USPC
Université Sorbonne
Paris Cité

PARIS DIDEROT

Thèse de doctorat de

L'Université de Constance et de

L'Université Sorbonne Paris Cité

Préparée à l'Université Paris Diderot

Ecole doctorale de Sciences Mathématiques de Paris Centre (ED386)

IMJ-PRG / Equipe de logique

Quasi-orders, C-groups, and the differentiel rank of a differential-valued field

Par Gabriel Lehéricy

Thèse de doctorat de Mathématiques

Dirigée par Salma Kuhlmann et Françoise Point

Présentée et soutenue publiquement à Paris le 12 septembre 2018

Président du jury :	Cluckers, Raf / Directeur de recherche / CNRS
Rapporteur :	Hils, Martin / Professeur / Université de Münster
Examinateurs :	Rolin, Jean-Philippe / Maître de conférences / Université de Bourgogne
	Servi, Tamara / Maître de conférences / Université paris Diderot
	Halupczok, Immanuel / Professeur / HHU Düsseldorf
	Kaiser, Tobias / Professeur / Université de Passau
Directeur de thèse :	Point, Françoise / Directeur de recherche / FNRS-FRS
Co-directeur de thèse :	Kuhlmann, Salma / Professeur / Université de Constance

Titre: Quasi-ordres, C-groupes, et rang différentiel d'un corps différentiel valué.

Résumé: Cette thèse a pour objet les ordres, les valuations et les C-relations sur les groupes, ainsi que les corps différentiels valués tels qu'étudiés par Rosenlicht. Elle accomplit trois objectifs principaux. Le premier est d'introduire et d'étudier une notion de quasi-ordre sur les groupes qui a pour but de réunir les ordres et les valuations dans un même cadre. Nous donnons un théorème de structure des groupes munis d'un tel quasi-ordre, ce qui nous permet ensuite de donner un "théorème de plongement de Hahn" pour ces groupes. Le second objectif de cette thèse est de décrire les C-groupes à l'aide des quasi-ordres. Nous donnons un théorème de structure pour les C-groupes, qui énonce que tout C-groupe est un "mélange" de groupes ordonnés et de groupes valués. Nous utilisons ensuite ce résultat pour caractériser les groupes C-minimaux à l'intérieur de la classe des C-groupes. Le troisième objectif de cette thèse est d'introduire et d'étudier une notion de rang différentiel d'un corps différentiel valué. Nous définissons cette notion par analogie avec les notions de rang exponentiel d'un corps exponentiel et de rang de différence d'un corps aux différences. Nous montrons que cette notion de rang n'est pas tout à fait satisfaisante, et introduisons donc une meilleure notion de rang appelée le rang différentiel déployé. Nous donnons ensuite une méthode pour définir une dérivation "de type Hardy" sur un corps de séries formelles généralisées, ce qui nous permet de construire des corps différentiels valués dont le rang différentiel et le rang différentiel déployé ont été arbitrairement choisis.

Mots clefs: ordres, quasi-ordres, valuations, C-groupes, corps différentiels valués, couples asymptotiques, C-minimalité.

Title: Quasi-orders, C-groups and the differential rank of a differential-valued field.

Abstract: This thesis deals with orders, valuations and C-relations on groups, and with differential-valued fields *à la* Rosenlicht. It achieves three main objectives. The first one is to introduce and study a notion of quasi-order on groups meant to encompass orders and valuations in a common framework. We give a structure theorem for groups endowed with such a quasi-order, which then allows us to give a "Hahn's embedding theorem" for these groups. The second objective of this thesis is to describe C-groups via quasi-orders. We give a structure theorem for C-groups, which basically states that any C-group is a "mix" of ordered groups and valued groups. We then use this result to characterize C-minimal groups inside the class of C-groups. The third objective of this thesis is to introduce and study a notion of differential rank for differential-valued fields. We define this notion by analogy with the exponential rank of an exponential field and with the difference rank of a difference field. We show that this notion of rank is not quite satisfactory, so we introduce a better notion of rank called the unfolded differential rank. We then give a method to define "Hardy-type" derivations on fields of generalized power series, which allows us to build differential-valued fields of arbitrary given differential rank and unfolded differential rank.

Keywords: orders, quasi-orders, valuations, C-groups, differential-valued fields, asymptotic couples, C-minimality.

Zusammenfassung

Diese Doktorarbeit handelt von auf Gruppen definierten Ordnungen, Bewertungen und C-Relationen, und von den von Rosenlicht eingeführten differentiell bewerteten Körpern. Hier werden drei Ziele verfolgt. Das erste Ziel ist das Einführen und Studieren von Quasiordnungen für abelsche Gruppen, die dazu dienen sollen, Bewertungen und Ordnungen unter einem gemeinsamen Begriff zu behandeln. Das zweite Ziel ist die Beschreibung der C-Gruppen mithilfe der Quasiordnungen und die Charakterisierung der C-minimalen Gruppen. Das dritte Ziel ist das Einführen und Studieren eines Begriffs von differentiellem Rang für differentiell bewertete Körper.

Obwohl Ordnungen und Bewertungen klassischerweise als unterschiedliche Themen behandelt werden, existieren merkwürdige Ähnlichkeiten zwischen diesen beiden Objekten. Aus diesem Grund wurde mehrmals versucht, eine Theorie zu entwickeln, die Ordnungen und Bewertungen vereinigen würde. In [Fak87] schlug Fakhruddin vor, dieses Problem mit Hilfe von Quasiordnungen (d.h. von binären Relationen, die reflexiv, transitiv und total sind) zu lösen. Fakhruddin hat in [Fak87] den Begriff "quasi-geordneter Körper" eingeführt, wobei die Quasiordnung bestimmte Bedingungen von Verträglichkeit mit den Körperoperationen erfüllt. Geordnete und bewertete Körper sind zwei Beispiele von solchen Strukturen (jede Bewertung induziert auf kanonische Weise eine Quasiordnung). Noch merkwürdiger ist der Hauptsatz in [Fak87], der sagt, dass jede Quasiordnung auf einem Körper entweder eine Ordnung oder eine Bewertung sein muss. Dann stellt sich aber die Frage, ob eine ähnliche Behauptung auch für abelsche Gruppen gilt. Solch eine Untersuchung wird dadurch motiviert, dass geordnete und bewertete Gruppen eine wichtige Rolle in der Theorie der mit einem Operator versehenen bewerteten Körper spielen. Falls zum Beispiel (G, ψ) das zu einem differentiell bewerteten Körper assoziierte asymptotische Paar ist, ist G eine geordnete Gruppe und ψ eine Bewertung auf G. Die Theorie der Quasiordnungen auf Gruppen bietet die Möglichkeit, gleichzeitig Eigenschaften von geordneten und von bewerteten Gruppen, sowohl wie auch die Interaktion zwischen ihnen, zu entdecken.

