# Ordinary and partial integro-differential equations with applications in glass-rheology

Patrick Kurth

*Petrópolis-RJ, September 2013*

# Introduction

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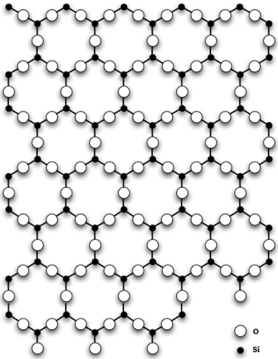

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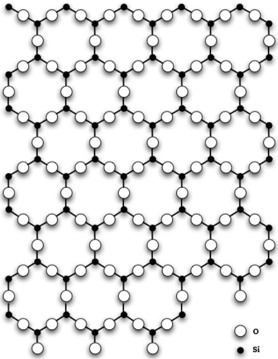
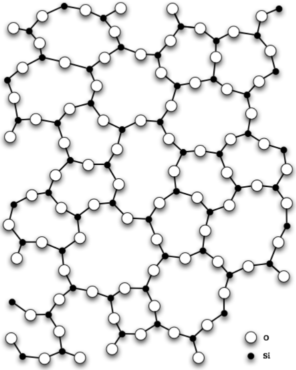



solid (silicon dioxide)




glass

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- $\lim_{t\to\infty} \phi(t) = 0$: material stays viscous
  $\lim_{t\to\infty} \phi(t) \neq 0$: glass-transition

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- Treating problems for certain partial integro-differential equations, e.g.

$$\partial_t u(t,x) - \triangle u(t,x) + \int_0^t F(u(t-s,x))\partial_s u(s,x)ds = 0, \quad +\text{IV}, +\text{BC}$$

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It follows: $\phi$ ist strongly monotonically decreasing.

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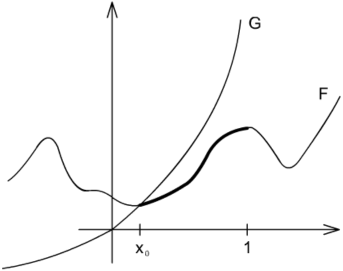

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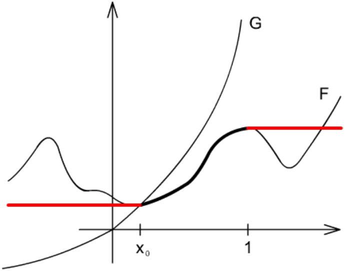

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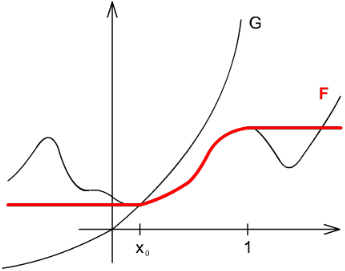

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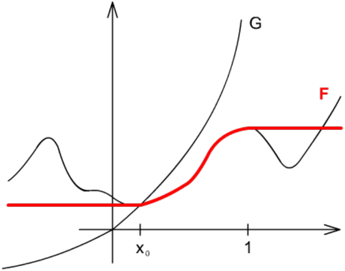


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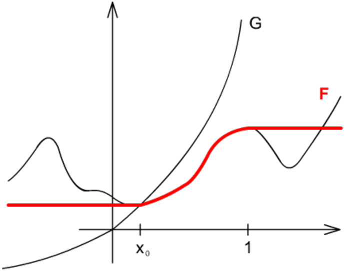


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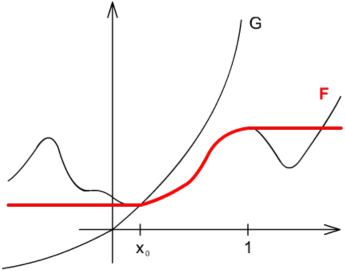


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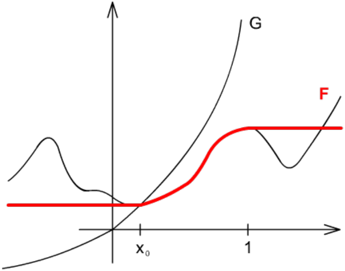


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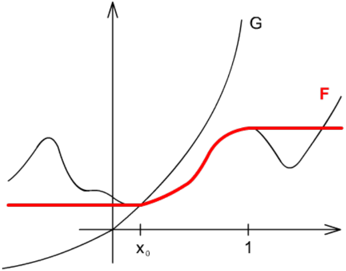


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In particular, $\phi$ is convergent.

