# Approximation of positive polynomials by sums of squares. A short overview.

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**Abstract.**

This overview is intended to provide an " atlas " of what is known about approximations of the cone of positive polynomials (on a semialgebraic set $K_S$) by various preorderings (or the corresponding module versions). These approximations depend on the description $S$ of $K_S$, the dimension of the semi-algebraic set $K_S$, intrinsic geometric properties of $K_S$ (e.g. compact or unbounded), and special properties of $K_S$ (symmetry, sparse representation)

**Contents.**



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## 0. INTRODUCTION.

In algebraic geometry, we consider ideals of the polynomial ring and algebraic varieties in affine space. In semi-algebraic geometry, we consider preorderings of the polynomial ring and semialgebraic sets in affine space.

**Notation and definitions:** Let $\mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the ring of polynomials in $n$ variables and real coefficients. A subset $M \subseteq \mathbb{R}[X]$ is a **quadratic module** if $1 \in M$, $M$ is closed under addition and multiplication by squares (i.e. $a^2 f \in M$, $\forall a \in \mathbb{R}[X]$ and $f \in M$).A **quadratic preordering** is a quadratic module which is also closed under multiplication. The smallest preordering of $\mathbb{R}[X]$ is the set of **sums of squares** of $\mathbb{R}[X]$, denoted by $\sum \mathbb{R}[X]^2$. Given a finite subset $S = \{f_1, ..., f_s\}$ of $\mathbb{R}[X]$, the smallest preordering containing $S$ (**preordering finitely generated by** $S$) is:

$$T_S = \{ \sum_{e \in \{0,1\}^s} \sigma_e f^e : \sigma_e \in \sum \mathbb{R}[X]^2, f_1, \cdots, f_s \in S\}$$

where $f^e := f_1^{e_1} \cdots f_r^{e_s}$, if $e = (e_1, \cdots, e_s)$. The smallest module containing $S$ (**module finitely generated by** $S$) is:

$$M_S = \{\sigma_0 + \sigma_1 f_1 + ... + \sigma_s f_s \ ; \sigma_e \in \sum \mathbb{R}[X]^2 \ .\}$$

Let $S = \{f_1, \cdots, f_s\} \subset \mathbb{R}[X]$, $S$ defines a **basic closed semialgebraic** subset of $\mathbb{R}^n$:

$$K = K_S = \{x \in \mathbb{R}^n \ : \ f_1(x) \geq 0, \ldots, f_s(x) \geq 0\}$$

Consider polynomials **positive semi-definite** on $K_S$:

$$\mathrm{Psd}(K_S) := \{f \in \mathbb{R}[X] : f(x) \geq 0 \text{ for all } x \in K_S\}$$

$\mathrm{Psd}(K_S)$ is a preordering in $\mathbb{R}[X]$ and $T_S \subseteq \mathrm{Psd}(K_S)$.

Hilbert's 17th Problem and Stengle's Positivstellensatz are concerned with the issue of representation of positive semi-definite polynomials; motivated by the question: when it true that $\mathrm{Psd}(K_S) = T_S$ ? More generally, we are concerned with the issue of approximating $\mathrm{Psd}(K_S)$ by "smaller" preorderings (modules):

$$T_S^{\dagger} = \{f \ : \ \forall \text{ real } \epsilon > 0 \ , \ f + \epsilon \in T_S\}.$$

$$T_S^{\ddagger} = \{f \ : \ \exists q \in \mathbb{R}[X] \ \text{ such that } \forall \text{ real } \epsilon > 0 \ , \ f + \epsilon q \in T_S\}.$$

$$\overline{T_S} := \{f \ : \ L(f) \geq 0 \ , \forall \text{ lin. funct. } L \neq 0 \text{ on } \mathbb{R}[X] \text{ s. t. } L(T_S) \geq 0\} \ .$$

We have:

$$T_S \subseteq T_S^{\dagger} \subseteq T_S^{\ddagger} \subseteq \overline{T_S} \subseteq \mathrm{Psd}(K_S) \ .$$

• $T_S$ is **saturated** if $\mathrm{Psd}(K_S) = T_S$.

• $(\dagger)_S$ holds if $T_S^\dagger = \mathrm{Psd}(K_S)$

• $(\ddagger)_S$ holds if $T_S^\ddagger = \mathrm{Psd}(K_S)$

• $S$ **solves the** $K_S$**– moment problem** if $\overline{T_S} = \mathrm{Psd}(K_S)$. Note that this is equivalent to saying that $T_S$ is *dense* in $\mathrm{Psd}(K_S)$.

