Well-posedness and asymptotic behaviour of ordinary integro-differential equations in glass rheology

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Introduction

Mode-coupling theory of glass-transition (MCT):

$$\phi(t) + \tau\dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0, \quad t \in [0,\infty), \quad \phi(0) = 1, \ (1)$$

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- w.l.o.g.. $\tau = 1$ by substitution $t = \tau \cdot \hat{t}$.

- problem (1) is a simplification of the following problem

$$\phi(t) + \dot{\phi}(t) + \ddot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0, \qquad (2)$$

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where we neglect the term $\ddot{\phi}(t)$ under the physically sensible assumption, that a limit of the solution exists.

- An example of F is $F(x) = ax^2 + bx$, $a, b > 0$.

Previous results

W. Götze, L. Sjögren: General Properties of Certain Non-linear Integro-Differential Equations, Journal of Mathematical Analysis and Applications 195, 230-250 (1995):

Theorem

Let $\delta > 0$ *and* $F : [0, 1+\delta) \to \mathbb{R}$ *an absolutely monotonic function, i.e.*

i) $F \in C^\infty([0, 1+\delta), \mathbb{R})$ *and*

ii) $\forall x \in [0, 1+\delta) : F^{(k)}(x) \geq 0 \ (k = 0, 1, 2, \dots)$.

Then the problem (1) has a unique solution $\phi \in C^\infty([0, \infty), \mathbb{R})$*, where* ϕ *is completely monotone, i.e.*

$$\forall t \in [0, \infty) : (-1)^k \phi^{(k)}(t) \geq 0, \quad (k = 0, 1, 2, \dots).$$

W. Götze, L. Sjögren:

Theorem

Let $g \in [0,1)$ be the maximal fixpoint of the equation

$$F(g) = \frac{g}{1-g}.$$

Then one has for the solution ϕ: $\lim_{t\to\infty} \phi(t) = g$.
If additionnally $F'(g) < \frac{1}{(1-g)^2}$ holds, then one has for all $n \in \mathbb{N}_0$

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Remark

In the work of Götze and Sjögren a result of exponential konvergence of the solution was presented.

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- Extending the results on additional time-dependent kernels. Examples in physics are

$$\phi(t) + \dot{\phi}(t) + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2(t-s)^2}\dot{\phi}(s)ds = 0,$$

$\gamma > 0$ *shear rate* and

$$\phi(t) + \dot{\phi}(t) + \int_0^t h(t)h(t-s)f(\phi(t-s))ds = 0.$$

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- Asymptotic behaviour in case of $F'(g) = \frac{1}{(1-g)^2}$

Well-posedness: bounded kernel-functions

We consider the problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0, \quad \phi(0) = 1, \tag{3}$$

where $F : \mathbb{R} \to \mathbb{R}$ is **bounded** and locally Lipschitz-continuous.

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norm on X: $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$. With that, X is a Banach space.

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problem (3) for $t \in [0, N]$ is equivalent to the following fixed point problem

$$\phi(t) = T\phi(t), \quad t \in [0, N]$$

where $T : X \to X, \phi \mapsto T\phi$, with

$$T\phi(t) = 1 + \int_0^t F(\phi(s)) - \phi(s) - \phi(t-s)F(\phi(s))ds. \tag{4}$$

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Lemma

Let $a, k > 0$ and

$$M_{a,k} := \{f \in X : f(0) = 1, |f(x)| \leq ae^{kx}, |f'(x)| \leq ae^{kx}, 0 \leq x \leq N\}.$$

Then one has

$$T(M_{a,k}) \subseteq M_{a,k}$$

for a, k sufficiently large.

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proof: Let $C := \sup_{x\in\mathbb{R}} |F(x)|$, $\phi \in M_{a,k}$.

$$e^{-kt}|T\phi(t)| \leq 1 + e^{-kt}\int_0^t C + |\phi(s)|e^{-ks}e^{ks} + C|\phi(s)|e^{-ks}e^{ks}ds$$

$$\leq 1 + NC + a\int_0^t e^{k(s-t)}ds + aC\int_0^t e^{k(s-t)}ds$$

$$\leq 1 + NC + \frac{1}{k}(a + aC)$$

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analogously (a, k see above): $\left|\frac{d}{dt}T\phi(t)\right| \leq ae^{kt}$ □

Remark

After restricting T on $M_{a,k}$ we will use boundedness of ϕ.