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*Then problem (1) with* $F$ *has a unique solution* $\phi \in C^1([0, \infty), \mathbb{R})$ *that is monotonically decreasing and convergent against the maximal fixed-point* $g \in [x_0, 1)$ *of* $F$ *with* $G$.

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W. Götze, L. Sjögren: General Properties of Certain Non-linear Integro-Differential Equations, Journal of Mathematical Analysis and Applications 195, 230-250 (1995):

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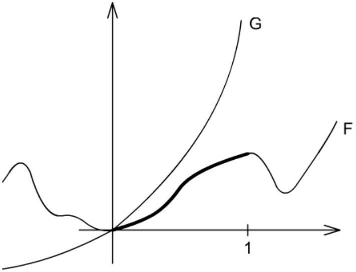
  ansatz: construction of an absolutely monotone function $H : [0,1] \to \mathbb{R}$ that fulfils $H'(0) < 1$ and $F(x) \le H(x) \le G(x)$ for all $x \in [0,1)$.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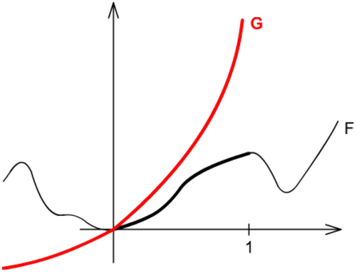
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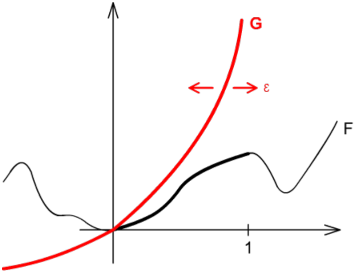
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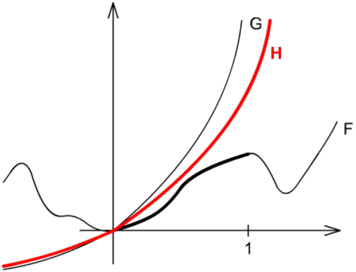
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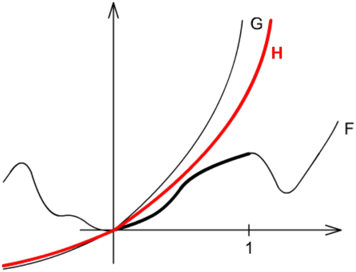


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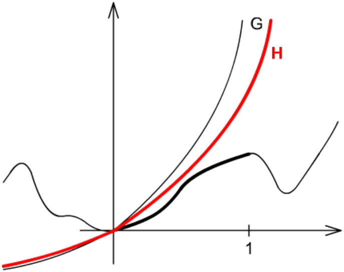


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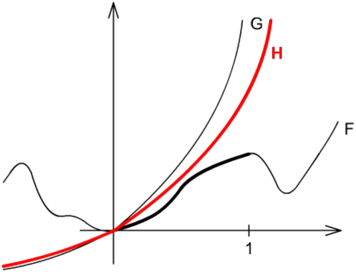


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$\exists \varepsilon \in (0,1), x_0 > 0 \ \forall n \in \mathbb{N}, x > x_0 :$

$$\int_{x_0}^{x} t^n F(\phi(t))dt \leq \int_{x_0}^{x} t^n H(\phi(t))dt \overset{\text{Götze,Sjögren}}{\leq} (1-\varepsilon)\int_{x_0}^{x} t^n \phi(t)dt$$

We need:

$H'(0) < 1 \Leftarrow F'(0) < 1$

## Theorem

*Let* $F : \mathbb{R} \to \mathbb{R}$ *and* $g < 1$ *the maximal fixed-point of* $F(x) = \frac{x}{1-x}$.
*Furthermore, let* $F$ *be continuously differentiable and monotonically increasing on* $[g, 1]$ *with* $F'(g) < \frac{1}{(1-g)^2}$.
*Then problem (1) with* $F$ *has a unique solution* $\phi \in C^1([0, \infty), \mathbb{R})$ *that is monotonically decreasing, satisfying*

$$\exists s_0 > 0 : \lim_{t \to \infty} e^{s_0 t} \left[\phi(t) - g\right] = 0.$$

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- In case of $g = 0$, $F'(0) = 1$, $F(x) \leq x$ one has $\Phi(t) \sim t^{-\frac{1}{2}}$.