Die Frage, ob die Fakhruddin'sche Dichotomie ein Analoges für abelsche Gruppen besitzt, war der Startpunkt des Kapitels 3 meiner Doktorarbeit. Da führe ich den Begriff "kompatible Quasiordnung" für abelsche Gruppen ein (Definition 3.1.1), was das Analoge der in [Fak87] betrachteten Quasiordnungen ist. Ich habe schnell festgestellt, dass die Fakhruddin'sche Dichotomie für Gruppen falsch ist: es gibt ja kompatible Quasiordnungen, die weder Ordnungen noch Bewertungen sind (siehe Beispiel 3.1.2). Der Hauptsatz

des Kapitels 3 ist ein Struktursatz (Satz 3.1.26) für mit einer kompatiblen Quasiordnung versehene abelsche Gruppen. Dieser Satz sagt ungefähr, dass jede kompatible Quasiordnung eine "Mischung" aus Ordnung und Bewertung ist. Anders gesagt ist jede abelsche Gruppe, die mit einer kompatiblen Quasiordnung versehen ist, eine Erweiterung einer bewerteten Gruppe durch eine geordnete Gruppe. Außerdem ist der "geordnete Teil" der Gruppe immer ein Anfangsstück der Gruppe.

Nachdem ich diesen Struktursatz bewiesen habe, habe ich versucht, manchen bekannten Ergebnissen aus der Theorie der geordneten Strukturen ein Analoges für quasi-geordnete Gruppen zu geben. Abschnitt 3.2 führt einen Begriff von Kompatibilität zwischen Quasiordnungen und Bewertungen ein, der das Analoge zum Begriff "konvexe Bewertung" ist. Im Satz 3.2.2 werden mehrere Charakterisierungen der Kompatibilität gegeben. Ich habe aber dann festgestellt, dass es keinen "Baer-Krull-Satz" für kompatible Quasiordnungen gibt (siehe Beispiel 3.2.5). Im Abschnitt 3.3 wird ein Produkt für mit kompatiblen Quasiordnungen versehenen Gruppen eingeführt (Definition 3.3.1). Mithilfe dieses Produkts wird dann der bekannte "Hahns-Einbettung-Satz" zu quasi-geordneten Gruppen verallgemeinert (siehe Satz 3.3.8).

Im letzten Teil des Kapitels 3 definiere ich den Begriff "q.o.-Minimalität", was eine Verallgemeinerung der o-Minimalität für quasi-geordnete Gruppen ist. Diese Definition beruht auf dem Begriff von "swiss cheese", den Holly in [Hol95] für bewertete Körper eingeführt hat. Danach wird gezeigt, dass jede kompatible Quasiordnung eine C-Relation induziert, und dass die q.o.-Minimalität einer Gruppe zu ihrer C-Minimalität äquivalent ist. Das öffnet den Weg für Kapitel 4, das sich mit C-Gruppen befasst.

Eine C-Relation (siehe [AN96]) ist eine drei-stellige Relation, die in der Menge der Zweige eines Baums interpretierbar ist. Genauer gesagt: Wenn T ein Baum ist, in dem jedes Paar (a, b) ein Infimum besitzt, und wenn M in der Menge aller Zweige von T enthalten ist, dann können wir folgenderweise eine C-Relation auf M definieren: wir sagen, dass $C(x, y, z)$ genau dann wahr ist, wenn der Schnittpunkt von x und y kleiner als der Schnittpunkt von y und z ist. Umgekehrt kann man zeigen, dass jede C-Relation auf irgendwelcher Menge sich auf diese Weise in der Menge aller Zweige eines Baums interpretieren lässt. In [MS96] wurden die Begriffe "C-Gruppe" und "C-Minimalität" eingeführt. Eine C-Gruppe ist eine mit einer C-Relation versehene Gruppe, so dass die C-Relation mit der Gruppenoperation verträglich ist. Die C-Minimalität ist ein Analoges der o-Minimalität, wobei die Ordnung durch eine C-Relation ersetzt wird. Bis jetzt gab es keine Beschreibung der C-Gruppen. Das Ziel des Kapitels 4 ist, die C-Gruppen zu beschreiben und die C-minimalen Gruppen in der Klasse aller C-Gruppen zu charakterisieren.

Am Ende des Kapitels 3 habe ich bemerkt, dass jede kompatible Quasiordnung eine C-Relation induziert. Ich habe aber früh festgestellt, dass die C-Relation in manchen abelschen C-Gruppen nicht so aus einer kompatiblen Quasiordnung stammt. Das bedeutet, dass die Klasse der kompatiblen Quasiordnungen dafür nicht geeignet ist, die ganze Klasse der C-Gruppen zu beschreiben. Dann habe ich aber herausgefunden, dass es eine andere Art Quasiordnung gibt, die für das Studieren von allen (sogar von nicht unbedingt abelschen) C-Gruppen geeignet ist; solche Quasiordnungen nenne ich C-Quasiordnungen (siehe Definition 4.1.2). Wenn $(G, ., 1, C)$ eine (nicht unbedingt abelsche) C-Gruppe ist, dann induziert C eine Quasiordnung auf G auf folgende Weise: $x \precsim y$ ist genau dann wahr, wenn $\neg C(x, y, 1)$. Eine C-Quasiordnung ist eine Quasiordnung, die auf diese Weise von der C-Relation einer C-Gruppe induziert wird. Das Bemerkenswerte daran ist, dass die C-Relation einer C-Gruppe durch die induzierte C-Quasiordnung völlig charakterisiert wird: aus einer C-Quasiordnung kann man die C-Relation wieder konstruieren, von der sie induziert wurde. Es gibt also eine bijektive Korrespondenz zwischen der Klasse aller C-Quasiordnungen und der Klasse aller mit der Gruppenoperation verträglichen C-Relationen.