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Definition

Let a, k be as above. We define with $\alpha > 0$ the metric on $M_{a,k}$

$$d_{\alpha+k}(f,g) := \max\left\{ \sup_{0\le x\le N} e^{-(\alpha+k)x}|f(x)-g(x)|, \sup_{0\le x\le N} e^{-(\alpha+k)x}|f'(x)-g'(x)| \right\}.$$

This metric is equivalent to the metric induced by $\|\cdot\|_X$,
i.e. $(M_{a,k}, d_{\alpha+k})$ is a complete metric space.

Lemma

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proof: Let $C := \sup_{x\in\mathbb{R}} |F(x)|$, L a Lipschitz constant of F on $[-ae^{kN}, ae^{kN}]$, a, k as above, $\alpha > 0$ and $\phi_1, \phi_2 \in M_{a,k}$, then one has

$$e^{-(\alpha+k)t}|T\phi_1(t) - T\phi_2(t)| \quad \leq \quad k_1(\alpha)d_{\alpha+k}(\phi_1, \phi_2)$$

with $k_1(\alpha) := \frac{1}{\alpha+k}(L + 1 + C + ae^{kN}L)$.

Analogously we have

$$e^{-(\alpha+k)t}\left|\frac{d}{dt}(T\phi_1)(t) - \frac{d}{dt}(T\phi_2)(t)\right| \quad \leq \quad k_2(\alpha)d_{\alpha+k}(\phi_1, \phi_2),$$

with $k_2(\alpha) = \frac{1}{\alpha+k}(1 + C + ae^{kN}L)$.

We now choose α large enough, that $k_i(\alpha) < 1$, $i = 1, 2$. □

Since $N > 0$ is arbitrary, we have the following

Theorem

Let $F : \mathbb{R} \to \mathbb{R}$ be bounded and locally Lipschitz continuous. Then the problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0, \quad \phi(0) = 1$$

has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$.

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Results for only bounded kernel-functions are not optimal because from the physical point of view unbounded polynomial functions are of interest.

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Lemma

Let $F : \mathbb{R} \to \mathbb{R}$ *be differentiable and* **monotonically increasing** *and let* $\phi \in C^1([0,\infty),\mathbb{R})$ *be a solution of*

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds, \quad \phi(0) = 1.$$

Then ϕ *is strictly monotonically decreasing.*

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Differentiating the integro-differential equation comes to

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In particular we have $\dot{\phi}(t') < 0$, contradiction! □

Remark

Formal limit $t \to \infty$ of

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holds $(\lim_{t\to\infty} \dot{\phi(t)} = 0)$ $\quad F(g) = \frac{g}{1-g}, \quad$ *where* $g := \lim_{t\to\infty} \phi(t)$.

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A fixed point g *of this equation (if exists) fulfils* $g \leq \phi(t) \leq 1$.

If this is fulfilled, it will be enough so regard F *only on the intervall* $[g, 1]$ *and to work with the following kernel-function instead of* F*:*

$$\tilde{F}(x) := \begin{cases} F(1), & x > 1 \\ F(x), & g \leq x \leq 1 \\ F(g), & x < g. \end{cases}$$

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If this is fulfilled, it will be enough so regard F only on the intervall $[g, 1]$ and to work with the following kernel-function instead of F:

$$\tilde{F}(x) := \begin{cases} F(1), & x > 1 \\ F(x), & g \leq x \leq 1 \\ F(g), & x < g. \end{cases}$$

To preserve differentiability, it will be necessary, to work with an approximation of $\tilde{F}$.

Lemma

Let $N > 0$, $\varepsilon > 0$, $F : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and bounded and let $\tilde{F} : \mathbb{R} \to \mathbb{R}$ be continuous and bounded, with $\|F - \tilde{F}\|_\infty < \varepsilon$.
Let $\phi, \tilde{\phi} : [0, N] \to \mathbb{R}$ be solutions of (1) for F resp. $\tilde{F}$.
Then there exists a constant $\kappa = \kappa(N, \varepsilon, F) > 0$:

$$\|\phi - \tilde{\phi}\|_\infty \leq \kappa \|F - \tilde{F}\|_\infty .$$

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Then there exists a constant $\kappa = \kappa(N, \varepsilon, F) > 0$:

$$\|\phi - \tilde{\phi}\|_\infty \leq \kappa \|F - \tilde{F}\|_\infty.$$

proof: Let L be a Lipschitz constant of F, $C = \sup_{x \in \mathbb{R}} |F(x)|$.

