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κ-bounded exponential groups and exp-log series fields without log-atomic elements.

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• **(1902-1912):** F. Hausdorff on foundations of *set theory*, and independently H. Hahn, develop universal constructions of totally ordered abelian groups and fields.
• **(1910):** G. H. Hardy monograph on *asymptotic scales for differentiable* real valued functions.
• **(1927)** E. Artin and O. Schreier develop the algebraic theory of *real closed fields*.
• **(1930)** A. Tarski establishes *quantifier elimination for and decidabilty* of T:= $\mathrm{Th}(\mathbb{R}, +, \cdot, <)$.
• **(1932)** W. Krull lays foundations for *valuation theory.*
•**(1954)** A. Seidenberg gives a *geometric interpretation for semi-algebraic sets* of Tarski's elimination result.

Today the celebrated *Tarski - transfer principle* is the foundation of modern *semi-algebraic geometry*, the Artin-Schreier theory that of *real algebra*, the work of Hausdorff-Hahn-Hardy that of *ordered algebraic structures and asymptotic analysis.* Combined with the work of Krull, this provides universal *power series constructions of non-Archimedean real closed fields*, i.e. of non-Archimedean models of T.

In his monograph Tarski asks for *analogues results* for $\mathrm{T}_{\exp}$=: $\mathrm{Th}(\mathbb{R}, +, \cdot, <, \exp)$, the elementary theory of the real exponential field.

This led L. van den Dries in the **1980-1990s** to develop *o-minimal geometry*, A. Wilkie in the **1990s** to prove the model completeness and o-minimality of $\mathrm{T}_{\exp}$, and together with A. Macintyre its decidabilty modulo the *real Schanuel conjecture*.

Our contributions since the 1990s provided universal power series constructions of non-Archimedean *exponential real closed fields*, i.e. of non-Archimedean models of $\mathrm{T}_{\exp}$. Inspired by the work of Hardy, we also studied differential operators on these models. The role of the *log-atomic monomials* in these constructions was recently revealed to us in connection to J. Conway's *field of surreal numbers*. This the main story I want to tell today.

PART I: Power Series Constructions of Real Closed Fields.

Construction of a Hahn Group:

• Let Γ any totally ordered set and $\mathbb{R}^\Gamma$ the set of all maps g from Γ to $\mathbb{R}$ such that the **support** $\{\gamma \in \Gamma \mid g(\gamma) \neq 0\}$ of g is well-ordered in Γ.

• Endowed with pointwise addition and the lexicographic order, $\mathbb{R}^\Gamma$ is a divisible ordered abelian group **(DOAG)**, called the **Hahn group of rank** Γ.

• $\mathbb{R}^\Gamma$ is archimedean iff Γ is a singleton.

• **Hahn's Embedding Theorem:** a DOAG group of rank Γ is (isomorphic to) a subgroup of $\mathbb{R}^\Gamma$.

Representation for the elements of Hahn groups:

• For every $\gamma \in \Gamma$, we denote by $\mathbf{1}_\gamma$ the map which sends γ to $\mathbf{1}$ and every other element to 0

• $\mathbf{1}_\gamma$ is the characteristic function of the singleton $\{\gamma\}$.

• For $g \in \mathbb{R}^\Gamma$ write

$$g = \sum_{\gamma \in \Gamma} g_\gamma \mathbf{1}_\gamma$$

with $g_\gamma := g(\gamma) \in \mathbb{R}$ and well-ordered support in Γ.

Construction of a Hahn field:

• For $G \neq 0$ an ordered abelian group, $\mathbb{R}((G))$ is the field of generalized **power series** with coefficients in $\mathbb{R}$ and exponents in G:

• A series $s \in \mathbb{R}((G))$ is written

$$s = \sum_{g \in G} s_g t^g$$

with $s_g \in \mathbb{R}$ and well-ordered support $\{g \in G \mid s_g \neq 0\}$.

• As an ordered abelian group, this is just the Hahn group $\mathbb{R}^G$, endowed with convolution (Cauchy) product.

• (Tarski' s recursive axiomatisation:) A totally ordered field K is said to be **real closed** if every positive element has a square root in K, and every polynomial in $K[x]$ of odd degree has a root in K.

• (W. Krull / S. MacLane/ ...) $\mathbb{R}((G))$ is real closed iff G is divisible.

• Let $(K, +, \cdot, 0, 1, <)$ be an ordered field, the **natural valuation** v on K is the valuation whose valuation ring the convex hull of $\mathbb{Z}$ in K.