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We start considering the following related linear problem:

$$\phi(t) + \dot{\phi}(t) + \int_0^t m(t-s)\dot{\phi}(s)ds = 0, \quad \phi(0) = 1, \tag{5}$$

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1. If $m'$ is exponentially decaying (resp. polynomial decaying) "fast enough", then $\phi$ converges exponentially (resp. polynomially) against zero.
2. Fixed-point arguments: Let $u$ be an element out of a suitable class of functions and $Tu$ be the solution of (5) with $m = F \circ u$. Schauders fixed-point theorem leads to a solution of the nonlinear problem.

## Theorem

*Let* $\varepsilon \in (0,1)$ *and* $f \in C^1\left(\left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right], \mathbb{R}\right)$ *twice differentiable in* $x = 0$, $f(0) = f'(0) = 0$ *and* $f(1) > -1$. *Then there exists a* $\kappa \in (0,1]$ *such that: Problem (1) with* $F := \kappa \cdot f$ *has a unique solution* $\phi \in C^1([0,\infty),\mathbb{R})$ *satisfying*

$$|\phi(t)| \leq \frac{4}{3+3\kappa f(1)} e^{-\frac{3+3\kappa f(1)}{4}t} \quad \textit{and} \quad |\dot{\phi}(t)| \leq e^{-\frac{3+3\kappa f(1)}{4}t}.$$

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## Theorem

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(i) $F(x) = \frac{27}{2624}(x^2 - x^4)$,

(ii) $F(x) = \pm\left(\frac{2}{3}\sqrt{21} - 3\right) x^2$.

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- The above presented methods can easily be extended on problems with more-parametric kernel-functions:

$$\dot{\phi}(t) + \phi(t) + \int_0^t F(\phi(t-s), t-s, s)\dot{\phi}(s)ds = 0, \quad \phi(0) = 1.$$

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Let $G \subseteq \mathbb{R}^n$ a bounded domain, $A = \sum_{i,j=1}^{n} -\partial_i a_{ij}(\cdot)\partial_j + a(\cdot)$ an elliptic operator with positive spectrum $\sigma(A) \subseteq [q, \infty)$ $(q > 0)$.

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$$u_t(t,x) + Au(t,x) + \int_0^t F(u(t-s,x))u_t(s,x)ds = 0, \tag{6}$$

$$\text{IC: } u(0,x) = u_0(x), \ x \in G$$

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where $C_{\mu,\gamma,p} \geq 0$ and $\alpha^i_{l,p} \in \mathbb{N}_0^n$ with $\alpha^i_{l,p} \leq \beta$ and $|\alpha^i_{l,p}| = i$.

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$$\|\partial^\beta(F(u))\| \leq \sum_\mu \sum_\gamma \sum_p C_{\mu,\gamma,p} \|F^{(\mu)}(u)\|_\infty \left\| \prod_{i=1}^{|\beta|} \prod_{l=1}^{\gamma_i} \partial^{\alpha^i_{l,p}} u \right\|$$

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Gagliardo-Nirenberg: $\frac{1}{s} = \frac{h}{mq} \Rightarrow \|u\|_{W^{h,s}(G)} \leq C\|u\|^{\frac{h}{m}}_{W^{m,q}(G)} \|u\|^{1-\frac{h}{m}}_{L^\infty(G)}$.

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It follows

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Still to prove: $F(u) \in D(A^k)$ (boundary values).

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One shows by iteration: $F(u) \in D\left(A^k\right)$

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$4k > n$, $F \in C^{2k}(\mathbb{R},\mathbb{R})$, $|F^{(i)}(x)| \leq v_1|x|^\alpha$ $(i = 1,\ldots,2k)$,
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*Let $u_0 \in D(A^{k+1})$ and $m \in C^1([0,\infty), D(A^k))$ with $m(0)(x) \geq -q + \varepsilon$ for a $\varepsilon > 0$ and for all $x \in G$. Furthermore, let $m_t(t)v \in D(A^k)$ for all $t \in [0,\infty)$ and for all $v \in D(A^k)$. In addition to that, let $\|m_t(t)\|_{D(A^k)} \leq \omega e^{-c_1 t}$ and $\lim_{t\to\infty} \|m(t)\|_{D(A^k)} = 0$, where $c_1 > \varepsilon$ and $\omega > 0$ such that $C_3\omega < \varepsilon(c_1 - \varepsilon)$ and $\frac{C_0}{C_1}\omega < \varepsilon(c_1 - \varepsilon)$. Then one has*

$$\|u_t(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}$$

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$$\text{as well as} \quad \|u_t(t)\| \leq \|u_0\|_{D(A)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon}t}$$