C-Quasiordnungen bilden das Untersuchungsobjekt des Kapitels 4. Die ganze Untersuchung der C-Gruppen wird nicht durch direkte Arbeit mit C-Relationen durchgeführt, sondern mit C-Quasiordnungen. Der Hauptsatz des Kapitels 4 ist der Satz 4.3.33, der ein Struktursatz für C-Gruppen ist. Es ist schon bekannt, dass geordnete und bewertete Gruppen zwei Beispiele von C-Gruppen sind (im Sinne, dass jede Ordnung und jede Bewertung eine C-Relation induziert). Der Satz 4.3.33 sagt ungefähr, dass diese zwei Beispiele die "elementaren Blöcke" aller C-Gruppen sind: jede C-Gruppe lässt sich aus geordneten und bewerteten Gruppen konstruieren.

Im Abschnitt 4.4 wende ich den Satz 4.3.33 auf C-Minimale Gruppen an. Mein Hauptsatz über C-minimalen Gruppen ist der Satz 4.4.37. Er sagt, dass jede C-minimale abelsche Gruppe ein endliches direktes Produkt von o-minimalen und C-minimalen bewerteten Gruppen ist.

Das letzte Kapitel dieser Doktorarbeit studiert differentiell bewertete Körper. Dabei meine ich differentielle Körper, die mit einer differentiellen Bewertung (im von Rosenlicht definierten Sinne, siehe [Ros80]) versehen sind. Im Kapitel 5 geht es darum, einen Begriff von differentiellem Rang für differentiell bewertete Körper zu entwickeln.

Der Rang (beziehungsweise, der Hauptrang) eines bewerteten Körpers (K, v) ist der Ordnungstyp der Menge aller Vergröberungen ("coarsenings") (beziehungsweise, aller Hauptvergröberungen) von v (wobei die Ordnung auf dieser Menge die Mengeninklusion ist). Der Rang wird auf drei Stufen charakterisiert: auf dem Körper K, auf der Bewertungsgruppe G von (K, v) (als die Menge aller konvexen Untergruppen von G) und auf der Bewertungskette Γ von G (als die Menge aller Endstücke von Γ). Mehrere Begriffe vom Rang für bewertete Körper mit einem Operator wurden auch definiert. Als Beispiele davon kann man den exponentiellen Rang eines exponentiellen Körpers nennen (siehe [Kuh00]) oder den σ-Rang eines Differenzkörpers (siehe [KMP17]). Die Idee ist immer, einen Rang zu haben, der Information über das Verhalten des Operators liefert. In [Kuh00] definiert die Autorin einen Begriff von exp-Kompatibilität für eine Bewertung auf einem exponentiellen geordneten Körper $(K, \leq, \exp)$. Sie definiert dann den exponentiellen Rang (beziehungsweise, den exponentiellen Hauptrang) von $(K, \exp)$ als den Ordnungstyp der Menge aller exp-kompatiblen Vergröberungen (beziehungsweise, aller exp-Hauptvergröberungen) der archimedischen Bewertung. Sie zeigt dann, dass in den von ihr betrachteten Körpern die Abbildung log eine Abbildung χ auf G induziert, und dass der exponentielle Rang von $(K, \exp)$ zur Menge aller χ-abgeschlossenen konvexen Untergruppen von G isomorph ist. Sie zeigt auch, dass χ eine Abbildung ζ auf Γ

induziert, und dass der exponentielle Rang auch zur Menge aller ζ-abgeschlossenen Endstücke von Γ isomorph ist. Eine ähnliche Arbeit wurde in [KMP17] für Differenzkörper durchgeführt. In [KMP17] wird ein Begriff von Kompatibilität zwischen einer Bewertung und dem Automorphismus σ eines Differenzkörpers (K, σ) eingeführt. Der σ-Rang (beziehungsweise, der σ-Hauptrang) eines bewerteten Differenzkörpers (K, v, σ) wird dann als der Ordnungstyp der Menge aller σ-kompatiblen Vergröberungen (beziehungsweise, aller σ-Hauptvergröberungen) von v definiert. Die Abbildung σ induziert die Abbildung σ_G auf G, und diese induziert die Abbildung σ_Γ auf Γ. Es wird in [KMP17] gezeigt, dass der σ-Rang von (K, v, σ) auch zur Menge aller σ_G-abgeschlossenen konvexen Untergruppen von G und zur Menge aller σ_Γ-abgeschlossenen Endstücke von Γ isomorph ist. Die Autoren von [KMP17] geben auch eine Charakterisierung des σ-Rangs bezüglich einer bestimmten von σ induzierten Äquivalenzrelation (siehe [KMP17, Theorem 5.3, Corollary 5.4, Corollary 5.5]).

Das Ziel des Kapitels 5 ist, durch Analogie mit den oben genannten Beispielen von Rängen einen Begriff von differentiellem Rang für differentiell bewertete Körper zu entwickeln. Dafür habe ich zunächst einen allgemeinen Begriff von ϕ-Rang und ϕ-Hauptrang entwickelt, wobei ϕ ein beliebiger Operator auf einem bewerteten Körper ist. Dieser Begriff von ϕ-Rang verallgemeinert die früheren oben genannten Begriffe von Rängen. Mit diesem Begriff konnte ich leicht zeigen, dass der ϕ-Rang auf den drei üblichen Stufen charakterisiert werden kann (Proposition 5.1.8). Im Abschnitt 5.3 wird die Theorie des ϕ-Rangs auf den Spezialfall von differentiell bewerteten Körpern angewandt. Ich definiere da den differentiellen Rang (beziehungsweise, den differentiellen Hauptrang) eines differentiell bewerteten Körpers als seinen ϕ-Rang (beziehungsweise, als seinen ϕ-Hauptrang), wobei ϕ die logarithmische Ableitung ist (Definition 5.3.1). Aus Proposition 5.1.8 folgt dann unmittelbar, dass der differentielle Rang auch auf dem zum Körper assoziierten asymptotischen Paar charakterisiert werden kann. Diese Tatsache erlaubt mir später, die Eigenschaften des differentiellen Rangs durch Arbeit auf dem asymptotischen Paar zu untersuchen, anstatt auf dem Körper direkt zu arbeiten. Ich gebe dann eine Charakterisierung des differentiellen Rangs mithilfe einer von der logarithmischen Ableitung induzierten Quasiordnung (Satz 5.3.5), was ein Analoges zu [KMP17, Theorem 5.3, Corollary 5.4, Corollary 5.5] ist. Für diesen Satz muss man sich aber auf “H-type” Körper beschränken.