As a consequence of the proof of the existence theorem for bounded kernel-functions we obtain

$$|\phi(t)| \leq a e^{kN} =: M, \quad |\tilde{\phi}(t)| \leq a_\varepsilon e^{k_\varepsilon N} =: M_\varepsilon, \quad t \in [0, \infty),$$

where

$$a = 2 + NC, k = 2a + 2aC,$$
$$a_\varepsilon = 2 + N(C + \varepsilon), k_\varepsilon = 2a + 2a(C + \varepsilon).$$

Remembering the fixed point equations

$$\phi = T_1\phi, \quad \tilde{\phi} = T_2\tilde{\phi},$$

where $T_1\phi(t) = 1 + \int_0^t F(\phi(s)) - \phi(s) - \phi(t-s)F(\phi(s))ds$ and

$$T_2\tilde{\phi}(t) = 1 + \int_0^t \tilde{F}(\tilde{\phi}(s)) - \tilde{\phi}(s) - \tilde{\phi}(t-s)\tilde{F}(\tilde{\phi}(s))ds.$$

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With this we have

$$|\phi(t) - \tilde{\phi}(t)| = \Bigg| \int_0^t \left[F(\phi(s)) - F(\tilde{\phi}(s))\right] + \left[F(\tilde{\phi}(s)) - \tilde{F}(\tilde{\phi}(s))\right]$$
$$+ \left[\tilde{\phi}(s) - \phi(s)\right] + \tilde{F}(\tilde{\phi}(s)) \left[\tilde{\phi}(t-s) - \phi(t-s)\right]$$
$$+ \phi(t-s)\left[\tilde{F}(\tilde{\phi}(s)) - F(\tilde{\phi}(s)) + F(\tilde{\phi}(s)) - F(\phi(s))\right] ds \Bigg|$$

$$\leq (N + MN)\|F - \tilde{F}\|_\infty + (L + 1 + C + \varepsilon + ML) \int_0^t |\phi(s) - \tilde{\phi}(s)| ds.$$

With Gronwall's inequality it follows

$$\|\phi - \tilde{\phi}\|_\infty \quad \leq \quad \kappa \|F - \tilde{F}\|_\infty,$$

with $\kappa = (N + MN) + (N + MN)\left(e^{(L+1+C+\varepsilon+ML)N} - 1\right)$. □

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Theorem

Let $F : \mathbb{R} \to \mathbb{R}$, *with*

i) $\exists x_0 < 1 : F(x_0) = \frac{x_0}{1-x_0}$,

ii) F *is differentiable, monotonically increasing and (locally-)Lipschitz continuous on* $[x_0, 1]$.

Then there exists a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ *of the problem*

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0, \quad \phi(0) = 1,$$

where ϕ *is monotonically decreasing, with* $x_0 \leq \phi(t) \leq 1$, $t \in [0,\infty)$.

proof: We define

$$\tilde{F}(x) := \begin{cases} F(1), & x > 1 \\ F(x), & x_0 \leq x \leq 1 \\ F(x_0), & x < x_0 \end{cases}, \quad x \in \mathbb{R}.$$

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Let $(F_n)_{n\in\mathbb{N}} \subseteq C^0(\mathbb{R},\mathbb{R})$ be a sequence of differentiable, bounded, monotonically increasing functions, with $\|F_n - \tilde{F}\|_\infty \stackrel{n\to\infty}{\longrightarrow} 0$.

proof: We define

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One has for all $N > 0$: $\|\phi_n - \phi\|_{C^0([0,N])} \leq \kappa(N)\|F_n - \tilde{F}\|_\infty \overset{n\to\infty}{\longrightarrow} 0$, i.e. ϕ is monotonically decreasing.

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With this and the integro-differential equation one obtains

$$\begin{aligned} \dot{\phi}(t) &= -\phi(t) - \int_0^t \tilde{F}(\phi(t-s))\dot{\phi}(s)ds \\ &\geq -(1+\tilde{F}(x_0))\phi(t) + \tilde{F}(x_0) \\ \overset{Gronwall}{\Rightarrow} \quad \phi(t) &\geq e^{-(1+\tilde{F}(x_0))t} + \int_0^t e^{-(1+\tilde{F}(x_0))(t-s)}\tilde{F}(x_0)ds \\ &\geq \frac{\tilde{F}(x_0)}{1+\tilde{F}(x_0)} \overset{\tilde{F}(x_0)=F(x_0)}{=} x_0. \end{aligned}$$

proof: We define

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There exists a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ for the problem (1) with $\tilde{F}$, where ϕ is monotonically decreasing.