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Let $4k > n$, $G \subseteq \mathbb{R}^n$ be a bounded domain with $C^{2k}$-boundary, $u_0 \in D(A^{k+1})$ and $F \in C^{2k+1}(\mathbb{R}, \mathbb{R})$ with $F^{(2k+1)}$ locally Lipschitz-continuous and $F(u_0(x)) > -q + \varepsilon$ for a $\varepsilon > 0$ and for all $x \in G$.

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$$v_1 C_3 C_4 \|Au_0\|_{D(A^k)}^{\alpha+1} \left(\frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1 - \varepsilon)}\right)^{\alpha} \leq \omega$$

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Furthermore, let

*v*) $\left|F^{(i)}(x)\right| \le v_1|x|^{\alpha}$, $i = 0, \ldots, 2k+1$, $x \in \mathbb{R}$.

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$$\mathcal{C} := \left\{ \begin{array}{l} u \in C^0([0,\infty), D(A^k)) \cap C^1([0,\infty), D(A^{k-1})) : u(0) \overset{D(A)}{=} u_0, \\ \|u(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} \frac{\varepsilon - c_1}{C_3\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} t}, \\ \|u_t(t)\|_{D(A^k)} \leq \|Au_0\|_{D(A^k)} e^{\frac{C_3\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} t}, \\ \|u(t)\| \leq \|u_0\|_{D(A)} \frac{\varepsilon - c_1}{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} t}, \\ \|u_t(t)\| \leq \|u_0\|_{D(A)} e^{\frac{\frac{C_0}{C_1}\omega - \varepsilon(c_1-\varepsilon)}{c_1-\varepsilon} t} \end{array} \right\}.$$

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$$\mathcal{T} : \mathcal{C} \longrightarrow \mathcal{C},\ v \mapsto \mathcal{T}(v),$$

where $\mathcal{T}(v)$ is the solution of the related linear problem with kernel-function $m = F \circ v$.

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Due to the smallness-conditions on $F$ (resp. $u_0$), $\mathcal{T}$ is well-defined.

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To this, let $u^0 \in \mathcal{C}$, $u^n := \mathcal{T}(u^{n-1})$ $(n \in \mathbb{N})$.

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## Theorem

*Let* $n \leq 3$ *and* $f \in C^3\left(\left[-\frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}, \frac{C_0}{C_1}\|Au_0\|_{D(A)}\frac{2}{q}\right], \mathbb{R}\right)$*, four times differentiable in* $x = 0$ *with* $f'''$ *locally Lipschitz-continuous and* $f(0) = f'(0) = f''(0) = f'''(0) = 0$*. Then there exists a* $\kappa > 0$ *such that the problem*

$$u_t(t,x) + Au(t,x) + \int_0^t F(u(t-s,x))u_t(s,x)ds = 0,$$

$$u(0,x) = u_0(x), \quad u|_{[0,\infty)\times\partial G} = 0,$$

*with* $F = \kappa \cdot f$ *has a unique solution* $u \in C^1([0,\infty), D(A))$ *such that* $u$ *and* $u_t$ *decay exponentially with respect to the norms* $\|\cdot\|_{D(A)}$ *and* $\|\cdot\|$*.*

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The techniques can be used to treat more easier problems for equations with kernels, that are independend of the space-variable $x$:

$$u_t(t,x) + Au(t,x) + \int_0^t F(u(t-s))u_t(s,x)ds = 0, \qquad (7)$$

$$\text{IC: } u(0,x) = u_0(x),\ \ x \in G$$

$$\text{BC: } u(t,x) = 0,\ \ x \in \partial G,$$

where $F : L^2(G) \to \mathbb{R}$.

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- Improvement of the conditions on the kernel-functions.
- Treating problems in unbounded domains, e.g. whole space, half-space or exterior domains.

Thanks for your attention.