Im Fall der exponentiellen Körper (beziehungsweise, der Differenzkörper) kann man den Rang dadurch charakterisieren, dass eine Bewertung w genau dann im exponentiellen Rang (beziehungsweise, im σ-Rang) enthalten ist, wenn exp (beziehungsweise σ) eine Exponentialfunktion (beziehungsweise ein Automorphismus) auf dem Restklassenkörper Kw induziert. Es stellt sich also die Frage, ob eine ähnliche Aussage auch im Fall der differentiell bewerteten Körper gilt. Die Proposition 5.3.6 gibt eine negative Antwort auf diese Frage. Ich habe jedoch eine völlige Charakterisierung der Bewertungen, die zum differentiellen Rang gehören, im Satz 5.3.11 gegeben.

Es schien mir aber, dass der von mir eingeführte Begriff von differentiellem Rang nicht befriedigend war, weil er keine Information über den Elementen des Körpers, deren Bewertungen nahe dem 0 liegen, gibt. Im Fall der Hardy-Körper zum Beispiel gibt der differentielle Rang keine Information über den Funktionen, die langsamer als alle Hintereinanderführungen von Logarithmen nach $+\infty$ wachsen. Ich habe dieses Problem durch das Einführen eines neuen Begriffs von Rang gelöst, den ich “entfalteten differentiellen Rang” genannt habe (siehe Definition 5.3.14). Der entfaltete differentielle Rang wird auf dem zu einem differentiell bewerteten Körper assoziierten asymptotischen Paar (G, ψ) definiert. Die Idee ist, eine Familie von Translationen von ψ zu betrachten, die ψ um 0 “entfaltet”. Ich definiere dann den entfalteten differentiellen Rang (beziehungsweise, den entfalteten differentiellen Hauptrang) als die Vereinigung aller ψ_a-Ränge (beziehungsweise, aller ψ_a-Hauptränge) für alle diese Translationen ψ_a von ψ. Dieses Verfahren ermöglicht uns, Information über das Verhalten von ψ nahe dem 0 zu bekommen, ohne die im differentiellen Rang enthaltene Information zu verlieren. Ich zeige in der Proposition 5.3.17, dass der entfaltete differentielle Rang immer entweder gleich dem differentiellen Rang oder gleich einem Endstück des differentiellen Rangs ist. Ich lege dann eine Verbindung zwischen dem entfalteten differentiellen Rang und dem exponentiellen Rang im Satz 5.3.22 offen: wenn ein differentiell bewerteter Körper auch mit einer Exponentialfunktion versehen ist, dann stimmen sein exponentieller Rang und sein entfalteter differentieller Rang überein.

Der letzte Teil des Kapitels 5 handelt von Ableitungen auf Körpern formaler Potenzreihen. In der klassischen Theorie der bewerteten Körper weiß man, dass jede total geordnete Menge als der Hauptrang eines Körpers formaler Potenzreihen auftaucht. Eine analoge Aussage gilt auch für Differenzkörper (siehe [KMP17]). Es stellt sich also die Frage, ob ein Analoges auch für differentiell bewertete Körper gilt. Die Schwierigkeit liegt daran, dass es nicht klar ist, wie man eine Ableitung auf einem beliebigen Körper formaler Potenzreihen definieren kann. Die Autoren von [KM12] und [KM11] haben dieses Problem angegangen. Ich habe manche ihrer Ideen benutzt, um die folgende Frage zu beantworten: gegeben seien ein asymptotisches Paar (G, ψ) und ein Körper k der Charakteristik 0; kann man dann eine Ableitung D auf dem Körper der formalen Potenzreihen $K := k((G))$ definieren, sodass K ein differentiell bewerteter Körper wird, dessen assoziiertes asymptotisches Paar (G, ψ) ist? Ich beantworte diese Frage im Satz 5.4.12, indem ich nötige und ausreichende Bedingungen auf (G, ψ) für die Existenz von D gebe. Auf dem Weg zum Beweis dieses Satzes gebe ich eine explizite Konstruktion von D (Formel (‡) im Abschnitt 5.4.1). Ich benutze dann den Satz 5.4.12, um zu zeigen, dass jedes Paar von total geordneten Mengen, wobei die erste ein Hauptendstück des zweiten ist, als das Paar (“differentieller Hauptrang”, “entfalteter differentieller Hauptrang”) eines Körpers formaler Potenzreihen realisiert werden kann (Satz 5.4.28).

Résumé

Cette thèse a pour objet les ordres, les valuations et les C-relations sur les groupes, ainsi que les corps différentiels valués tels qu'étudiés par Rosenlicht. Trois objectifs principaux y sont accomplis. Le premier de ces objectifs est d'introduire et d'étudier une notion de quasi-ordre sur les groupes, qui a pour but de regrouper l'étude des valuations et des ordres dans un cadre commun. Le deuxième est de décrire la structure des C-groupes à l'aide de quasi-ordres, et de caractériser parmi eux ceux qui sont C-minimaux. Le troisième est l'introduction et l'étude d'une notion de rang différentiel pour les corps différentiels valués.

Bien que les ordres et les valuations soient classiquement traités comme deux sujets différents, il existe cependant de remarquables similarités entre ces deux objets. C'est pour cette raison que plusieurs tentatives ont été faites d'établir une théorie qui unifierait les ordres et les valuations. Dans [Fak87], Fakhruddin proposa de résoudre ce problème à l'aide des quasi-ordres, c'est-à-dire des relations binaires qui sont réflexives, transitives et totales. Pus précisément, dans [Fak87], il a introduit la notion de corps quasi-ordonné, où le quasi-ordre vérifie des relations de compatibilité avec les opérations du corps. Les corps ordonnés et les corps valués sont deux exemples d'une telle structure (en effet, toute valuation induit naturellement un quasi-ordre). Plus remarquablement, il a aussi montré ce qu'on appelle la dichotomie de Fakhruddin, c'est-à-dire que, dans tout corps quasi-ordonné, le quasi-ordre est soit un ordre, soit une valuation. Il est alors naturel de se demander si un résultat analogue existe pour les groupes. Cette question est d'autant plus motivée par le fait que les groupes valués et les groupes ordonnés jouent un rôle important dans la théorie des corps valués munis d'un opérateur. Par exemple, si (G, ψ) est le couple asymptotique associé à un corps différentiel valué, alors G est ordonné, et de plus ψ est une valuation sur G. L'intérêt de développer une théorie des groupes quasi-ordonnés est de pouvoir établir des résultats intéressants concernant les groupes ordonnés et les groupes valués, et d'explorer les possibles interactions entre ordres et valuations.

