ as a subfield of $k_\gamma((A_\gamma))$, where $k_\gamma = k((D_\gamma)) \cap F$ as before. Then $S_f(\gamma)$ is just the support of f_γ viewed as an element of $k_\gamma((A_\gamma))$.

Corollary 4.4 *Assume that $\Gamma_f = \{\gamma\}$, $S_f(\gamma)$ is finite, and f is prime as an element of $k_\gamma[A_\gamma^{\le 0}]$. Then f is irreducible in $k((G^{\le 0}))$.*

Proof: Since A_γ is archimedean, f remains irreducible in $k_\gamma((A_\gamma^{\le 0}))$ by Berarducci's result [B; Theorem 11.2]. Now apply Corollary 4.3. $\square$

Corollary 4.5 *Let f be prime in $k[G^{\le 0}]$ and suppose that* support f *generates an archimedean subgroup of G. Then f remains irreducible in $k((G^{\le 0}))$ and, consequently, in* $\mathrm{Neg}\,(F) \oplus k$*, for any truncation closed F, $k(G) \subset F \subset k((G))$.*

Proof: Let H be the subgroup generated by support f. Since H is archimedean, all nonzero elements of H have the same valuation, say γ. Moreover, we can pick a complement A_γ for D_γ in C_γ in such a way that $H \subset A_\gamma$. Since f is prime in $k[G^{\le 0}]$, the constant term $f_0 \neq 0$. By Corollary 4.4, it suffices to show that f is prime in $k_\gamma[A_\gamma^{\le 0}]$. By the choice of A_γ, $f \in k[A_\gamma^{\le 0}]$. By Lemma 3.4, f is prime in $k[A_\gamma^{\le 0}]$, so $R := k[A_\gamma^{\le 0}]/(f)$ is a domain. It follows that $R((D_\gamma))$ is also a domain. But $R((D_\gamma)) = \left(k[A_\gamma^{\le 0}]/(f)\right)((D_\gamma)) = k[A_\gamma^{\le 0}]((D_\gamma))/(f) = k((D_\gamma))[A_\gamma^{\le 0}]/(f) = k_\gamma[A_\gamma^{\le 0}]/(f)$. We conclude that f is prime in $k_\gamma[A_\gamma^{\le 0}]$. $\square$

Example 4.6 Consider $f = a_1 t^{g_1} + a_2 t^{g_2} + 1$ where $g_1, g_2 \in G^{<0}$ are $\mathbb{Q}$-linearly independent and $0 \neq a_1, a_2 \in k$. As we know, f is prime in $k[G^{\le 0}]$. It remains irreducible in $k((G^{\le 0}))$ iff $v(g_1) = v(g_2)$ and g_1, g_2 are $\mathbb{Q}$-valuation independent.

Proof: If the conditions on g_1, g_2 are satisfied, then the subgroup $\langle g_1, g_2 \rangle$ is archimedean by Corollary 1.3, so f remains irreducible in $k((G^{\leq 0}))$ by Corollary 4.5. Now if $v(g_1) \neq v(g_2)$, then by the same argument as in Example 4.1, f does not remain irreducible in $k((G^{\leq 0}))$. If $v(g_1) = v(g_2) = \gamma$, but g_1, g_2 are not $\mathbb{Q}$-valuation independent, then there exist $m, n \in \mathbb{N}$ such that $v(g) > \gamma$ where $g = ng_1 - mg_2$. In this case, let $x = t^{\frac{1}{3m}g_1}$, then $f = a_1 x^{3m} + a_2' x^{3n} + 1$ where $a_2' = a_2 t^{-\frac{1}{m}g} \in k_\gamma = k((D_\gamma))$. Clearly, if k is real closed, this polynomial factors in $k_\gamma[x]$, so f does not remain irreducible in $k((G^{\leq 0}))$. □