**Remark 0.1** (i) $\overline{T_S}$ is the **closure** of $T_S$ in $\mathbb{R}[X]$ (for the finest locally convex topology on $\mathbb{R}[X]$).

Denote by $P_d$ the (finite dimensional) vector space consisting of all polynomials in $\mathbb{R}[X]$ of degree $\leq 2d$, and by $T_d = T_S \cap P_d$. The set $T_d$ is obviously a bf cone in $P_d$, i.e., $T_d + T_d \subseteq T_d$ and $\mathbb{R}^+ T_d \subseteq T_d$. Denote by $\overline{T}_d$ the closure (in the Euclidean topology) of $T_d$ in $P_d$. Then:

(ii) $T_S^\ddagger = \cup_{d \geq 0} \overline{T}_d$.

(iii) The containments (end of page 4) may be strict. The conjecture that $T_S^\ddagger \neq \overline{T_S}$ was given in [K–M] and recently proved by T. Netzer.

(iv) All the above, except for $\mathrm{Psd}(K_S)$ *depend in general on the choice of the description* $S$ *of* $K = K_S$.

## 1. SATURATION.

In [S1] Scheiderer showed:

**Theorem 0.2** *If dim*$(K_S) \geq 3$*, then there exists a polynomial* $p(X) \in \mathbb{R}[X]$ *such that* $p(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *but* $p \notin T_S$ *(so* $T_S$ *cannot be saturated).*

Scheiderer's result is intrinsic; under this hypothesis on $K = K_S$, independently of the chosen description $S$, and whether $K_S$ is compact or unbounded, the preordering $T_S$ cannot be saturated.

In the same paper, he also shows another intrinsic result:

**Theorem 0.3** *If* $n = 2$ *and* $K_S$ *contains a cone of dimension 2, then there exists a polynomial* $p(X) \in \mathbb{R}[X]$ *such that* $p(x) \geq 0$ *for all* $x \in \mathbb{R}^n$ *but* $p \notin T_S$ *(so* $T_S$ *cannot be saturated).*

**Low dimensional sets:**

This left open the question formulated in [K-M]: what if $K_S \subseteq \mathbb{R}^2$ does not contain a cone of dimension 2? Are there compact/noncompact examples of such $K_S$ for which $T_S$ is saturated? More recently, Scheiderer developed in a series of papers [S2], [S3], [S4] several *local global principles* to determine when a polynomial $f \geq 0$ on $K_S$ belongs to the *quadratic module* $M_S$. His results generalize both Schmüdgen's and Putinar's Striktpositivstellensätze. With these tools, he was able to produce the example that we were looking for:

**Example 0.4** The modules generated by:
$S_1 = \{1+x, 1-x, 1+y, 1-y\}$ (compact $K_S$) and

$S_2 = \{x, 1-x, y, 1-xy\}$ (noncompact $K_S$)

are saturated.

*In [K-M-S], we studied saturated preorderings (modules) for subsets of the real line. We discuss the case $n = 1$.* To state [K-M-S; Theorem 2.2]. We need to define some notions.

If $K \subseteq \mathbb{R}$ is a non-empty closed semi-algebraic set. Then $K$ is a finite union of intervals. It is easily verified that $K = K_{\mathcal{N}}$, for $\mathcal{N}$ the set of polynomials defined as follows:

•If $a \in K$ and $(-\infty, a) \cap K = \emptyset$, then $X - a \in \mathcal{N}$.

•If $a \in K$ and $(a, \infty) \cap K = \emptyset$, then $a - X \in \mathcal{N}$.

•If $a, b \in K$, $(a, b) \cap K = \emptyset$, then $(X - a)(X - b) \in \mathcal{N}$.

• $\mathcal{N}$ has no other elements except these.

We call $\mathcal{N}$ **the natural set of generators** for $K$.

