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The Moment Problem for Continuous Linear Functionals

The Multidimensional Moment Problem.

• Let $V := \mathbb{R}[x] := \mathbb{R}[x_1, \cdots, x_n]$ be the real vector space of polynomials in n variables and real coefficients.

• Given ℓ a linear functional on V, consider its moment sequence (evaluations on the monomial basis):

$$s(\alpha) := \ell(x^\alpha)\,;\alpha \in \mathbb{N}^n$$

Multidimensional Moment Problem:

Given a closed subset $K \subseteq \mathbb{R}^n$, give necessary and sufficient conditions on $s(\alpha); \alpha \in \mathbb{N}^n$ so that ℓ corresponds to a finite positive Borel measure μ on K.

• Define the *cone of nonnegative polynomials* on K by

$$\mathrm{Psd}(K) = \{f \in \mathbb{R}[x] \ : \ \forall x \in K \ f(x) \geq 0\}.$$

Theorem (Haviland)

Let $K \subset \mathbb{R}^n$ closed, and $\ell : V \to \mathbb{R}$ a nonzero linear functional. The following are equivalent:

(i) $\ell(f) \geq 0$ for all $f \in \mathrm{Psd}(K)$

(ii) $\exists$ a positive Borel measure μ on K such that

$$\ell(f) = \int_K f d\mu \, , \forall \, f \in V$$

The main challenge in applying Haviland's Theorem is verifying its condition (i). Schmüdgen analysed this problem for a special class of closed subsets:

• $K \subseteq \mathbb{R}^n$ is a *basic closed semialgebraic set* if there exist a finite set of polynomials $S = \{g_1, \ldots, g_s\}$ such that

$$K = K_S := \{x \in \mathbb{R}^n \ : \ g_i(x) \geq 0, \ i = 1, \ldots, s\}.$$

• Consider the *finitely generated preordering* $T_S \subset \mathrm{Psd}(K)$:

$$T_S := \{ \sum_{e \in \{0,1\}^s} \sigma_e \underline{g}^e \ : \ \sigma_e \text{ is a sos for all } e \in \{0,1\}^s\},$$

where $e = (e_1, \cdots, e_s) \in \{0,1\}^s$, and

$\underline{g}^e := \ g_1^{e_1} \ldots g_s^{e_s}$.

In 1991 Schmüdgen improved condition (i) of Haviland's Theorem and proved the following:

Theorem Assume that $K = K_S$ is a *compact* basic closed semi-algebraic set, and $\ell : V \to \mathbb{R}$ a nonzero linear functional. The following are equivalent:

(i) $\ell(f) \geq 0$ for all $f \in T_S$

(ii) $\ell(h^2 \underline{g}^e) \geq 0 \quad \forall h \in \mathbb{R}[x]$ and $e \in \{0,1\}^s$

(iii) $\exists$ a positive Borel measure μ on K such that

$$\ell(f) = \int_K f d\mu \,, \forall\, f \in V$$

Remark: For any $g \in \mathbb{R}[x]$, let S_g be the matrix of which $\alpha\beta$-th coefficient is $\ell(x^{\alpha+\beta} g)$ (for $\alpha, \beta \in \mathbb{N}^n$).

Condition (ii) reduces the problem to verifying that the 2^s Moment matrices $\{S_{\underline{g}^e}\,;\ e \in \{0,1\}^s\}$ are psd.

We explain this result as a single topological statement, which in turn allows to get stronger results (with additinal constraints imposed on the moment sequence).

Closures of Cones in Locally Convex Topologies.

• Fix τ a locally convex (Hausdorff) topological vector space topology on V. Denote V_τ the corresponding topological space.

• Fix $C \subseteq V$ a cone (i.e. closed under addition and scalar multiplication by positive reals).

• Fix $K \subseteq \mathbb{R}^n$ closed.

From Haviland and Hahn–Banach, one deduces:

Fact: The following are equivalent:

(1) $\overline{C} \supseteq \mathrm{Psd}(K)$ in V_τ

(2) for a *continuous* (w.r.t. τ) linear functional ℓ ;

$\ell(C) \geq 0$ implies $\exists\ \mu$ on K such that:

$$\ell(f) = \int_K f d\mu \, , \forall\, f \in V$$

From now on, we focus on solutions $(\tau\, , C\, , K)$ *for the inclusion (1).*

Example:

• $\tau = \varphi :=$ the finest locally convex (Hausdorff) topology, (all linear functionals are continuous).