With this and the integro-differential equation one obtains

$$x_0 \leq \phi(t) \leq 1, \quad t \in [0,\infty).$$

$\Rightarrow \quad \tilde{F}(\phi(t)) = F(\phi(t)), \quad t \in [0,\infty)$, i.e.. ϕ is a solution of (1) with F.

□

Examples

One can now treat the problem with the following functions (physical background not ensured)

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1. $F(x) = sin(x)$, monotone on $[0, 1]$,
2. $F(x) = \sqrt{x}$,
3. Polynomial functions with negative coefficients, e.g. $F(x) = -x^2 + 2x$

Asymptotic behaviour

Asymptotic behaviour

Theorem

Let $F : \mathbb{R} \to \mathbb{R}$, *with*

i) $\exists x_0 < 1 : F(x_0) = \frac{x_0}{1-x_0}$,

ii) F *is differentiable, monotonically increasing and (locally-)Lipschitz continuous on* $[x_0, 1]$.

Then the solution ϕ *of the problem (1) converges to the maximum intercept point of* F *with* G, *where* $G(x) = \frac{x}{1-x}$, $x \in (-\infty, 1)$.

proof: $\exists g \in \mathbb{R} : \phi(t) \overset{t\to\infty}{\longrightarrow} g$

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One has for $0 < t_1 < t$

$$\begin{aligned} & \left| \int_0^t F(\phi(t-s))\dot{\phi}(s)ds - F(g)(g-1) \right| \\ \leq & \left| \int_0^{t_1} F(\phi(t-s))\dot{\phi}(s)ds - F(g)(g-1) \right| + \left| \int_{t_1}^t F(\phi(t-s))\dot{\phi}(s)ds \right| \\ \equiv & \; I_1 + I_2 \end{aligned}$$

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$$I_1 \stackrel{t\to\infty}{\longrightarrow} \left| \int_0^{t_1} F(g)\dot{\phi}(s)ds - F(g)(g-1) \right| = |F(g)||\phi(t_1) - g|$$

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and for the second

$$\begin{aligned} I_2 & \leq \int_{t_1}^t |F(\phi(t-s))||\dot{\phi}(s)|ds \overset{\dot{\phi}\leq 0}{\leq} C\int_{t_1}^t -\dot{\phi}(s)ds \\ & \leq -C\left(|\phi(t)-g| - |\phi(t_1)-g|\right) \overset{t\to\infty}{\longrightarrow} C|\phi(t_1) - g|. \end{aligned}$$

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We choose t_1 large enough, that we have for arbitrary $\varepsilon > 0$

$$\lim_{t\to\infty} I_1 + I_2 < \varepsilon.$$

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We choose t_1 large enough, that we have for arbitrary $\varepsilon > 0$

$$\lim_{t\to\infty} I_1 + I_2 < \varepsilon.$$

$$\Rightarrow \quad \lim_{t\to\infty} \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = F(g)(g-1).$$

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Since $\phi \in C^1([0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} \phi(t) = g$, there exists a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0,\infty) : t_n \stackrel{n\to\infty}{\longrightarrow} \infty, \phi(t_n) \stackrel{n\to\infty}{\longrightarrow} g, \dot{\phi}(t_n) \stackrel{n\to\infty}{\longrightarrow} 0.$

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$$\overset{Integro-DE}{\Rightarrow} \quad \int_0^{t_n} F(\phi(t_n-s))\dot{\phi}(s)ds = -\phi(t_n) - \dot{\phi}(t_n) \quad \overset{n\to\infty}{\longrightarrow} \quad -g$$

$$\Rightarrow \quad F(g) = \frac{g}{1-g} = G(g)$$

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Since for all $x_0 < 1$, with $F(x_0) = G(x_0)$: $\phi(t) \geq x_0$, $t \in [0,\infty)$, ϕ converges to the maximum intercept point of F and G.