La question de savoir si la dichotomie de Fakhruddin admet un analogue pour les groupes a été le point de départ du chapitre 3 de ma thèse. J'y introduis la notion de quasi-ordre compatible défini sur un groupe abélien (où "compatible" signifie "compatible avec l'opération du groupe", voir Définition 3.1.1). Cette notion est l'exacte analogue des quasi-ordres considérés par Fakhruddin sur les corps dans [Fak87]. J'ai rapidement établi que la dichotomie de Fakhruddin était fausse dans le cas des groupes: il existe des

quasi-ordres compatibles qui ne sont ni des ordres, ni des valuations (voir Exemples 3.1.2). Le résultat principal du chapitre 3 (Théorème 3.1.26) est un théorème de structure pour les groupes abéliens munis d'un quasi-ordre compatible. Il établit que tout quasi-ordre compatible est un "mélange" d'ordre et de valuation. Plus précisément, un groupe abélien muni d'un quasi-ordre compatible est une extension d'un groupe valué par un groupe ordonné. De plus, la partie ordonnée est un segment initial du groupe.

Une fois ce théorème de structure des quasi-ordres compatibles établi, il est alors naturel de vouloir étendre des résultats connus de la théorie des structures ordonnées au cas des groupes quasi-ordonnés. Dans la section 3.2, j'introduis la notion de valuation compatible avec un quasi-ordre. Cette notion est l'analogue de la notion bien connue de valuation convexe pour un corps ordonné (voir par exemple [EP05, chapitre 2]). J'y donne notamment une caractérisation de le compatibilité entre une valuation et un quasi-ordre, qui est analogue à celle donnée pour les valuations convexe sur un corps quasi-ordonné dans [KMP17, Théorème 2.2]. J'établis en revanche qu'il n'existe pas de "théorème de Baer-Krull" pour les groupes abéliens munis d'un quasi-ordre compatible (voir Exemple 3.2.5). Dans la section 3.3, j'introduis une notion de produit pour les groupes quasi-ordonnés (Définition 3.3.1), ainsi qu'une notion d'archimédianité. Cela me permet de prouver un "théorème de plongement de Hahn" pour les quasi-ordres compatibles (Théorème 3.3.8), qui généralise le théorème de plongement de Hahn classique.

Enfin, dans la dernière partie du chapitre 3, j'introduis une notion de q.o.-minimalité pour les groupes quasi-ordonnés, qui est une généralisation de l'o-minimalité, où l'ordre est remplacé par un quasi-ordre compatible. L'objectif était de définir une notion de minimalité pour les groupes quasi-ordonnés qui généralise l'o-minimalité, mais qui donne également une classe de groupes intéressante pour le cas des groupes valués. Pour cela, j'ai utilisé la notion de "swiss cheese" introduite par Holly dans [Hol95]. Je définis une notion de "swiss cheese" pour les groupes quasi-ordonnés par analogie aux "swiss cheese" de Holly, puis je définis un groupe q.o.-minimal comme un groupe quasi-ordonné dont tout sous-ensemble définissable est une union finie de ces "swiss cheese", et tel que cette propriété est préservée par équivalence élémentaire. Je montre alors que tout quasi-ordre compatible induit une C-relation, et que la notion de q.o.-minimalité que j'ai définie est alors équivalente à la C-minimalité (voir Propositions 3.4.1 et 3.4.4). Ceci ouvre la voie au chapitre 4, qui a pour objet les C-groupes.

Une C-relation est une relation ternaire interprétable dans l'ensemble des branches d'un arbre dans lequel toute pair admet un infimum. Plus précisément, si T est un tel arbre, on définit une C-relation C sur l'ensemble des branches de T en disant que $C(x, y, z)$ est vrai si et seulement si l'intersection de y avec z est strictement supérieure à l'intersection de x avec z. Réciproquement, on peut montrer que tout ensemble muni d'une C-relation est interprétable dans l'ensemble des branches d'un arbre muni de la C-relation décrite ci-dessus. Dans [MS96], les auteurs ont introduit la notion de C-groupe et de C-minimalité. Un C-groupe est un groupe muni d'une C-relation compatible avec l'opération du groupe. La C-minimalité est un analogue de l'o-minimalité, dans lequel l'ordre est remplacé par une C-relation. Le but du chapitre 4 est de décrire les C-groupes et de caractériser parmi eux les groupes C-minimaux.

A la fin du chapitre 3, j'ai remarqué que les groupes munis d'un quasi-ordre compatible sont naturellement des C-groupes. J'ai rapidement établi que la réciproque était fausse: il existe des C-groupes dont la C-relation ne provient pas d'un quasi-ordre compatible. Cependant, j'ai également remarqué qu'on pouvait tout de même étudier les C-groupes à l'aide de certains quasi-ordres, que j'appelle des C-quasi-ordres. Plus précisément, si $(G, ., 1, C)$ est un C-groupe, alors C induit naturellement un quasi-ordre sur G défini ainsi: on dit que $x \precsim y$ si et seulement si $\neg C(x, y, 1)$. Un C-quasi-ordre est un quasi-ordre ainsi induit par une C-relation. Le fait remarquable est qu'un C-quasi-ordre détermine entièrement la C-relation qui l'induit. Il y a donc une correspondance bijective entre les C-quasi-ordres et les C-relations compatibles, ce qui signifie que l'étude des C-groupes se ramène à l'étude des C-quasi-ordres. Les C-quasi-ordres constituent l'objet central du chapitre 4; toute l'étude des C-groupe est poursuivie non pas en travaillant directement avec des C-relations, mais en utilisant les C-quasi-ordres. Il est à noter qu'un C-quasi-ordre n'est en général pas un quasi-ordre compatible, bien qu'il existe un lien entre les deux.

Le résultat le plus important du chapitre 4 est le théorème 4.3.33, qui est un théorème de structure pour les groupes munis d'un C-quasi-ordre. On sait que les groupes ordonnés et les groupes valués sont des exemples de C-groupes, dans le sens où un ordre ou une valuation induit naturellement une C-relation. Le théorème 4.3.33 énonce que les groupes ordonnés et le groupes valués sont les "blocs élémentaires" de la classe des C-groupes, c'est-à-dire que tout C-groupe peut se construire à partir de groupes ordonnés et de groupes valués. Dit autrement, les C-groupes sont un "mélange" de groupes ordonnés et de groupes valués. Ceci établit une analogie avec les quasi-ordres compatibles, qui sont également des "mélanges" d'ordre et de valuation. Il faut cependant noter une différence importante: dans le cas des quasi-ordres compatibles, le "mélange" est particulièrement simple car la partie ordonnée est un segment initial du groupe. Dans le cas des C-groupes, ce "mélange" peut être beaucoup plus arbitraire (voir exemple 4.3.1(d)).