• $C := T_S$

• $K := K_S$ compact.

Schmüdgen's result can be reformulated as:

$$\overline{T_S} = \mathrm{Psd}(K) \text{ in } V_\varphi$$

• Call a linear functional ℓ *positive semi definite* (psd) if

$$\ell(h^2) \geq 0 \text{ for all } h \in \mathbb{R}[x] \quad (*)$$

• So ℓ is psd if and only if its moment matrix $S_!$, (of which coefficients are the moments $s(\alpha + \beta) = \ell(x^{\alpha+\beta})$ is psd.

In the following, we shall study situations where the 2^s *conditions (ii) in Schmüdgen can be replaced by just the first among them, namely condition (*).*

The Moment Problem for Continuous Positive Semidefinite Linear Functionals.

Below, for $1 \leq p \leq \infty$:

$V_p :=\ V$ endowed with the ℓ_p–norm topology (**on the coefficients of polynomials**).

Theorem (Berg et al.):

$$\overline{\sum V^2} = \text{ Psd } [-1,1]^n \text{ in } V_1 .$$

Corollary Let ℓ be a linear functional such that its moment sequence $\{s(\alpha)\}_{\alpha \in \mathbb{N}^n}$ is bounded, and its moment matrix $[s(\alpha+\beta)]_{\alpha\,,\,\beta \in \mathbb{N}^n}$ is psd. Then

$$\exists \mu \text{ on } [-1,1]^n \text{ such that } \ell(f) = \int f d\mu \ \forall \ f \in V .$$

Remark: Compare to Schmüdgen: We can describe the compact basic closed semi-algebraic unit hypercube by $2n$ linear inequalities. For an arbitrary linear functional, we would a priori check the psd-ness of 2^{2n} moment matrices.

Weighted ℓ_p Topologies.

Let $r = (r_1, \dots, r_n)$ be a n-tuple of positive real numbers.

- For $1 \leq p < \infty$,

$$\ell_{p,r}(\mathbb{N}^n) := \{s \in \mathbb{R}^{\mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r_1^{\alpha_1} \dots r_n^{\alpha_n} < \infty\}$$

is a Banach space with respect to the norm

$$\|s\|_{p,r} = (\sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^p r_1^{\alpha_1} \dots r_n^{\alpha_n})^{\frac{1}{p}}.$$

- For $p = \infty$

$$\ell_{\infty,r}(\mathbb{N}^n) := \{s \in \mathbb{R}^{\mathbb{N}^n} : \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{\alpha_1} \dots r_n^{\alpha_n} < \infty\}$$

is a Banach space with respect to the norm

$$\|s\|_{\infty,r} = \sup_{\alpha \in \mathbb{N}^n} |s(\alpha)| r_1^{\alpha_1} \dots r_n^{\alpha_n}.$$

Let us describe the continuous linear functionals on $\ell_{p,r}(\mathbb{N}^n)$. Below, we let q be the conjugate of p.

Proposition. Let $1 \leq p < \infty$.

If $p > 1$, then $\ell_{p,r}(\mathbb{N}^n)^* = \ell_{q,r^{-\frac{q}{p}}}(\mathbb{N}^n)$.

If $p = 1$, then $\ell_{1,r}(\mathbb{N}^n)^* = \ell_{\infty,r^{-1}}(\mathbb{N}^n)$.

Here $r^{-\frac{q}{p}} := (r_1^{-\frac{q}{p}}, \cdots, r_n^{-\frac{q}{p}})$, similarly for r^{-1}.

Now let $f \in V$. Assume that

$$f \geq 0 \text{ on } \prod_{i=1}^{n} [-r_i, r_i] .$$

Then the polynomial

$$\tilde{f}(\underline{X}) = f(r_1 X_1, \cdots, r_n X_n)$$

is a nonnegative polynomial on $[-1, 1]^n$.

Combining this observation with Berg's result we get:

Fix $r = (r_1, \cdots, r_n)$ with $r_i > 0$ for $i = 1, \cdots, n$.