□

Remark

We define

$$F(x) := \begin{cases} \frac{x}{1-x} - 1, & x < \frac{1}{2} \\ 0, & x \geq \frac{1}{2} \end{cases} .$$

F is bounded and monotonically increasing, with $F(x) < G(x)$ for all $x \in \mathbb{R}$.
Then there exists a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ of (1) with F, where ϕ monotonically decreasing, with $\lim_{t\to\infty} \phi(t) \to -\infty$,
i.e. there are divergent solutions.

We now aim to prove polynomial convergency of solutions.
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Lemma

Let $F : [0,1] \to \mathbb{R}$ be continuously differentiable with the following conditions

i) $F(x) < \frac{x}{1-x}, x \neq 0,$
ii) $F(0) = 0,$
iii) $F'(0) < 1.$

Then one has a $\varepsilon \in (0,1)$, s.t. for all $x \in [0,1)$

$$F(x) \leq \frac{x}{1-x} - \varepsilon x =: H(x).$$

H is an absolutely monotone function.

Theorem

Let $F : [0,1] \to \mathbb{R}$ be continuously differentiable and monotonically increasing with the following conditions

i) $F(x) < \frac{x}{1-x}, x \neq 0,$

ii) $F(0) = 0,$

iii) $F'(0) < 1.$

Let ϕ be the solution of (1) with kernel-function F (convergent to 0). Then there exists the improper integrals for all $n \in \mathbb{N}_0$

$$\int_0^\infty t^n \phi(t) dt \quad und \quad \int_0^\infty t^n F(\phi(t)) dt.$$

proof: With previous lemma there exists $\varepsilon_0 \in (0,1)$, s.t. for all $x \in [0,1)$

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proof: With previous lemma there exists $\varepsilon_0 \in (0,1)$, s.t. for all $x \in [0,1)$

$$F(x) \leq \frac{x}{1-x} - \varepsilon_0 x =: H(x).$$

H is absolutely monotone, with $H'(0) < 1$. Then there exists $\varepsilon \in (0,1)$ and $x_0 > 0$, s.t. for all $n \in \mathbb{N}_0$, $x > x_0$

$$\int_{x_0}^{x} t^n F(\phi(t))dt \leq \int_{x_0}^{x} t^n H(\phi(t))dt \overset{\text{Götze\&Sjögren}}{\leq} (1-\varepsilon) \int_{x_0}^{x} t^n \phi(t)dt.$$

proof: Hence we have for all $n \in \mathbb{N}_0$

$$\int_{x_0}^{x} t^n F(\phi(t))dt \leq (1-\varepsilon) \int_{x_0}^{x} t^n \phi(t)dt. \qquad (5)$$

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Integrating the integro-differential equation from x_0 to x, we obtain

$$\int_{x_0}^{x} \phi(t)dt + \int_{x_0}^{x} \dot{\phi}(t)dt + \int_{x_0}^{x} \frac{d}{dt} \left(\int_0^t F(\phi(s))\phi(t-s) - F(\phi(s))ds \right) dt = 0.$$

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Since for all $t \in [0,\infty)$ $F(\phi(t)) \geq 0$ and $\phi(t) \geq 0$, we conclude

$$\begin{aligned} \int_{x_0}^{x} \phi(t)dt \quad &\leq \phi(x_0) - \phi(x) + \int_{x_0}^{x} F(\phi(t))dt + \int_0^{x_0} F(\phi(s))\phi(x_0 - s)ds \\ &\stackrel{\phi\geq 0,(5)}{\leq} \phi(x_0) + (1-\varepsilon)\int_{x_0}^{x} \phi(t)dt + \int_0^{x_0} F(\phi(s))\phi(x_0 - s)ds. \end{aligned}$$

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Since for all $t \in [0,\infty)$ $F(\phi(t)) \geq 0$ and $\phi(t) \geq 0$, we conclude

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$$\Rightarrow \quad \int_{x_0}^{x} \phi(t)dt \leq \frac{1}{\varepsilon}\phi(x_0) + \frac{1}{\varepsilon}\int_0^{x_0} F(\phi(x_0 - s))\phi(s)ds \overset{\text{indep. of } x}{<} \infty.$$

proof: We conclude

$$\int_0^\infty \phi(t)dt < \infty \quad \text{and} \quad \int_0^\infty F(\phi(t))dt < \infty.$$

proof:
Multiplying the integro-DE with t^n and integrating from x_0 to x we obtain by a similar calculation

$$\int_{x_0}^{x} t^n \phi(t) dt \leq C(x_0) + n \int_{x_0}^{x} t^{n-1} \phi(t) dt + \int_{x_0}^{x} t^n F(\phi(t)) dt + n \int_{x_0}^{x} \int_{0}^{t} t^{n-1} F(\phi(s)) \phi(t-s) ds dt,$$

where $C(x_0)$ only depends on integrals with integration limits 0 and x_0.