Dans la section 4.1, je donne une axiomatisation des C-quasi-ordres (Proposition 4.1.7). J'explique ensuite le lien entre les C-quasi-ordres et les quasi-ordres compatibles étudiés au chapitre précédent. Enfin, je décris les C-quasi-ordres "de type ordre", c'est-à-dire les C-quasi-ordres induits par une C-relation qui est elle-même induite par un ordre. Dans la section 4.2, je montre que tout C-quasi-ordre induit naturellement un C-quasi-ordre sur le quotient du groupe par un sous-groupe convexe. Cela me permet de démontrer un "théorème de Baer-Krull" pour les C-quasi-ordres (Théorème 4.2.11), que je mets ensuite en relation avec le théorème de Baer-Krull classique pour les valuations convexes. La section 4.3 est entièrement dédiée à la preuve du théorème de structure des C-groupes (le théorème 4.3.33). La section 4.4 est dédiée à l'étude des groupes C-minimaux. On sait grâce au théorème 4.3.33 que tout C-groupe peut se décomposer en parties ordonnées et en parties valuées (les "composantes fondamentales", voir Remarque 4.3.35). On peut alors se demander s'il est possible de caractériser la C-minimalité du groupe en fonction des propriétés modèle-théorique de ces "composantes fondamentales". C'est l'objet de la section 4.4. Je commence par réinterpréter les théorème de [MS96] sur les C-groupes dans le langage des C-quasi-ordres. Les résultats de [MS96] disent essentiellement que, dans un groupe C-minimal, les parties valuées ne peuvent pas alterner indéfiniment avec les parties ordonnées (voir Théorème 4.4.7). J'établis ensuite que le produit valuationel, qui est une notion de produit que je définis pour les C-quasi-ordres, préserve les équivalences élémentaires (Théorème 4.4.13). Mon résultat principal concernant les groupes C-minimaux est le théorème 4.4.37. Il énonce que tout C-groupe abélien C-minimal est un produit fini de groupes o-minimaux et de groupes C-minimaux valués. Je termine le chapitre 4 en donnant un exemple de C-groupe C-minimal qui n'est ni un groupe ordonné, ni un groupe valué (Exemple 4.4.43).

Le dernier chapitre de cette thèse a pour but objet les corps différentiels valués et la notion de rang différentiel définie pour ces corps. J'entends par "corps différentiel valué" un corps différentiel muni d'une valuation différentielle telle que définie par Rosenlicht dans [Ros80].

Le rang (respectivement, le rang principal) d'un corps valué (K, v) est un invariant important de (K, v). Il est défini comme le type d'ordre de l'ensemble des sous-anneaux (respectivement, des sous-anneaux principaux) de K qui contiennent l'anneau de valuation $\mathcal{O}_v$. Il possède plusieurs caractérisations. On peut en effet le caractériser au niveau du corps lui-même, mais également au niveau du groupes des valeurs G (comme l'ensemble des sous-groupes convexes de G) ainsi qu'au niveau de la chaîne des valeurs Γ de G (comme l'ensemble des segments finaux de Γ). Récemment, plusieurs notions de rang définies pour des corps valués munis d'un opérateur ont vu le jour. On peut citer le rang exponentiel d'un corps exponentiel (voir [Kuh00]) et le σ-rang d'un corps muni d'un automorphisme σ (voir [KMP17]). L'idée essentielle est d'avoir une notion de rang qui rend compte du comportement de l'opérateur. Dans [Kuh00], S. Kuhlmann définit une notion de compatibilité entre l'exponentielle d'un corps exponentiel $(K, \leq, \exp)$ et un anneau de valuation sur K. Elle définit alors le rang exponentiel d'un corps exponentiel $(K, \leq, \exp)$ comme le type d'ordre de l'ensemble des anneaux compatibles avec l'exponentielle qui contiennent l'anneau de valuation $\mathcal{O}_v$, où v est la valuation archimédienne sur K. Elle montre ensuite que, dans les corps exponentiels qu'elle considère, le logarithme induit naturellement une application χ sur le groupe des valeurs G de (K, v), et que χ induit une application ζ sur la chaîne des valeurs Γ de G. Enfin, elle montre que le rang exponentiel est isomorphe à l'ensemble des sous-groupes convexes de G qui sont clos par χ, ainsi qu'à l'ensemble des segments finaux de Γ qui sont clos par ζ. Dans [KMP17], les auteurs présentent des travaux similaires pour le cas des corps munis d'un automorphisme. Ils définissent une notion de compatibilité entre l'automorphisme σ d'un corps et un anneau de valuation. Ils définissent ensuite le σ-rang d'un corps valué aux différences (K, v, σ) comme le type d'ordre de l'ensemble des anneaux contenant $\mathcal{O}_v$ qui sont compatibles avec σ. Ils montrent ensuite que ce rang est isomorphe à l'ensemble des sous-groupes convexes de G qui sont σ_G-clos, où σ_G est l'application induite par σ sur G, ainsi qu'à l'ensemble des segments finaux de Γ qui sont σ_Γ-clos, où σ_Γ est l'application induite par σ_G sur Γ. Ils donnent également une caractérisation du σ-rang via certaines relations d'équivalence induites par σ (voir [KMP17, Theorem 5.3, Corollary 5.4, Corollary 5.5]).

Prenant inspiration des travaux de [Kuh00] et [KMP17], le but du chapitre 5 est de définir et d'étudier une notion de rang différentiel pour les corps différentiels valués.

