O-MINIMAL – Let L be a first order language containing a binary relation symbol $<$ and M an L-**structure** in which $<$ is interpreted as a total order (cf. **order (on a set)**). Then M is called o-minimal if every parametrically definable subset of M is a finite union of intervals of M. An *interval* of M is a subset of the form $\{c \in M : a \Box_1 c \Box_2 b\}$ for some $a, b \in M \cup \{-\infty, \infty\}$, where $\Box_1, \Box_2$ stand for $<$ or $\leq$. For $n \geq 1$, a subset A of the cartesian product M^n is called *parametrically definable* if there is some L-formula $\phi(x_1, ..., x_n, y_1, ..., y_k)$ and $b_1, ..., b_k \in M$ such that $A = \{(a_1, ..., a_n) \in M^n : \phi(a_1, ..., a_n, b_1, ..., b_k)$ is true in $M\}$. An **elementary theory** is called o-minimal if every model of it is o-minimal.

This notion was introduced by L. v. d. Dries in [2], while studying the expansion $(\mathbf{R}, \exp)$ of the ordered field $\mathbf{R}$ of the real numbers by the **real exponential function**. V. d. Dries observed that the sets parametrically definable in cartesian products M^n for an o-minimal expansion M of $\mathbf{R}$ share many of the geometric properties of the **semi-algebraic sets**. For example, a semi-algebraic set has only finitely many connected components, each of them semi-algebraic (cf. [1]), and v. d. Dries showed that this result remains true if one replaces "semi-algebraic" by "parametrically definable in an o-minimal expansion of $\mathbf{R}$". This is a *finiteness theorem*, and v. d. Dries sets it as a goal to explain the other finiteness phenomena in real algebraic and real analytic geometry as consequences of o-minimality (cf. [3]).

In [6], J. F. Knight, A. Pillay and C. Steinhorn prove the following results. o-minimality is preserved under elementary equivalence. An **ordered group** is o-minimal if and only if it is divisible Abelian. An **ordered ring** is o-minimal if and only if it is a **real closed field**. Any parametrically definable unary function in an o-minimal structure is piecewise strictly monotone or constant, and continuous.

The real closed field $\mathbf{R}$ is o-minimal. The expansion of $\mathbf{R}$ by restricted analytic functions (cf. **Model theory of the real exponential function**) is o-minimal (cf. [4]), as a consequence of Gabrielov's theorem that the complement of a subanalytic set is subanalytic (cf. [5]). It follows from work of A. Wilkie ([7]) that $(\mathbf{R}, \exp)$ is o-minimal. A. Macintyre, v. d. Dries and D. Marker establish in [4] the o-minimality of $\mathbf{R}$ expanded by the restricted analytic functions and the exponential function. For a research account on o-minimal structures, see [3].

References

[1] G. E. COLLINS: *Quantifier Elimination for Real closed Fields by Cylindrical Algebraic Decomposition, Automata Theory and Formal Language*, 2nd G. I. Conference Kaiserslautern, Springer, Berlin (1975), 134-183.

[2] L. VAN DEN DRIES: *Remarks on Tarski's problem concerning* $(\mathbf{R}, +, \cdot, \exp)$, in: Logic Colloquium '82, eds. G. Lolli, G. Longo and A. Marcja, North-Holland (1984), 97–121.

[3] L. VAN DEN DRIES: *o-minimal structures*, in: Logic: From Foundations to Applications, European Logic Colloquium. Eds. W. Hodges, M. Hyland, C. Steinhorn and J. Truss, Oxford (1996), 137-185.

[4] L. VAN DEN DRIES, A. J. MACINTYRE and D. MARKER: *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. **140** (1994), 183-205.

[5] A. GABRIELOV: *Projections of semi-analytic sets*, Functional Analysis and its Applications **2** (1968) 282-291.

[6] J. F. KNIGHT, A. PILLAY and C. STEINHORN: *Definable Sets in Ordered Structures I, II*, Trans. Amer. Math. Soc. **295** (1986), 565-605.

[7] A. J. WILKIE: *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc.

S. Kuhlmann

AMS 1995 Subject Classification: 03C45, 03C60, 06F99

MODEL THEORY OF THE REAL EXPONENTIAL FUNCTION – A branch of **model theory** studying the **elementary theory** $\mathrm{Th}(\mathbf{R}, \exp)$ of the ordered field $\mathbf{R}$ of the real numbers with the **real exponential function** exp. It is motivated by Tarski's question ([7], p. 45) whether $\mathrm{Th}(\mathbf{R}, \exp)$ is decidable.