Theorem 1 (i) Let $1 \leq p < \infty$. Then

$$\overline{\sum V^2} = \text{ Psd } (\prod_{i=1}^{n} [-r_i^{\frac{1}{p}}, r_i^{\frac{1}{p}}]) \text{ in } V_{p,r} \ .$$

(ii)

$$\overline{\sum V^2} = \text{ Psd } (\prod_{i=1}^{n} [-r_i \ , r_i]) \text{ in } V_{\infty,r} \ .$$

Here, for $1 \leq p \leq \infty$:

$V_{p,r} := V$ endowed with the $\ell_{p,r}$–norm topology (on the coefficients of polynomials).

Corollary 1 Let $\ell : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional such that its moment matrix $s(\alpha+\beta)$ is psd and its moment sequence $s(\alpha)$ satisfies

$$\sup{}_{\alpha\in\mathbb{N}^n} |s(\alpha)| r_1^{-\alpha_1} \cdots r_n^{-\alpha_n} < \infty\,.$$

Then ℓ there exists a positive Borel measure μ on $K = \Pi_{i=1}^n [-r_i, r_i]$ such that

$$\ell(f) = \int_K f\ d\mu \quad \forall f \in \mathbb{R}[x]\,.$$

Corollary 2 Let $1 < p < \infty$.
Let $\ell : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional such that its moment matrix $s(\alpha+\beta)$ is psd and its moment sequence $s(\alpha)$ satisfies

$$\sum_{\alpha\in\mathbb{N}^n} |s(\alpha)|^q r_1^{-\frac{q}{p}\alpha_1} \cdots r_n^{-\frac{q}{p}\alpha_n} < \infty.\,.$$

Then there exists a positive Borel measure μ on $K = \Pi_{i=1}^n [-r_i^{-\frac{1}{p}}, r_i^{-\frac{1}{p}}]$ such that

$$\ell(f) = \int_K f\ d\mu \quad \forall f \in \mathbb{R}[x]\,.$$

Recent Developments.

I. Lasserre's Factoriel weighted ℓ_1-norm :
Recently, Lasserre proved that there exists a norm $\|\|_w$ on $\mathbb{R}[x]$ such that for any basic closed semi-algebraic set K_S (with non-empty interior), the closure of the preordering T_S with respect to $\|\|_w$ is equal to $\mathrm{Psd}(K_S)$. The $\|\|_w$ is explicitly defined by

$$\|\sum_\alpha f_\alpha x^\alpha\|_w := \sum_\alpha |f_\alpha| w(\alpha),$$

where $w(\alpha) := (2\lceil|\alpha|/2\rceil)!$ and $|\alpha| = |(\alpha_1, \ldots, \alpha_n)| = \alpha_1 + \cdots + \alpha_n$.

Compare to Schmüdgen, no compactness required, but continuity constraints on the moment sequence, etc $\cdots$

II. Closure of the cone of SO-2d:

Recently, Ghasemi - Marshall - Wagner used Jacobi-Putinar Archimedean Positivstellensatz to establish Berg et al results above on closures of $\sum \mathbb{R}[x]^2$ to closure of (the strictly smaller cone) of sums of 2d-powers:

$$\sum \mathbb{R}[x]^{2d} \subset \sum \mathbb{R}[x]^2 \,.$$

Remarks

1. The closure remains the same, for all d. For the conditions on the moment sequence, we just need $\ell(h^{2d}) \geq 0$. Interpretation.

2. All our results on closures of $\sum \mathbb{R}[x]^2$ in weighted p-norms carry over to $\sum \mathbb{R}[x]^{2d}$.

3. Berg et al. used techniques from Harmonic analysis on semi-groups.

Results hold more generally so-called absolute values, not just ℓ_1 norms.

III. Closure of the cone of SO-2d in $\mathbb{R}$- algebras:

Let R be an $\mathbb{R}$-algebra with 1 and $K \subseteq$ Hom $(R\,,\,\mathbb{R})$, closed with respect to the product topology. We consider R endowed with the topology τ_K, induced by the family of seminorms $\rho_\alpha(a) := |\alpha(a)|$, for $\alpha \in K$ and $a \in R$. In case K is compact, we also consider the topology induced by $\|a\|_K := \sup_{\alpha\in K} |\alpha(a)|$ for $a \in R$. If K is Zariski dense, then those topologies are Hausdorff. We prove that the closure of the cone of sums of $2d$-powers, $\sum R^{2d}$ with respect to those two topologies is equal to the cone Psd(K).

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