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where $C(x_0)$ only depends on integrals with integration limits 0 and x_0.
Following estimate was presented in the work of Götze&Sjögren

$$n \int_{x_0}^{x} \int_{0}^{t} t^{n-1} F(\phi(s)) \phi(t-s) ds dt \\ \leq n \sum_{i=0}^{n-1} \binom{n-1}{i} \int_{0}^{x} t^{n-1-i} \phi(t) dt \int_{0}^{x} t^{i} F(\phi(t)) dt.$$

proof: We conclude

$$\begin{aligned}
\int_{x_0}^{x} t^n \phi(t)dt \leq\ & C(x_0) + n\int_{x_0}^{x} t^{n-1}\phi(t)dt + (1-\varepsilon)\int_{x_0}^{x} t^n\phi(t)dt \\
& + n\sum_{i=0}^{n-1}\binom{n-1}{i}\int_{x_0}^{x} t^{n-1-i}\phi(t)dt \int_{x_0}^{x} t^i F(\phi(t))dt \\
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& + n\sum_{i=0}^{n-1}\binom{n-1}{i}\int_{0}^{x_0} t^{n-1-i}\phi(t)dt \int_{0}^{x_0} t^i F(\phi(t))dt.
\end{aligned}$$

proof: We conclude

$$\begin{aligned}\int_{x_0}^{x} t^n\phi(t)dt \quad &\leq \quad C(x_0) + n\int_{x_0}^{x} t^{n-1}\phi(t)dt + (1-\varepsilon)\int_{x_0}^{x} t^n\phi(t)dt \\ &+ n\sum_{i=0}^{n-1}\left(\begin{array}{c} n-1 \\ i \end{array}\right)\int_{x_0}^{x} t^{n-1-i}\phi(t)dt\int_{x_0}^{x} t^i F(\phi(t))dt \\ &- n\sum_{i=0}^{n-1}\left(\begin{array}{c} n-1 \\ i \end{array}\right)\int_{0}^{x_0} t^{n-1-i}\phi(t)dt\int_{x_0}^{x} t^i F(\phi(t))dt \\ &- n\sum_{i=0}^{n-1}\left(\begin{array}{c} n-1 \\ i \end{array}\right)\int_{x_0}^{x} t^{n-1-i}\phi(t)dt\int_{0}^{x_0} t^i F(\phi(t))dt \\ &+ n\sum_{i=0}^{n-1}\left(\begin{array}{c} n-1 \\ i \end{array}\right)\int_{0}^{x_0} t^{n-1-i}\phi(t)dt\int_{0}^{x_0} t^i F(\phi(t))dt.\end{aligned}$$

With a proof by induction, we obtain

$$\int_{x_0}^{x} t^n\phi(t)dt \overset{\text{indep. of } x}{<} \infty \quad \text{and} \quad \int_{x_0}^{x} t^n F(\phi(t))dt \overset{\text{indep. of } x}{<} \infty.$$

proof: We conclude the existence of the following improper integrals

$$\int_0^\infty t^n \phi(t) dt \quad \text{and} \quad \int_0^\infty t^n F(\phi(t)) dt$$

for all $n \in \mathbb{N}_0$. $\square$

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for all $n \in \mathbb{N}_0$. □

Corollary (proof Götze&Sjögren)

Let $F : [0,1] \to \mathbb{R}$ be continuously differentiable and monotonically increasing with the following conditions

i) $F(x) < \frac{x}{1-x}, x \neq 0,$
ii) $F(0) = 0,$
iii) $F'(0) < 1.$

Let ϕ be the solution of (1) with kernel-function F (convergent to 0). Then one has for all $n \in \mathbb{N}_0$

$$\lim_{t\to\infty} t^n \phi(t) = 0.$$

For the last result, the restriction $F'(0) < 1$ was needed. The following theorem is an input for the discussion of asymptotic behaviour in case of $F'(0) = 1$:

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Theorem

Let $F : [0,1] \to \mathbb{R}$ be differentiable and monotonically increasing, with

$$\exists c \in (0,1]\ \forall x \in [0,1] : 0 \leq F(x) \leq c \cdot x.$$

Then one has for the solution ϕ of (1) with F (convergent to 0)

$$\phi(t) \leq c^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}}, \quad t \in [0,\infty).$$

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By variation of constants, we obtain

$$\begin{aligned}
\phi(t) &= e^{-t} - e^{-t}\int_0^t e^s \int_0^s F(\phi(s-\tau))\dot{\phi}(\tau)d\tau ds \\
&\leq e^{-t} - e^{-t}\int_0^t e^s \int_0^s c\phi(s-\tau)\dot{\phi}(\tau)d\tau ds \\
&\overset{(6)}{=} e^{-t} - e^{-t}\int_0^t e^s \left(\frac{d}{ds}c\int_0^s \phi(s-\tau)\phi(\tau)d\tau \quad - c\phi(s)\right) ds \\
&\overset{\text{part.int.},c\leq 1}{\leq} e^{-t} + e^{-t}\int_0^t e^s \cdot c\int_0^s \phi(s-\tau)\phi(\tau)d\tau ds \\
&\qquad - c\int_0^t \phi(t-s)\phi(s)ds + e^{-t}\int_0^t e^s\phi(s)ds.
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We conclude

$$e^t\phi(t)+e^t\cdot c\int_0^t \phi(t-s)\phi(s)ds \leq 1+\int_0^t e^s\phi(s)+e^s\cdot c\int_0^s \phi(s-\tau)\phi(\tau)d\tau ds.$$

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one has

$$\tilde{\phi}(t) + (1-g)\dot{\tilde{\phi}}(t) + \int_0^t \tilde{F}(\tilde{\phi}(s))\dot{\tilde{\phi}}(t-s)ds = 0, \quad \tilde{\phi}(0) = 1.$$

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where $\tilde{\psi}(t) \stackrel{t\to\infty}{\longrightarrow} 0 \Leftrightarrow \phi(t) \stackrel{t\to\infty}{\longrightarrow} g$.

Time-dependent kernel-functions

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We remember the physically relevant problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2(t-s)^2}\dot{\phi}(s)ds = 0, \tag{7}$$

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Thus, it is useful, to discuss kernel-functions of the type

$$F(x,s) = f(x)\cdot g(s), \quad f:\mathbb{R}\to\mathbb{R}, \quad g:[0,\infty)\to\mathbb{R}$$

The problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t f(\phi(t-s))g(t-s)\dot{\phi}(s)ds = 0, \quad \phi(0) = 1$$

is equivalent to the following fixed point problem

$$\phi(t) = 1 + \int_0^t f(\phi(s))g(s) - \phi(s) - f(\phi(s))g(s)\phi(t-s)ds.$$

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One can treat this problem by using Banach fixed point theorem analogously to the previous chapter.

Hence, we conclude the following existence theorem for bounded kernel-functions:

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ *be bounded and locally Lipschitz continuous and* $g : [0, \infty) \to \mathbb{R}$, $s \mapsto g(s)$ *continuous.*
Then the problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t f(\phi(t-s))g(t-s)\dot{\phi}(s)ds = 0, \quad \phi(0) = 1$$

has a unique solution $\phi \in C^1([0, \infty), \mathbb{R})$.

$$\phi(t) + \dot{\phi}(t) + \int_0^t f(\phi(t-s))g(t-s)\dot{\phi}(s)ds = 0 \tag{8}$$

We are interested in monotone solutions under some restrictions on f and g. We will go ahead analogously to the previous chapter:

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1. Differentiating (8) comes to

$$[1 + f(1)g(0)]\,\dot{\phi}(t) + \ddot{\phi}(t)$$
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ϕ is monotonically decreasing if one of the following cases is valid

$$\begin{array}{ccc} f' \geq 0,\ g \geq 0 & & f \geq 0,\ g' \leq 0 \\ \text{or} & \text{and} & \text{or} \\ f' \leq 0,\ g \leq 0 & & f \leq 0,\ g' \geq 0 \end{array}$$

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2. Formal limit $t \to \infty$ of (9) comes to

$$f(x_0) \cdot \lim_{t\to\infty} g(t) = \frac{x_0}{1-x_0}, \text{ with } \lim_{t\to\infty} \phi(t) = x_0.$$

   A maximum fixed point of this equation (if ex.) is a candidate for a limit of ϕ.