This example shows that in Corollary 4.5 the condition that support f generates an archimedean subgroup cannot be weakened to say just that Γ_f is a singleton. On the other hand, consider $f' = t^{2g_1} + t^{g_1+g_2} + t^{g_1-g} + t^{g_2+g} + 1$ where $g_1, g_2 \in G^{<0}$ are $\mathbb{Q}$-valuation independent and $v(g_1) = v(g_2) = \gamma < v(g)$. If $g \neq 0$, then the subgroup generated by support f' is not archimedean, but f' is still irreducible in $k((G^{\leq 0}))$. Indeed, choose A_γ such that $g_1, g_2 \in A_\gamma$. Then f' is prime in $k_\gamma[A_\gamma^{\leq 0}]$: $f' = t^{2g_1} + \alpha^{-1}(\alpha t^{g_2} + 1)t^{g_1} + (\alpha t^{g_2} + 1)$ where $\alpha = t^g \in k_\gamma$, so by rewriting f' in terms of $x := t^{\frac{1}{m}g_1}$ and $y := t^{\frac{1}{m}g_2}$ ($m \in \mathbb{N}$), we obtain an irreducible polynomial $x^{2m} + \alpha^{-1}(\alpha y^m + 1)x^m + (\alpha y^m + 1)$ (cf. Remark 3.6). Therefore, by Corollary 4.4, f' is irreducible in $k((G^{\leq 0}))$.

We do not know in general in Corollary 4.5 if f actually remains prime in $\mathrm{Neg}\,(F) \oplus k$, but we can show this in the case when the field F is "small".

Theorem 4.7 *Assume that for any $f \in \mathrm{Neg}\,(F) \oplus k$ and any $\gamma \in \Gamma_f$, the set $S_f(\gamma)$ is finite. Then every irreducible element of $\mathrm{Neg}\,(F) \oplus k$ is prime.*

Proof: Let f be irreducible in $\mathrm{Neg}\,(F) \oplus k$. Then $\Gamma_f = \{\gamma\}$ for some $\gamma \in \Gamma$. By Theorem 4.2, f is irreducible in $\mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma$. By our assumption on F, $\mathrm{Neg}_{k_\gamma}(F_\gamma) \oplus k_\gamma = k_\gamma[A_\gamma^{\leq 0}]$. By Corollary 3.2, f is prime in $k_\gamma[A_\gamma^{\leq 0}]$. Hence by Theorem 4.2, f is prime in $\mathrm{Neg}\,(F) \oplus k$. □

One important case when the sets $S_f(\gamma)$ are finite is $F = k(G)^r$, the field of algebraic power series. (Recall that by [M–R], $k(G)^r$ is a truncation closed subfield of $k((G))$.) In the following corollaries we assume that k is a real closed field.

Corollary 4.8 *Let $F = k(G)^r$. Then every irreducible element of $\mathrm{Neg}\,(F) \oplus k$ is prime.*

Proof: By Theorem 4.7, it suffices to show that for any $f \in k((G^{\leq 0}))$ which is algebraic over $k(G)$ and any $\gamma \in \Gamma_f$, the set $S_f(\gamma)$ is finite. (As a matter of fact, the set Γ_f is also finite, but we do not need this property here.) By truncation closedness of F, we can assume without loss of generality that $f = f_\gamma$.

We claim that f is algebraic over $k(C_\gamma)$. Indeed, since f is algebraic over $k(G)$, there exists a nonzero polynomial $p(x) \in k[G][x]$ such that $p(f) = 0$. Choose a complement E_γ for C_γ in G. Then $k[G] = k[C_\gamma][E_\gamma]$ and so we can write $p(x) = \sum_{g \in E_\gamma} p_g(x) t^g$ (finite sum) where $p_g(x) \in k[C_\gamma][x]$. Since $p(x)$ is nonzero, there exists $g_0 \in E_\gamma$ such that $p_{g_0}(x)$ is nonzero. Since $p(f) = 0$ and the set $\{t^g \mid g \in E_\gamma\}$ is linearly independent over $k((C_\gamma))$, it follows that $p_{g_0}(f) = 0$, which proves the claim.