Notation:

Value group: $G = v(K) := \{v(x) \mid x \in K, x \neq 0\}$.
Valuation ring:, $R_v := \{x \mid x \in K \text{ and } v(x) \geq 0\}$.
Valuation ideal: $I_v := \{x \mid x \in K \text{ and } v(x) > 0\}$.
Group of positive units:
$U_v^{>0} := \{x \mid x \in R_v, x > 0, v(x) = 0\}$.
Residue field: $k = \overline{K} := R_v/I_v$.

Kaplansky Embedding's Theorem (1942): Let K be real closed field with residue field k and value group G. Then K is (isomorphic to) a subfield of a field of $k((G))$.

Direct sum (respectively product) lexicographic decompositions: Let K be real closed field with value group G, then:

$(K, +, <) = \mathbb{A} \oplus R_v$

$(K^{>0}, \cdot, <) = \mathbb{B} \times U_v^{>0}$

The divisible ordered abelian groups $\mathbb{A}$ (additive) and $\mathbb{B}$ (multiplicative) are unique up to isomorphism. More precisely $\mathbb{B} \simeq G$. The rank of $\mathbb{A}$ is (isomorphic to) $G^{<0}$ as linearly ordered sets.

Example: Let G be a DOAG and set $\mathbb{K} := \mathbb{R}((G))$.

• The natural valuation on $\mathbb{R}((G))$ is $v(s) = \min \operatorname{support} s$ for $s \in \mathbb{R}((G))$, the value group is G.

• The valuation ring $\mathbb{R}((G^{\geq 0}))$ consists of the series with non-negative exponents, the valuation ideal $\mathbb{R}((G^{>0}))$ of the series with positive exponents. The residue field is $\mathbb{R}$

• The **constant term** of a series s is the coefficient s_0. The units of $\mathbb{R}((G^{\geq 0}))$ are the series in $\mathbb{R}((G^{\geq 0}))$ with a non-zero constant term.

- **Additive Decomposition:**

Given $s \in \mathbb{R}((G))$, we truncate it at its constant term and re-write it as the sum of two series, one with strictly negative exponents, and the other with non-negative exponents. Thus an additive lexicographic complement $\mathbb{A}$ in $(\mathbb{K}, +, <)$ to the valuation ring is the Hahn group $\mathbb{R}^{G^{<0}}$. We call it the **canonical complement to the valuation ring** and denote it by $\mathbb{R}((G^{<0}))$.

• **Multiplicative Decomposition:**

Given $s \in \mathbb{R}((G))^{>0}$, we factor out the monomial of smallest exponent $g \in G$ and re-write $s = t^g u$ as a product, where u a unit with a positive constant term. Thus a multiplicative lexicographic complement $\mathbb{B}$ in $(\mathbb{K}^{>0}, \cdot, <)$ to the subgroup $U_v^{>0}$ of positive units is the group consisting of the (monic) monomials t^g. We call it the **canonical complement to the positive units** and denote it by **Mon** $\mathbb{K}$. Note that, $\operatorname{Mon}\mathbb{K}$ is order isomorphic to G through the isomorphism $t^g \mapsto -g$.

PART II: Power Series Constructions of Real Closed Exponential Fields.

A real closed field K is an **exponential field** if there exists a map

$$\exp : (K, +, 0, <) \longrightarrow (K^{>0}, \cdot, 1, <)$$

which is an isomorphism of ordered groups. A map exp with these properties will be called an **exponential** on K. A **logarithm** on K is the compositional inverse $\log = \exp^{-1}$ of an exponential. We always require w.l.o.g. that the exponential be v**-compatible**:

$$\exp(R_v) = U_v^{>0}.$$

We are only interested in exponentials (logarithms) satisfying the **growth axiom** scheme: **(GA)**:

$$\forall n \in \mathbb{N}: \; x > \log(x^n) = n\log(x) \text{ for all } x \in K^{>0} \setminus R_v \;.$$

Via the natural valuation v, this is equivalent to

$$v(x) < v(\log(x)) \text{ for all } x \in K^{>0} \setminus R_v \;. \tag{1}$$

A logarithm log is a **(GA)-logarithm** if it satisfies (1).

Left Exponentiation is imposed on us:

Let $(K, +, \cdot, \exp)$ be a real closed field endowed with a v-compatible exponential exp. We see that exp must restrict to a left exponential, i.e. an isomorphism $\exp : \mathbb{A} \to \mathbb{B}$.