Pour cela, j'ai commencé par introduire une notion générale de "ϕ-rang" et de "ϕ-rang principal" d'un corps valué muni d'un opérateur ϕ qui généralise les notions de rang exponentiel et de σ-rang. Avec quelques hypothèses raisonnables sur ϕ, j'ai pu facilement montrer que, comme dans le cas classique, le ϕ-rang peut se caractériser à trois niveaux différents: au niveau du corps, au niveau du groupe des valeurs et au niveau de la chaîne des valeurs (voir Proposition 5.1.8). J'ai ensuite appliqué cette notion de ϕ-rang au cas des corps différentiels valués: j'ai défini le rang différentiel (respectivement, le rang différentiel principal) de (K, v, D) comme le ϕ-rang (respectivement, le ϕ-rang principal) de (K, v), où ϕ est la dérivée logarithmique. En appliquant la Proposition 5.1.8, j'obtiens directement que le rang différentiel peut être caractérisé à trois niveaux différents (Théorème 5.3.3), comme pour le rang classique. En particulier, le rang différentiel est isomorphe à l'ensemble des sous-groupes convexes de G qui sont clos par ψ (ce que j'appelle également le ψ-rang de G), où ψ est l'application induite par la dérivée logarithmique sur G. Ceci me permet ensuite d'étudier le rang différentiel en étudiant le couple asymptotique (G, ψ) plutôt qu'en travaillant directement sur le corps K. Je donne ensuite une caractérisation du rang différentiel en fonction d'un certain quasi-ordre induit par la dérivée logarithmique (Théorème 5.3.5). Ce résultat est l'analogue des résultat dans [Kuh00] et [KMP17] qui expriment le rang exponentiel (respectivement, le σ-rang) via une relation d'équivalence induite par exp (respectivement, par σ). Il faut cependant noter que, dans le cas des corps différentiels, cette caractérisation n'est valable que lorsque le couple asymptotique est de type H (voir section 2.4).

Dans le cas exponentiel (respectivement, dans le cas d'un automorphisme), le rang peut être caractérisé par le fait qu'un anneau de valuation $\mathcal{O}_w$ appartient au rang exponentiel (respectivement, au σ-rang) si et seulement si exp (respectivement, σ) induit naturellement une exponentielle (respectivement, un automorphisme) sur le corps résiduel Kw. On peut alors se demander si un résultat analogue est vrai dans le cas différentiel. J'ai répondu à cette question par la négative dans la Proposition 5.3.6. En effet, pour que $\mathcal{O}_w$ appartienne au rang différentiel, la condition que D induise une dérivation sur Kw n'est pas suffisante. Je donne cependant une caractérisation complète des anneaux qui appartiennent au rang différentiel dans le théorème 5.3.11.

Il m' est apparu que la notion de rang différentiel ainsi définie n'était pas entièrement satisfaisante, car elle ne rend pas compte du comportement de la dérivée logarithmique sur les éléments du corps qui ont une valuation proche de 0. Dans le cas des corps de Hardy par exemple, cela se traduit par le fait que le rang différentiel ne voit pas les fonctions qui croissent lentement vers $+\infty$ telles que les translogarithmes. J'ai résolu ce problème en introduisant une autre notion de rang, appelée rang différentiel déployé, qui étend le rang différentiel. Le rang différentiel déployé est défini sur le couple asymptotique (G, ψ) d'un corps différentiel valué. L'idée est de considérer une famille de translatées de ψ qui permet de "dérouler" ψ autour de 0. Je définis alors le rang différentiel déployé (respectivement, le rang différentiel déployé principal) comme l'union de tous les ψ_a-rang (respectivement, de tous les ψ_a-rangs principaux) de G pour toutes ces translatées ψ_a de ψ (voir Définition 5.3.14). En procédant ainsi, on obtient les informations qu'on souhaite concernant les éléments du groupe proches de 0, tout en conservant l'information contenue dans le rang différentiel. Plus précisément, je montre que le rang différentiel est soit égal au rang différentiel déployé, soit égal à un segment final principal du rang différentiel déployé (Proposition 5.3.17). Je donne ensuite une caractérisation des valuations qui appartiennent au rang différentiel déployé (Proposition 5.3.19). Enfin, je mets en lien le rang différentiel déployé et le rang exponentiel: le corollaire 5.3.22 énonce que, si on considère un corps différentiel valué qui est également un corps exponentiel, alors le rang exponentiel est égal au rang différentiel déployé.

Dans la théorie classique des corps valués, on sait que tout ordre linéaire peut être réalisé comme le rang principal d'un certain corps valué. Des résultats analogues existent pour le rang exponentiel et le σ-rang. Dans le cas des corps pures comme dans le cas des corps avec automorphisme, on peut même réaliser cet ordre linéaire comme le rang d'un certain corps de séries généralisées. On peut alors se demander si un résultat analogue existe dans le cas différentiel. La dernière partie du chapitre 5 est dédiée à ce problème. Dans le cas des corps de séries "classiques" avec exposants dans $\mathbb{Z}$, il existe une manière naturelle de définir une dérivation. En revanche, la situation devient plus compliquée lorsqu'on considère des corps de séries dont les exposants sont dans un groupe ordonné arbitraire. Dans [KM12] et [KM11], les auteurs ont travaillé sur cette question. J'ai utilisé certaines de leurs idées pour répondre à la question suivante: Étant donné un couple asymptotique (G, ψ) et un corps k de caractéristique 0, peut-on définir sur le corps de séries généralisées $k((G))$ une dérivation qui en fait un corps différentiel valué dont le couple asymptotique est (G, ψ)? La réponse est donné par le théorème 5.4.12, qui donne une condition nécessaire et suffisante sur les "composantes" C_λ de la valuation ψ pour qu'une telle dérivation existe. En prouvant ce théorème, je donne même une construction explicite de cette dérivation (formule (‡) de la Section 5.4.1). Enfin, j'utilise ce résultat pour montrer que toute paire d'ensembles totalement ordonnés dont l'un est segment final principal de l'autre peut être réalisée comme la pair ("rang différentiel principal", "rang différentiel déployé principal ") d'un corps de séries généralisées muni d'une dérivation (Théorème 5.4.28).

Acknowledgements

I would like to thank my supervisors Salma Kuhlmann and Françoise Point, firstly for suggesting the topic of my PhD and secondly for their constant guidance and support which brought an immense contribution to the writing of the present thesis and taught me how to do mathematical research. I thank them for the many hours they spent discussing my papers, for introducing me to the research community by helping me participate in various conferences, but also for helping me overcome some administrative struggles which were necessary in getting my doctoral degree.

I also want to thank several people whom I met at various events during my PhD and who contributed to giving my thesis its current shape. I would first like to thank Françoise Delon for our discussion about my poster at the CIRM, which prompted my work on C-groups. I am also extremely grateful to Allen Gehret for drawing a nice picture representing asymptotic couples at the CIRM, which greatly helped my understanding of the matter, to Lou van den Dries and Matthias Aschenbrenner who enhanced my understanding of differential-valued fields, to Mickaël Matusinski for his discussions about Hardy-type derivations on power series, and finally to David Bradley-Williams, whose lecture on the works of Adeleke and Neumann gave me a new perspective on C-relations.

I am also thankful to Raf Cluckers and Martin Hils for reading and correcting my thesis. I thank Martin Hils for pointing out a very elegant simplification of several proofs concerning C-minimal groups in chapter 4.