Now let $K = k((D_\gamma))$. Then $f \in K((A_\gamma^{\leq 0}))$. Since $k(C_\gamma) \subset K(A_\gamma)$, f is algebraic over $K(A_\gamma)$. But A_γ is archimedean, so by Proposition 2.4, the support of f, viewed as an element of $K((A_\gamma))$, is finite. As noted earlier, this support is precisely $S_f(\gamma)$. □

Example 4.9 Let $F = k(G)^r$. Let g_1, $g_2 \in G^{<0}$ be $\mathbb{Q}$-valuation independent and $v(g_1) = v(g_2)$. Consider $f = a_1 t^{g_1} + a_2 t^{g_2} + 1$ where $0 \neq a_1, a_2 \in k$. Then f is prime in $\mathrm{Neg}\,(F) \oplus k$.

Proof: Combine Corollaries 4.5 and 4.8. □

Corollary 4.10 *Assume that for every $\gamma \in \Gamma$ we have $\dim_{\mathbb{Q}} A_\gamma \geq 2$. Then for any truncation closed F, $k(G) \subset F \subset k((G))$, the ordered ring $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ has a cofinal set of irreducible elements with finite support.*

Proof: Let $s \in \mathrm{Neg}\,(F) \oplus \mathbb{Z}$ and let $g = \min(\mathrm{support}\, s) \in G^{\leq 0}$. Take $g_1 < g$ and set $\gamma = v(g_1)$. Choose A_γ such that $g_1 \in A_\gamma$. By hypothesis, we can find $g_2 \in A_\gamma$ linearly independent from g_1. Without loss of generality, $g_2 < 0$. Then $f = t^{g_1} + t^{g_2} + 1$ is irreducible in $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ by Corollary 4.5. Clearly, $f > s$. □

Corollary 4.11 *Assume that for every $\gamma \in \Gamma$ we have $\dim_{\mathbb{Q}} A_\gamma \geq 2$. Let $F = k(G)^r$. Then the ordered ring $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ has a cofinal set of primes with finite support.*

Proof: Combine Corollaries 4.10 and 4.8. □

As another application of Corollary 4.3, we obtain the following improvement of [P; Theorem 4.1] and [P; Theorem 4.2].

Theorem 4.12 *Let $s \in k((G^{<0}))$ be such that all elements of $\mathrm{support}\, s$ have the same valuation $\gamma \in \Gamma$. Assume also that the order type of the set $\pi_\gamma(\mathrm{support}\, s) \subset A_\gamma^{<0}$ is ω and the least upper bound is 0. Then $s + 1$ is prime in $k((G^{\leq 0}))$.*

Proof: By Corollary 4.3, it suffices to show that $f := s + 1$ is prime in $k_\gamma((A_\gamma^{\leq 0}))$ where $k_\gamma = k((D_\gamma))$. Since A_γ is archimedean and the set $S_f(\gamma) = \pi_\gamma(\mathrm{support}\, s)$ is precisely the support of s viewed as an element of $k_\gamma((A_\gamma))$, we can apply Pitteloud's result [P; Theorem 4.1] to conclude that f is indeed prime in $k_\gamma((A_\gamma^{\leq 0}))$. □

Corollary 4.13 *Let $s \in k((G^{<0}))$ be such that the support of s is of order type ω and is contained in an archimedean subgroup A of G. Assume that s is not divisible by any monomial t^a with $a \in A^{<0}$. Then $s + 1$ is prime in $k((G^{\leq 0}))$.*

Example 4.14 For any $g \in G^{<0}$, the series

$$p_g := \sum_{n=1}^{\infty} t^{\frac{1}{n}g} + 1$$

is prime in $k((G^{\leq 0}))$.

Example 4.15 Let $g, g' \in G^{<0}$ be such that $v(g') > v(g)$. Then the series

$$p_{g,g'} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} t^{\frac{1}{n}g + \frac{1}{m}g'} + 1$$

is prime in $k((G^{\leq 0}))$. Note that the support of $f := p_{g,g'} - 1$ has order type ω^2, but $S_f(\gamma)$, where $\gamma = v(g)$, has order type ω, so Theorem 4.12 applies.