This observation has two mighty consequences:

Theorem 1 [KKS]: Let $G \neq 0$ be a DOAG and set $\mathbb{R}((G))$. Then $\mathbb{R}((G^{<0}))$ is *never* isomorphic to G, so $\mathbb{R}((G))$ cannot admit a v-compatible exponential.

Theorem 2 [S.K.]: Let $(K, +, \cdot, \exp)$ be a real closed field endowed with a v-compatible exponential exp. Let G be its value group and Γ its rank. Then Γ must be isomorphic as linearly ordered set to $G^{<0}$.

Theorem 2 follows immediately from the decomposition theorems.

Call a DOAG G an **exponential group** if its rank Γ is isomorphic as linearly ordered set to $G^{<0}$.

Theorems 1 and 2 are two major obstacles to construction of exponential fields via power series: 1. We cannot use the full field $\mathbb{R}((G))$*, we must try to use an appropriate subfield instead 2. We cannot use our favorite DOAG as value group, instead we have to first learn to construct exponential groups.*

Characterization of countable exponential groups [S.K]:

A countable DOAG $G \neq 0$ is an exponential group if and only if G is isomorphic to the Hahn sum $\oplus_{\mathbb{Q}} C$ for some countable archimedean group $C \neq 0$.

For the uncountable case we must work much harder.

κ**–bounded Hahn Groups and Fields.** Fix a regular uncountable cardinal κ.

• The κ**-bounded Hahn group** $(\mathbb{R}^\Gamma)_\kappa \subseteq \mathbb{R}^\Gamma$ consists of all maps of which support has cardinality $< \kappa$.

• The κ**-bounded power series field** $\mathbb{R}((G))_\kappa \subseteq \mathbb{R}((G))$ consists of all series of which support has cardinality $< \kappa$. It is a valued subfield of $\mathbb{R}((G))$. We denote by $\mathbb{R}((G^{\geq 0}))_\kappa$ its valuation ring. Note that $\mathbb{R}((G))_\kappa$ contains the monic monomials.

We denote by $\mathbb{R}((G^{<0}))_\kappa$ the complement to $\mathbb{R}((G^{\geq 0}))_\kappa$.

Proposition 0.1 *Set* $K = \mathbb{R}((G))_\kappa$. *Then* $(K, +, 0, <)$ *decomposes lexicographically as the sum:*

$$(K, +, 0, <) = \mathbb{R}((G^{<0}))_\kappa \oplus \mathbb{R}((G^{\geq 0}))_\kappa \ . \qquad (2)$$

$(K^{>0}, \cdot, 1, <)$ *decomposes lexicographically as the product:*

$$(K^{>0}, \cdot, 1, <) = \mathrm{Mon}\,(K) \times U_v^{>0} \qquad (3)$$

• **Our goal** is to define an exponential (logarithm) on $\mathbb{R}((G))_\kappa$ (for appropriate choice of G).

Proposition 0.1 allows us to achieve our goal in two main steps; by defining the logarithm on $\mathrm{Mon}\,(K)$ and on $U_v^{>0}$.

Theorem 0.2 *Let* Γ *be a chain, set* $G := (\mathbb{R}^\Gamma)_\kappa$ *and* $K := \mathbb{R}((G))_\kappa$*. Assume that*

$$l : \Gamma \to G^{<0}$$

is an isomorphism of chains. Then l *uniquely defines a logarithm*

$$log \ : (K^{>0}, \cdot, 1, <) \longrightarrow (K, +, 0, <)$$

as follows: given $a \in K^{>0}$*, write* $a = t^g r(1+\varepsilon)$*,* $g = \sum_{\gamma\in\Gamma} g_\gamma \mathbf{1}_\gamma$*,* $r \in \mathbb{R}^{>0}$*,* ε *infinitesimal, and set*

$$\log(a) := -\sum_{\gamma\in\Gamma} g_\gamma t^{l(\gamma)} + \log r + \sum_{i=1}^{\infty} (-1)^{(i-1)} \frac{\varepsilon^i}{i} \qquad (4)$$

This logarithm satisfies

$$v(\log t^g) = l(\min \operatorname{support} g) \tag{5}$$

Moreover, log *satisfies* **GA** *and is a model of* $T_{\exp}$ *if and only if*

$$l(\min \operatorname{support} g) > g \qquad \textit{for all } g \in G^{<0}\,. \tag{6}$$

Thus the theorem states that the general necessary condition that the value group be an exponential group is also sufficient in this particular context. Therefore our next homework is to construct κ*-bounded exponential groups.*

The κ-th closure $(\Gamma_\kappa, \iota_\kappa, G_\kappa)$ of a triplet (Γ_0, ι_0, G_0):

• **Input:** Γ_0 any non-empty chain, $G_0 = (\mathbb{R}^{\Gamma_0})_\kappa$, and ι_0 a **group section**, that is

$$\iota_0 : \Gamma_0 \to G_0^{<0}$$

is an embedding of linearly ordered sets satisfying

$$\min \operatorname{support}\left(\iota_0(\gamma)\right) = \gamma \text{ for all } \gamma \in \Gamma_0 \, .$$

• **Output:** A chain Γ_κ, the corresponding κ-bounded group $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$, an *isomorphism* of linearly ordered sets

$$\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

making G_κ the requested κ-bounded exponential group.