We first consider the non-compact case:

**Theorem 0.5** *Assume that $K = K_S \subseteq \mathbb{R}$ is not compact. Then $T_S$ is saturated if and only $S$ contains the natural set of generators of $K$ (up to scalings by positive reals).*

For the compact case, we also have a criterion. Assume that $K_S$ has no isolated points :

**Theorem 0.6** *Let $K_S$ be compact, $S = \{g_1, \cdots, g_s\}$. Then $T_S$ is saturated if and only if, for each endpoint $a \in K_S$, there exists $i \in \{1, \cdots, s\}$ such that $x - a$ divides $g_i$ but $(x - a)^2$ does not.*

*What about the module version?*

In [K-M-S] we asked whether $M_S = T_S$ if $K_S \subseteq \mathbb{R}$ is compact. Scheiderer provided a positive answer using his local-global criteria. In [F] another elementary proof of this fact is given. Thus the above theorem is a criterion for the quadratic module $M_S$ to be saturated.

## 2. THE DAGGER CONDITION.

In [Sc1] Schmüdgen proved the following intrinsic result:

**Theorem 0.7** *If $K_S$ is compact, then $(\dagger)_S$ holds for $T_S$.*

A quadratic module $M$ is **archimedean** if for all $f \in \mathbb{R}[X]$, there exists an integer $n \geq 1$ such that $n - f$ and $n + f \in M$. Putinar proved the following result:

**Theorem 0.8** *If $M_S$ is archimedean, then $(\dagger)_S$ holds for $M_S$.*

**Remark 0.9** (i) If $T_S$ (or $M_S$) is archimedean then $K_S$ is compact.
(ii) Wörman showed that if $K_S$ is compact then $T_S$ is archimedean (providing a proof of Schmüdgen's Theorem via the Kadison-Dubois Theorem).
(iii) If $K_S$ is compact, $M_S$ need *not* be archimedean.

*What if $K_S$ is not compact?*

Apart from the non-compact examples of dimension $\leq 2$ presented in the previous section, no non-compact examples in dimension $\geq 3$ are known. This motivated considering $(\ddagger)$ instead, as we shall see in the next section.

## 3. THE DOUBLEDAGGER CONDITION.

**Non-compact examples by dimension extension.**

In [K-M] we construct a large number of non-compact examples where $(\ddagger)$ holds.

Let $S \subseteq \mathbb{R}[X]$ finite and set $p = 1 + \sum_{i=1}^{n} X_i^2$.

Denote by $\mathbb{R}[X, Y]$ the polynomial ring in $n+1$ variables $X = X_1, \ldots, X_n, Y$ and consider the finite set

$S' = S \cup \{1 - pY, -(1 - pY)\}$ in $\mathbb{R}[X, Y]$.

Then $K_{S'}$ consists of those points on the hypersurface

$$H = \{(x, y) \in \mathbb{R}^{n+1} \mid p(x)y = 1\}$$

in $\mathbb{R}^{n+1}$ which map to $K_S$ under the projection $(x, y) \mapsto x$.

**Theorem 0.10** $(\ddagger)'_S$ *holds.*

**Cylinders with compact base.**

We continue to denote by $\mathbb{R}[X, Y]$ the polynomial ring in $n + 1$ variables $X_1, \dots, X_n, Y$.

Consider a subset $S = \{g_1, \dots, g_s\}$ of $\mathbb{R}[X, Y]$ where the polynomials $g_1, \dots, g_s$ involve only the variables $X_1, \dots, X_n$.

So $K_S$ has the form $K \times \mathbb{R}$, $K \subseteq \mathbb{R}^n$. We further assume that $K$ is compact.

We describe this situation by saying that $K_S$ is a cylinder with compact cross-section. In [K-M] we prove:

**Theorem 0.11** *If $K_S$ is a cylinder with compact cross-section, then $(\ddagger)_S$ holds.*

*More precisely, let $f \in \mathbb{R}[X, Y]$ is such that $f \geq 0$ on $K_S$. Let $d \geq 1$ so that the degree of $f$ as a polynomial in $Y$ is $\leq 2d$. Set*

$$q(Y) := 3 + Y + 3Y^2 + Y^3 + \dots + 3Y^{2d}\,.$$

*Then for all $\epsilon > 0$, $f + \epsilon q(Y) \in T_S$.*

**Closed Polyhedra.**

In [K-M-S] we develop a "fiber criterion" for $(\ddagger)$ to hold on *subsets* of cylinders. In particular, we get an application to generalized polyhedra.

Assume that $K_S$ is the basic closed semi-algebraic set in $\mathbb{R}^m$, $m \geq 1$, defined by $S = \{\ell_1, \dots, \ell_s\}$, where $\ell_1, \dots, \ell_s$ are linear, so $K_S$ is a **closed polyhedron**. If $K_S$ is compact then, by [J–P], $(\dagger)$ holds for $M_S$.