Corollary 4.16 *The ordered ring $k((G^{<0})) \oplus \mathbb{Z}$ has a cofinal set of prime elements with infinite support.*

Proof: Let $s \in k((G^{<0})) \oplus \mathbb{Z}$ and let $g = \min(\text{support}\, s) \in G^{\leq 0}$. Pick $g' < g$, then $p_{g'}$ is prime by Example 4.14 and $p_{g'} > s$. □

5 Exponential Integer Parts of Exponential Fields

In this Section, we apply the results of the previous sections to study EIP's of exponential fields. Let $(K, +, \cdot, 0, 1, <)$ be a real closed field (RCF). We say that K is an **exponential field** if there exists an exponential on K, that is, a map exp such that

(EXP) $\exp : (K, +, 0, <) \simeq (K^{>0}, \cdot, 1, <)$ (that is, exp is an isomorphism of ordered groups).

We shall further consider exponentials that satisfy the **growth axiom scheme**:

(GA) for all $x \in K,\ x \geq n^2 \implies \exp(x) > x^n$ $\quad (n \geq 1)$,

Thus $(K, \exp)$ is a model for the fragment

$$T_{EXP} = RCF + EXP + GA$$

of Th$(\mathbb{R}, \exp)$:= the elementary theory of the ordered field of the reals with exponentiation.

A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential.

As before, we consider the residue field k embedded into K. By appropriately modifying exp (see [K; Lemma 1.18]), we can always assume that

$$\exp(\mathcal{O}_K) = \mathcal{U}_K^{>0} \quad \text{and} \quad \exp(\mathcal{M}_K) = 1 + \mathcal{M}_K \quad \text{and} \quad \exp(k) = k^{>0}\ .$$

Theorem 5.1 *Every truncation integer part of an exponential field has a cofinal set of irreducibles.*

Proof: We can apply Corollary 4.10 because the condition on G is always fulfilled when G is the value group of an exponential field (see [K; Proposition 1.22]). $\square$

An **exponential integer part** (EIP) of K is an integer part Z such that for any $z \in Z^{>0}$, we have $2^z \in Z^{>0}$ (where by definition $2^x := \exp(x \log 2)$ for all $x \in K$). The following easy proposition tells us when a truncation IP is actually an EIP.

Proposition 5.2 *Let F be a truncation closed subfield of $k((G))$ that contains $k(G)$. Assume that F admits an exponential* $\exp$ *such that* $\exp(\mathrm{Neg}\,(F)) \subset \mathrm{Mon}\,(F)$ *where* $\mathrm{Mon}\,(F) := \mathrm{Mon}\, k((G))$. *Then the canonical integer part $Z_F = \mathrm{Neg}\,(F) \oplus \mathbb{Z}$ is an exponential integer part.*

Proof: Let $z = a + z_0 \in Z_F^{\geq 0}$, $a \in \mathrm{Neg}\,(F)$, $z_0 \in \mathbb{Z}$. If $a = 0$ then $z_0 > 0$ and $2^{z_0} \in \mathbb{Z}$. If not, then $a > 0$ and $2^{a+z_0} = 2^{z_0} 2^a = 2^{z_0} \exp(a \log 2)$. Since $\log 2 \in k$ and $\mathrm{Neg}\,(F)$ is a k-vector space, $a \log 2 \in \mathrm{Neg}\,(F)$. So $\exp(a \log 2) = t^g$ for some $g \in G$. Since $a \log 2 > 0$, $t^g > 1$, so $g < 0$. Therefore, $2^{a+z_0} = 2^{z_0} t^g \in \mathrm{Neg}\,(F) \subset Z_F$ as required. $\square$

Theorem 5.3 *Every exponential field has an exponential integer part with a cofinal set of irreducibles.*

Proof: In [R] it is shown that given an exponential field K, there is a truncation closed embedding φ of K into $k((G))$ such that $\exp(\mathrm{Neg}\,(F)) \subset \mathrm{Mon}\,(F)$ where $F := \varphi(K)$, so $\mathrm{Neg}\,(F) \oplus \mathbb{Z}$ is an EIP of F. It follows that K has a truncation IP which is an EIP, thus the assertion follows by Theorem 5.1. $\square$

In the following example, we describe a procedure to construct countable non-archimedean models of T_{EXP} which are algebraic power series fields, and investigate their canonical IP's. We recall that a real closed field K is an ordered vector space over the residue field k, and all archimedean components of $(K, +, 0, <)$ have dimension 1 over k.