• We shall construct by transfinite induction on $\mu \leq \kappa$ a chain Γ_μ together with an embedding of ordered chains

$$\iota_\mu : \Gamma_\mu \to G_\mu^{<0}$$

where $G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa$. We shall have $\Gamma_\nu \subset \Gamma_\mu$ and $\iota_\nu \subset \iota_\mu$ if $\nu < \mu$.

• Assume that for all $\alpha < \mu$ we have already constructed Γ_α, $G_\alpha := (\mathbb{R}^{\Gamma_\alpha})_\kappa$, and the embedding

$$\iota_\alpha : \Gamma_\alpha \to G_\alpha^{<0} \ .$$

First assume that $\mu = \alpha + 1$ is a successor ordinal. Since Γ_α is isomorphic to a subchain of $G_\alpha^{<0}$ through ι_α, we can take $\Gamma_{\alpha+1}$ to be a chain containing Γ_α as a subchain and admitting an isomorphism $\iota_{\alpha+1}$ onto $G_\alpha^{<0}$ which extends ι_α .

More precisely,

$$\Gamma_{\alpha+1} := \Gamma_\alpha \cup (G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha)) ,$$

endowed with the **patch ordering**: if $\gamma_1, \gamma_2 \in \Gamma_{\alpha+1}$ both belong to Γ_α, compare them there, similarly if they both belong to $G_\alpha^{<0}$. If $\gamma_1 \in \Gamma_\alpha$ but $\gamma_2 \in G_\alpha^{<0}$ we set $\gamma_1 < \gamma_2$ if and only if $\iota_\alpha(\gamma_1) < \gamma_2$ in G_α. Then $\iota_{\alpha+1}$ is defined in the obvious way: $\iota_{\alpha+1}|_{\Gamma_\alpha} := \iota_\alpha$ and $\iota_{\alpha+1}|_{(G_\alpha^{<0} \setminus \iota_\alpha(\Gamma_\alpha))} :=$ the identity map. Note that

$$\iota_{\alpha+1}(\Gamma_{\alpha+1}) = G_\alpha^{<0}. \tag{7}$$

Thus $\iota_{\alpha+1}$ is an embedding of $\Gamma_{\alpha+1}$ into $G_{\alpha+1}^{<0}$.

If μ is a limit ordinal we set

$$\Gamma_\mu := \bigcup_{\alpha<\mu} \Gamma_\alpha \ , \ \ \iota_\mu := \bigcup_{\alpha<\mu} \iota_\alpha \quad \text{and} \quad G_\mu := (\mathbb{R}^{\Gamma_\mu})_\kappa \ .$$

Note that by construction and (7)

$$\iota_\mu(\Gamma_\mu) = \bigcup_{\alpha<\mu} G_\alpha^{<0} \tag{8}$$

and $\bigcup_{\alpha<\mu} G_\alpha \subset G_\mu$.
This completes the construction of $\Gamma_\kappa := \bigcup_{\alpha<\kappa} \Gamma_\alpha$, $\iota_\kappa := \bigcup_{\alpha<\kappa} \iota_\alpha$ and $G_\kappa := (\mathbb{R}^{\Gamma_\kappa})_\kappa$.

We now claim that

$$G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$$

Once the claim is established, we conclude from (8) that $\iota_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$ is an isomorphism, as required. Let $g \in G_\kappa$ and $\kappa > \delta :=$ card(support g). Now support $g := \{\gamma_\mu ; \mu < \delta\} \subset \Gamma_\kappa$, so for every $\mu < \delta$ choose $\alpha_\mu < \kappa$ such that $\gamma_\mu \in \Gamma_{\alpha_\mu}$. Clearly card$(\{\alpha_\mu ; \mu < \delta\}) \leq \delta < \kappa$ so $\{\alpha_\mu ; \mu < \delta\}$ cannot be cofinal in κ since κ is **regular**, therefore it is bounded above by some $\alpha \in \kappa$. It follows that support $g \subset \Gamma_\alpha$, so $g \in G_\alpha$ as required.