A. Wilkie showed in [8] that $\mathrm{Th}(\mathbf{R}, \exp)$ is model complete. Combining this with Hovanskii's finiteness theorem ([5]), it follows that this theory is **o-minimal**. In fact, Wilkie first studies expansions (cf. **Structure**) of $\mathbf{R}$ by a Pfaffian chain of functions (see also [2]): Fix $m, l \in \mathbf{N}$ and an open set $U \subset \mathbf{R}^m$ containing the closed unit box $[0, 1]^m$. A *Pfaffian chain* of functions on U is a sequence $G_1,...,G_l : U \to \mathbf{R}$ of analytic functions for which there exist polynomials $P_{i,j} \in \mathbf{R}[z_1, ..., z_{m+i}]$ (for $i = 1, ..., l$, $j = 1, ..., m$) such that

$$\frac{\partial G_i}{\partial x_j}(x) = P_{i,j}(x, G_1(x), ..., G_i(x))$$

for all $x \in U$. Wilkie shows that the expansion of $\mathbf{R}$ by a Pfaffian chain of functions restricted to the closed unit box has a model complete theory. In particular, the expansion $(\mathbf{R}, \exp|_{[0,1]})$ of $\mathbf{R}$ by the restricted exponential function has a model complete theory. Wilkie then deduces the model completeness of $\mathrm{Th}(\mathbf{R}, \exp)$ from this last result. An alternative proof of the model completeness, and an axiomatization of $\mathrm{Th}(\mathbf{R}, \exp)$ over $\mathrm{Th}(\mathbf{R}, \exp|_{[0,1]})$ was found by J. P. Ressayre 1991 (see [3] for a generalization of Ressayre's result).

In [6], A. Macintyre and Wilkie show that $\mathrm{Th}(\mathbf{R}, \exp)$ is decidable provided that the real version of Schanuel's conjecture (cf. **Algebraic independence**) is true.

The theory $\mathrm{Th}(\mathbf{R}, \exp)$ does not admit **elimination of quantifiers**. In fact, an expansion of $\mathbf{R}$ by a family of total real analytic functions (see [1]) admits elimination of quantifiers if and only if each function is *semi-algebraic*, i.e., has a semi-algebraic graph (cf. **semi-algebraic set**). However, let An denote the family of *restricted real analytic functions*, i.e., functions $f : \mathbf{R}^n \to \mathbf{R}$ for all $n \in \mathbf{N}$ which are given on $[-1, 1]^n$ by a power series converging on a neighborhood of $[-1, 1]^n$ and are set to 0 outside of $[-1, 1]^n$. It is shown in [3] that the expansion $(\mathbf{R}, \exp, \log, f)_{f \in \mathrm{An}}$ admits elimination of quantifiers. The authors also give a complete axiomatization of $\mathrm{Th}(\mathbf{R}, \exp, f)_{f \in \mathrm{An}}$, and establish that it is o-minimal. In [4] they construct a model of this theory which is not Archimedean (cf. **Archimedean ring**) and use it to solve a problem raised by G. H. Hardy: they show that the compositional inverse of the function $(\log x)(\log \log x)$ is not asymptotic at $+\infty$ to a composition of semi-algebraic functions, log and exp.

References

[1] L. van den Dries: *Remarks on Tarski's problem concerning* $(\mathbf{R}, +, \cdot, \exp)$, Logic Colloquium '82, eds. G. Lolli, G. Longo and A. Marcja, North-Holland (1984), 97–121.

[2] L. van den Dries: *Tarski's problem and Pfaffian Functions*, Logic Colloquium '84, eds. J. B. Paris, A. J. Wilkie and G. M. Wilmers, North-Holland (1986), 59–90.

[3] L. van den Dries, A. J. Macintyre and D. Marker: *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. **140** (1994), 183-205.

[4] L. van den Dries, A. J. Macintyre and D. Marker: *Logarithmic-exponential power series*, J. London Math. Soc.

[5] A. Hovanskii: *On a class of systems of transcendental equations*, Soviet Math. Dokl. **22** (1980) 762-765.

[6] A. J. Macintyre and A. J. Wilkie: *On the decidability of the real exponential field*, Kreisel 70th Birthday Volume, ed. P. G. Odifreddi, CLSI (1995).

[7] Tarski, A. and McKinsey, J. C. C.: *A decision method for elementary algebra and geometry*, Univ. California Press, Berkeley (1951).

[8] A. J. Wilkie: *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc.

S. Kuhlmann

AMS 1995 Subject Classification: 03C60, 12J10, 12J15