Example 5.4 : Countable Fields of Algebraic Power Series with Exponentiation.

Let E be a countable exponential subfield of $(\mathbb{R}, \exp)$ (for example, take E to be the smallest real closed, exp-closed and log-closed subfield of $\mathbb{R}$). Set $G = \oplus_{\mathbb{Q}} E$ the lexicographic sum over $\mathbb{Q}$ of copies of $(E, +, 0, <)$, that is,

$$G := \{\sum_{q \in \mathbb{Q}} g_q 1_q \mid g_q \in E, g_q = 0 \text{ for all but finitely many } q \in \mathbb{Q}\}$$

endowed with pointwise addition and lexicographic order (here, 1_q denotes the characteristic function on the singleton $\{q\}$). Set $\mathbf{E} = E(G)^r$, then $\mathbf{E}$ is countable and real closed. Further $\mathbf{E}$ is a truncation closed subfield of $E((G))$ [M–R].

We want an exponential exp (or equivalently, a logarithm log) on $\mathbf{E}$ such that $(\mathbf{E}, \exp)$ is a model of T_{EXP}, and such that the canonical IP $Z_{\mathbf{E}} = \mathrm{Neg}\,(\mathbf{E}) \oplus \mathbb{Z}$ is an EIP.

Observe first that

$$(\mathbf{E}, +, 0, <) = \mathrm{Neg}\,(\mathbf{E}) \oplus \mathcal{O}_{\mathbf{E}} \; . \tag{5}$$

$$(\mathbf{E}^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(\mathbf{E}) \times \mathcal{U}_{\mathbf{E}}^{>0}. \tag{6}$$

We shall define log on $\mathbf{E}^{>\mathbf{0}}$ as follows. We first define log on $\mathcal{U}_{\mathbf{E}}^{>0} = E^{>0} \times (1+\mathcal{M}_{\mathbf{E}})$. By [K; Theorem 2.22] there exists a $\mathbb{Q}$-valuation basis B of $\mathcal{M}_{\mathbf{E}}$ such that the set $\{1+b \,|\, b \in B\}$ is a $\mathbb{Q}$-valuation basis of $1+\mathcal{M}_{\mathbf{E}}$. Define $\log(1+b) := b$ and extend log to $1+\mathcal{M}_{\mathbf{E}}$ by linearity. One proves that log is an isomorphism of ordered groups with $\log(1+\mathcal{M}_{\mathbf{E}}) = \mathcal{M}_{\mathbf{E}}$ [K; Lemma 2.21]. For $u \in \mathcal{U}_{\mathbf{E}}^{>0}$, write $u = ry$ where $r \in E^{>0}$, $y \in 1+\mathcal{M}_{\mathbf{E}}$, and define $\log u := \log r + \log y$ (where $\log r \in \mathbb{R}$ is the natural logarithm). Using these definitions, it is easily seen that log is an isomorphism of ordered groups with $\log(\mathcal{U}_{\mathbf{E}}^{>0}) = \mathcal{O}_{\mathbf{E}}$.

Now we define log on $\mathrm{Mon}\,(\mathbf{E})$. By [K; Proposition 2.16], fix an isomorphism of chains $s : \mathbb{Q} \to G^{<0}$ such that for all $g \in G^{<0}$

$$s(v(g)) > g \; . \tag{7}$$

Note that $\{1_q \,|\, q \in \mathbb{Q}\}$ is a strongly independent E-basis of G. Now $\mathrm{Neg}\,(\mathbf{E})$ is a countable ordered E-vector space with 1-dimensional archimedean components, so it admits a strongly independent E-basis.