*What if $K_S$ is not compact?*

If $K_S$ contains a cone of dimension 2 then, by [K–M]
(‡) fails for $T_S$.

In [K–M] we asked whether (‡) holds in the remaining case, i.e., when $K_S$ is not compact and does not contain a cone of dimension 2.

In [K–M–S] we settle this question completely:

**Theorem 0.12** *Let $P$ be a closed polyhedron in $\mathbb{R}^m$ defined by a finite set $S$ of linear polynomials.*

*(i) If $P$ is compact then (†) holds for $M_S$.*

*(ii) If $P$ is not compact but does not contain a 2-dimensional cone then (‡) holds for $M_S$.*

*(iii) If $P$ contains a 2-dimensional cone then (‡) fails for $T_S$.*

## 4. THE DENSITY CONDITION.

All the previous examples, compact or not, satisfying one of the previous conditions considered, satisfy the density condition. In [Sc2], Schmüdgen gives other methods to produce examples where the density condition holds. In [K-M] we gave an intrinsic condition for the density condition *to fail*:

**Theorem 0.13** *The density condition fails whenever $n \geq 2$ and $K_S$ contains a cone of dimension 2.*

In [P–S] a stronger intrinsic condition is given (if $K_S$ contains a "nasty curve" then the density condition fails). The following example is particularly interesting:

**Example 0.14** Consider

$$K := \{(x, y) \in \mathbb{R}^2 : -1 \leq (x^2 - 1)(y^2 - 1) \leq 0\}$$

in the plane $\mathbb{R}^2 = V(\mathbb{R})$ (see figure 1).

Arguing using the Powers-Scheiderer condition, one shows now that $K$-moment problem is not finitely solvable.

*Note however that the given set is very special; it displays interesting symmetries. This motivates the next section.*

## 5. SPECIAL SITUATIONS.

**Invariant Sets.**

We can extend the results of the previous sections in another direction.

The idea is to fix a distinguished subset $B \subset \mathbb{R}[X]$ and to attempt the various approximations *only for polynomials in B*. That is, we want to study the inclusions

$$T_S \cap B \subseteq \overline{T_S} \cap B \subseteq \mathrm{Psd}(K_S) \cap B\,.$$

In [C–K–S], we investigated the particularly privileged situation when $B$ **is the subring of invariant polynomials** with respect to some action of a group on the polynomial ring $\mathbb{R}[X]$.

Let us revisit the last example of the last section:

**Example 0.15** $K$ is $G$-invariant, where $G = D_4$ the dihedral group of order eight acting on $\mathbb{R}^2$ in the natural way (as the symmetry group of a square centered at the origin).

The ring of invariants is $\mathbb{R}[x,y]^G = \mathbb{R}[u,v]$ with

$$u = x^2 + y^2, \quad v = x^2y^2,$$

and the orbit variety $W = \mathbb{R}^2//G$ is itself an affine plane.

The image of $\pi\colon V(\mathbb{R}) \to W(\mathbb{R})$ is

$$Z = \pi(\mathbb{R}^2) = \{(u,v) \in \mathbb{R}^2\colon u \geq 0,\ v \geq 0,\ u^2 \geq 4v\}.$$

Since $(x^2-1)(y^2-1) = v - u + 1$, we have

$$\pi(K) = \{(u,v) \in \mathbb{R}^2\colon v \geq 0,\ 1 \leq u - v \leq 2\}.$$

This is a (half-) strip in the $(u,v)$-plane (see figure 2):

The moment problem for $\pi(K)$ is solved by the preordering $N$ in $\mathbb{R}[u,v] = W(\mathbb{R})$ generated by $v$, $u-v-1$ and $2-u+v$ (by [K-M-S]). This means that the $G$-invariant $K$-moment problem is solvable (i.e. *invariant* linear functional non-negative on the finitely generated preordering is represented by an invariant measure).

**Positive polynomials on fibre products.**

Throughout this section, a real algebraic, affine variety $V \subseteq \mathbb{R}^d$ is the common zero set of a finite set of polynomials.

The algebra of regular functions on $V$ (the coordinate ring of $V$) is $\mathbb{R}[V] = \mathbb{R}[X]/I(V)$, where $I(V)$ is the radical ideal associated to $V$.