Thus we are able to formulate the following theorems

Theorem (without proof)

Let $g : [0,\infty) \to \mathbb{R}$*, s.t. the limit* $\bar{g} := \lim_{t\to\infty} g(t)$ *exists.*

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Then the problem (9) with f *and* g *has a unique solution* $\phi \in C^1([0, \infty), \mathbb{R})$*, where* ϕ *is monotonically decreasing, with* $x_0 \leq \phi(t) \leq 1$ *for all* $t \in [0, \infty)$*.*

Theorem (without proof)

i) *Under the conditions of the previous theorem, the solution* ϕ *converges to the maximum fixed point* $x_0 < 1$ *of the equation*

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$$f'(x_0) \cdot \bar{g} < \frac{1}{(1 - x_0)^2},$$

then one has for all $n \in \mathbb{N}$ *at*

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iii) *If instead of (ii) one has*

$$f(x)g(s) \leq \frac{c}{(1 - x_0)^2} + \frac{x_0}{1 - x_0} - c\frac{x_0}{(1 - x_0)^2},$$

with $c \in (0, 1]$ *f.a.* $(x, s) \in [x_0, 1] \times [0, \infty)$*,*
then one has

$$|\phi(t) - x_0| \leq (1 - x_0)^{\frac{3}{2}} c^{-\frac{1}{2}} t^{-\frac{1}{2}}, \quad t \in [0, \infty).$$

Applying these results to the physical example, we arrive at the following result:

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Corollary

Let $f : [0,1] \to \mathbb{R}$ *be non-negative, differentiable, monotonically increasing and (locally-)Lipschitz continuous, then the problem*

$$\phi(t) + \dot{\phi}(t) + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2(t-s)^2}\dot{\phi}(s)ds = 0, \quad \phi(0) = 1$$

has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$, *where* ϕ *is monotonically decreasing, with*

$$\lim_{t\to\infty} t^n \phi(t) = 0.$$

Remark

To treat the second physical example

$$\phi(t) + \dot{\phi}(t) + \int_0^t h(t)h(t-s)f(\phi(t-s))\dot{\phi}(s)ds = 0, \quad \phi(0) = 1,$$

h' needs to be locally bounded. This is obvious when one regards the following equivalent fixed point problem

$$\phi(t) = 1 + \int_0^t h^2(s)f(\phi(s)) - \phi(s) - h(t)h(s)f(\phi(s))\phi(t-s)ds$$
$$+ \int_0^t \int_0^s h'(s)h(r)f(\phi(r))\phi(s-r)drds.$$

The limit of the solution ist then given by the maximal fixed point of the following equation

$$f(x_0)\overline{h}^2 = \frac{x_0}{1-x_0},$$

where $\overline{h} := \lim_{t\to\infty} h(t)$.

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- Improvement of the asymptotic-results in case of $F'(g) = \frac{1}{(1-g)^2}$
- Well-Posedness and asymptotic behaviour in case of unbounded, non-monotone kernel-functions. A background is the following example in physics

$$\phi(t) + \dot{\phi}(t) + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2 sin^2(\omega(t-s))} \dot{\phi}(s)ds = 0, \quad \phi(0) = 1$$

Open questions

- Treatment of ordinary integro-differential equations with complex-valued kernel-functions. A background is the following example of a coupled system in physics

$$\dot{\phi}_1(t) + \omega_1 \phi_1(t) + \omega_1 \int_0^t \frac{f_1(\overline{\phi_1(t-s)}, \phi_2(t-s))}{1 - ik_1 F} \dot{\phi}_1(s) ds \overset{\mathbb{C}}{=} 0,$$

$$\dot{\phi}_2(t) + \omega_2 \phi_2(t) + \omega_2 \int_0^t \frac{f_2(\phi_2(t-s), Re(\phi_1(t-s)))}{1 + (k_2 F)^2} \dot{\phi}_2(s) \overset{\mathbb{R}}{=} 0,$$

$$f_1, f_2 \sim \text{linear}, \omega_1 \in \mathbb{C}, \omega_2 \in \mathbb{R}, f_1 \in \mathbb{C}, f_2 \in \mathbb{R}, \phi_1 \in \mathbb{C}, \phi_2 \in \mathbb{R}$$

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- Treatment of **partial** integro-differential equations

Thanks for your attention