Fix such a basis $\{b_{s(q)} \,|\, q \in \mathbb{Q}\}$ with $V(b_{s(q)}) = s(q)$ and $b_{s(q)} < 0$ for all $q \in \mathbb{Q}$. Define $\log(t^{1_q}) := b_{s(q)}$ for all $q \in \mathbb{Q}$, and extend log to $\mathrm{Mon}\,(\mathbf{E})$ by linearity:

$$\log(t^g) = \sum_{q\in\mathbb{Q}} g_q b_{s(q)} \text{ for } g = \sum_{q\in\mathbb{Q}} g_q 1_q.$$

It is easily verified that log is an isomorphism of ordered groups with $\log(\mathrm{Mon}\,(\mathbf{E})) = \mathrm{Neg}\,(\mathbf{E})$.

We extend log to $\mathbf{E}^{>0}$ using the above definitions via the decomposition (6). The inequality (7) implies that $\exp := \log^{-1}$ satisfies (GA) [K; Theorem 2.5]. Thus $(\mathbf{E}, \exp)$ is a model of T_{EXP}. By Proposition 5.2, $Z_{\mathbf{E}} = \mathrm{Neg}\,(\mathbf{E}) \oplus \mathbb{Z}$ is an EIP. By Corollary 4.11, it has a cofinal set of primes (with finite support).

Example 5.5 : Exponential-Logarithmic Power Series Fields.

We describe the canonical integer part of the Exponential-Logarithmic Power Series Fields (EL-series fields for short). The EL-series fields are models of $\mathrm{Th}(\mathbb{R}, \exp)$ constructed in [K; Chapter 5 p. 79].

The EL-series field $\mathbb{R}((\Gamma))^{EL}$ is a countable increasing union of power series fields with exponents in properly chosen value groups G_n:

$$\mathbb{R}((\Gamma))^{EL} = \bigcup_{n\in\mathbb{N}} \mathbb{R}((G_n)).$$

The field $\mathbb{R}((\Gamma))^{EL}$ is a proper subfield of $\mathbb{R}((G_\omega))$ where

$$G_\omega := \bigcup_{n\in\mathbb{N}} G_n.$$

In fact, $\mathbb{R}((\Gamma))^{EL}$ is truncation closed and contains $\mathbb{R}(G_\omega)$ (so its value group is G_ω). Furthermore,

$$\mathrm{Neg}\,(\mathbb{R}((\Gamma))^{EL}) = \bigcup_{n\in\mathbb{N}} \mathrm{Neg}\,\mathbb{R}((G_n)).$$

Therefore, the truncation integer part of $\mathbb{R}((\Gamma))^{EL}$ is

$$Z_{EL} = \bigcup_{n\in\mathbb{N}} Z_n$$

where $Z_n := \mathbb{R}((G_n^{<0})) \oplus \mathbb{Z}$ is the canonical integer part of $\mathbb{R}((G_n))$. The construction of $\mathbb{R}((\Gamma))^{EL}$ guarantees that

$$\exp\left(\mathrm{Neg}\,\left(\mathbb{R}((\Gamma))^{EL}\right)\right) = \mathrm{Mon}\,\left(\mathbb{R}((\Gamma))^{EL}\right).$$

By Proposition 5.2 we have that Z_{EL} is indeed an EIP of $\mathbb{R}((\Gamma))^{EL}$. We now show that Z_{EL} has a cofinal set of primes (with infinite support):

Proposition 5.6 *The canonical exponential integer part Z_{EL} of $\mathbb{R}((\Gamma))^{EL}$ has a cofinal set of primes with infinite support.*

Proof: Let $g \in G_\omega^{<0}$. Then $g \in G_n$ for some $n \in \mathbb{N}$ and the series $p_g = \sum_{l=1}^{\infty} t^{\frac{g}{l}} + 1 \in \mathbb{R}((G_n))$ is prime in Z_m for all $m \geq n$ by Example 4.14. Therefore, by Lemma 2.8, p_g remains prime in $Z_{EL} = \bigcup_{n\in\mathbb{N}} Z_n$. The set $\{p_g \,|\, g \in G_\omega^{<0}\}$ is clearly cofinal in $\mathbb{R}((G_\omega))$ and thus in $\mathbb{R}((\Gamma))^{EL}$. □

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