The non-negativity set of a subset $S \subset \mathbb{R}[V]$ is

$$K(S) = \{x \in V; f(x) \geq 0, \quad f \in S\}.$$

Let $I$ be a non-empty set, endowed with a partial order relation $i \leq j$. A *projective system* of algebraic varieties indexed over $I$ consists of a family of varieties (affine in our case) $V_i,\ i \in I$, and morphisms $f_{ij} : V_j \longrightarrow V_i$ defined whenever $i \leq j$, and satisfying the compatibility condition

$$f_{ik} = f_{ij} f_{jk} \text{ if } i \leq j \leq k.$$

The topological projective limit $V = \text{proj.lim}(V_i, f_{ij})$ is the universal object endowed with morphisms

$$f_i : V \longrightarrow V_i$$

satisfying the compatibility conditions

$$f_i = f_{ij} f_j, \quad i \leq j.$$

A *directed projective system* carries the additional assumption on the index set that for every pair $i, j \in I$ there exists $k \in I$ satisfying $i \leq k$ and $j \leq k$.

A finite partially ordered set $I = \{i_0, ..., i_n\}$ is a *rooted tree* if the order structure is generated by the inequalities

$$i_1 \geq i_0 \text{ and for every } k > 1, i_k \geq i_{j(k)} \text{ with } j(k) < k.$$

In [K–P] We are concerned with finite projective systems of algebraic varieties. The main result is the following:

**Theorem 0.16** *Let $(V_i, f_{ij})$ be a finite projective system of real affine varieties, indexed over a rooted tree. Let $Q_i \subset \mathbb{R}[V_i]$ be archimedean quadratic modules, subject to the coherence condition $f_{ij}^* Q_i \subseteq Q_j$. Let $p \in \sum_i f_i^* \mathbb{R}[V_i]$ be an element which is positive on the set $\cap_{i \in I} f_i^{-1} K(Q_i)$. Then $p \in \sum_i f_i^* Q_i$.*

We also consider fibre products of affine real varieties: Let $Z = X_1 \times_Y X_2$ be the fibre product of affine real varieties. Specifically

$$f_i; X_i \longrightarrow Y, \quad i = 1, 2,$$

are given morphisms and

$$Z = \{(x_1, x_2) \in X_1 \times X_2;\ f_1(x_1) = f_2(x_2)\}.$$

This is still an algebraic variety, with the ring of regular functions

$$\mathbb{R}[X_1 \times_Y X_2] = \mathbb{R}[X_1] \otimes_{\mathbb{R}[Y]} \mathbb{R}[X_2].$$

Denote by $u_i : Z \longrightarrow X_i,\ i = 1, 2$, the projection maps, so that: $f_1 u_1 = f_2 u_2$.

**Proposition 0.17** *With the above notation, let $Q_i \subseteq \mathbb{R}[X_i],\ i = 1, 2$, be archimedean quadratic modules.*
*If an element $p \in u_1^* \mathbb{R}[X_1] + u_2^* \mathbb{R}[X_2]$ is strictly positive on the set $u_1^{-1} K(Q_1) \cap u_2^{-1} K(Q_2)$, then $p \in u_1^* Q_1 + u_2^* Q_2$.*

The proposition applies to the case of fibre products of affine spaces to recover a result of [L].

Specifically, let $X_1 = \mathbb{R}^{n_1} \times \mathbb{R}^m$, $X_2 = \mathbb{R}^m \times R^{n_2}$ and $Y = \mathbb{R}^m$, while $f_1, f_2$ are the corresponding projection maps onto $Y$. Denote by $x_1, y, x_2$ the corresponding tuples of variables. Then one immediately identifies

$$Z = \mathbb{R}^{n_1} \times \mathbb{R}^m \times \mathbb{R}^{n_2}$$

and the proposition yields:

**Corollary 0.18** *Let $Q_{x_1,y}, Q_{y,x_2}$ be archimedean quadratic modules in the respective sets of variables. Let*
*$\Pi := (K(Q_{x_1,y}) \times \mathbb{R}^{n_2}) \cap (\mathbb{R}^{n_1} \times K(Q_{y,x_2})) \subseteq Z$.*

*If a polynomial $p(x_1, y, x_2) = p_1(x_1, y) + p_2(y, x_2)$ is positive on $\Pi$, then $p \in Q_{x_1,y} + Q_{y,x_2}$.*

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