Finally we take care of **(GA)** by exploiting **right shifts** of Γ_0 to improve the group exponential ι_κ:

Proposition 0.3 *Assume that* $\sigma_\kappa \in \mathrm{Aut}\,(\Gamma_\kappa)$ *is such that* $\sigma_\kappa|_{\Gamma_\mu} \in \mathrm{Aut}\,(\Gamma_\mu)$ *for all* $\mu \in \kappa$ *and* $\sigma_\kappa(\gamma) > \gamma$ *for all* $\gamma \in \Gamma_0$. *Then the isomorphism*

$$l := \iota_\kappa \circ \sigma_\kappa : \Gamma_\kappa \to G_\kappa^{<0}$$

satisfies (6).

We are now ready to summarize the procedure of constructing the **Exponential-Logarithmic field of κ-bounded series over** $(\Gamma_0, \iota_0, G_0, \sigma_0)$ where σ_0 a right shift automorphism of Γ_0.

• Construct $\Gamma_\kappa, \quad G_\kappa, \quad \iota_\kappa, \quad$ and σ_κ.

• Set $K := \mathbb{R}((G_\kappa))_\kappa$ and $l := \iota_\kappa \circ \sigma_\kappa$. Note that l is surjective and satisfies (6).

• Denote by log the surjective **GA** logarithm induced on $K^{>0}$ by l and set $\exp = \log^{-1}$.

• $(K, \exp)$ is a model of $T_{\mathrm{an,exp}}$.

Group sections and log-atomic monomials.

Set K:= $\mathrm{EL}(\kappa, \Gamma_0, \iota_0, G_0, \sigma_0)$.

- A monomial t^g is **log-atomic** if $\log_n(t^g)$ is a monomial for all $n \in \mathbb{N}$.
- Log-atomic elements are fundamental when defining transserial derivations on K.
- Group sections always exist and can vary widely.
- It turns out that the set of log-atomic monomials depends on the choice of the group section.

For the computations below recall that

$$\log(t^g) = \sum_{\gamma\in\Gamma} -g_\gamma t^{l(\gamma)}$$

for $g = \sum_{\gamma\in\Gamma} g_\gamma 1_\gamma$.

• Below denote by $\sigma^n(\gamma)$ the **n-th iterate** of $\sigma := \sigma_0$.

The basic section: is defined by $\gamma \mapsto -\mathbf{1}_\gamma$. The corresponding $l : \Gamma \to G_0^{<0}$ is defined by $\gamma \mapsto -\mathbf{1}_{\sigma(\gamma)}$. By induction

$$\log{}_n(t^{-\mathbf{1}_\gamma}) = t^{-\mathbf{1}_{\sigma^n(\gamma)}}\ .$$

So $t^{-\mathbf{1}_\gamma}$ is log-atomic for all $\gamma \in \Gamma_0$.

The 2-element support section:

• Since Γ_0 has no last element, for every $\gamma \in \Gamma_0$ let $\tau(\gamma) > \gamma$. Consider the section $\gamma \mapsto -\mathbf{1}_\gamma + \mathbf{1}_{\tau(\gamma)}$.

• We claim that the corresponding log has no log-atomic elements.

• We prove this for t^g with $g \in G_\kappa = \bigcup_{\alpha<\kappa} G_\alpha$, by transfinite induction on α.

• We will need the following simple observation:

A necessary condition for log-atomic: Assume that t^g is log-atomic and set $\log_n(t^g) := t^{g_n}$. Then g_n must have singleton support for all $n \in \mathbb{N}_0$.

• $\alpha = 0$: Let $g \in G_0$ have singleton support, say $g = -\mathbf{1}_\gamma$. Compute $\log(t^g) = t^{l(\gamma)}$. But $l(\gamma) = -\mathbf{1}_{\sigma(\gamma)} + \mathbf{1}_{\tau(\sigma(\gamma))}$. So t^g is not log-atomic.

• α is a limit ordinal: clear by induction hypothesis.

• Consider now $g \in G_{\alpha+1}$. As before g has singleton support $\gamma \in \Gamma_{\alpha+1}$, say $g = -\mathbf{1}_\gamma$, compute $\log(t^g) = t^{l(\gamma)}$. But $l(\gamma) \in G_\alpha^{<0}$ by (7), so $\log(t^g) = t^{l(\gamma)}$ is not log-atomic, a fortiori t^g is not log-atomic.