Contents

Chapter 1

Introduction

This thesis deals with orders, valuations and C-relations on groups, and with differential-valued fields *à la* Rosenlicht. It achieves three main objectives. First, it introduces and studies a notion of quasi-order on groups meant to encompass orders and valuations in a common framework. Secondly, it describes C-groups via quasi-orders and characterizes the C-minimal groups amongst them. Finally, it introduces and studies a notion of differential rank for differential-valued fields. I will now recall the background and motivation and present my results on each of these topics.

Quasi-orders as a uniform approach to valuations and orders

Background

Although ordered and valued structures are classically treated as different subjects, they still bear significant similarities. For this reason, several attempts have been made to establish a unifying theory of orders and valuations. In [Fak87], Fakhruddin proposed to approach this problem with quasi-orders, i.e. binary relations which are reflexive, transitive and total. In [Fak87], Fakhruddin defined a quasi-ordered field as a field with a quasi-order compatible with the field operations (see Definition 2.7.6 below). It is easy to see that ordered fields and valued fields are two examples of quasi-ordered fields. Fakhruddin then showed a converse, stating that any quasi-ordered field is either an ordered field or a valued field (See Theorem 2.7.8 below); this result is known as Fakhruddin's dichotomy. Fakhruddin's results show that the theory of quasi-ordered fields is a good way of unifying the theory of both valued and ordered fields. It is then a natural question to ask if Fakhruddin's results can be transposed to the case of groups. The work done in [KMP17] gives an example of a result on quasi-ordered fields yielding interesting results for both ordered and valued fields. In [KMP17], Kuhlmann, Matusinski and Point defined the notion of compatibility between a valuation and a quasi-order on a field, and gave several characterizations of it (see Theorem 2.7.9 below). Applied to the case where the quasi-order is an order, this theorem gives conditions for a valuation to be convex; applied to the case where the quasi-order comes from a valuation v, this theorem gives conditions for a valuation to be a coarsening of v.

This prompts us to try to develop an analogous theory for groups, in order to unify the theory of both ordered and valued groups in a common framework. In particular, the following questions naturally arise: Is there an analog of Fakhruddin's dichotomy for groups? If not, what is the structure of the group analog of Fakhruddin's quasi-ordered fields? Is there a group analog of Kuhlmann, Matusinski and Point's characterization of compatibility between a valuation and a quasi-order? Another natural idea that arises when developing a theory of quasi-ordered groups is to generalize well-known results about ordered structures to the case of quasi-ordered structures. One example that comes to mind is Hahn's embedding theorem (Theorem 2.2.6 below). Another example is the Baer-Krull theorem (Theorem 2.3.1 below). The Baer-Krull theorem is a statement about ordered fields and had until now no group analog. Establishing a group version of the Baer-Krull theorem with quasi-orders instead of orders has the advantage of being useful both for the study of ordered groups and for the study of valued groups. Finally, it would be interesting to develop a notion of model-theoretic minimality for quasi-ordered groups which generalizes o-minimality.

My results

I introduced the notion of compatible quasi-ordered abelian groups (q.o.a.g.), which is the group analog of Fakhruddin's quasi-ordered fields. I quickly established that Fakhruddin's dichotomy fails in the case of groups (see Examples 3.1.2). I then described the structure of an arbitrary compatible quasi-ordered abelian group. I basically showed that a compatible q.o.a.g. is a "mix" of ordered and valued groups, in the sense that a compatible q.o.a.g. is an extension of a valued group by a totally ordered group (see Theorem 3.1.29). The key idea to obtain this result is to divide the elements of the group into two categories, respectively called o-type and v-type elements. By definition, v-type elements are equivalent to their inverse (for the equivalence relation induced by the quasi-order), whereas o-type elements are not. Remarkably, this simple property is sufficient to determine the behavior of the quasi-order around an element. More precisely, I showed that the set of o-type elements is actually an ordered abelian subgroup and that the quasi-order behaves like a valuation on the set of v-type elements.

I established a group analog of Kuhlmann, Matusinski and Point's characterization of compatibility between a quasi-order and a valuation (see Proposition 3.2.2). However, I quickly established that there was no equivalent of the Baer-Krull theorem for compatible quasi-ordered groups (see Example 3.2.5 and its preceding paragraph). I defined a notion of product for the category of compatible q.o.a.g.'s, and I showed that the product of an ordered group by a valued group preserves elementary equivalence (Theorem 3.3.13). I also defined a notion of archimedeanity for compatible quasi-orders which generalizes the classical notion of archimedeanity for orders. These notion of product and of archimedeanity then allowed me to give a generalization of Hahn's embedding theorem for quasi-ordered groups (see Theorem 3.3.8).

I then tackled the problem of defining a notion of minimality for compatible quasi-ordered groups in analog of o-minimality. The idea was to find a notion which generalizes o-minimality but also gives an interesting class of structures when applied to valued groups. The difficulty in this task is to find the right definition for the "basic" definable sets, i.e. the class of sets which should play the role that intervals play in the case of o-minimality. My approach was to choose the "basic" definable sets as the "swiss cheeses" introduced by Holly in [Hol95]. It turned out that the notion of minimality that we obtain this way is actually equivalent to C-minimality for a C-relation induced by the quasi-order. In particular, I showed that any compatible q.o.a.g. naturally induces a compatible C-relation (see Proposition 3.4.1), so we can view compatible quasi-ordered abelian groups as C-groups. However, it quickly turned out that some compatible C-relations are not induced by a compatible quasi-order. This prompted me to study C-groups further with the help of another class of quasi-orders, which I call C-quasi-orders.

It is worth noting that the techniques used to obtain these results on quasi-ordered groups generalize well-known methods used for ordered groups. In particular, the notion of convexity plays an important role, as well as the notion of quasi-order induced on a quotient.

Quasi-orders as a tool to study C-groups

Background

A C-relation is a ternary relation which is interpretable in the set of branches of a tree. C-minimality was introduced in [MS96] as an analog of o-minimality where the order is replaced with a C-relation. In [MS96], Macpherson and Steinhorn also introduced the notion of C-group, which consists in a group endowed with a C-relation compatible with the group operation. Later, Delon generalized the definition of C-relation in [Del11] to include totally ordered structures. The C-relations studied by Macpherson and Steinhorn are now called dense C-relations. In Delon's context, o-minimality and strong minimality both become special cases of C-minimality. Totally ordered and valued groups are both examples of C-groups (in the sense that both total orders and valuations naturally induce a C-relation). Macpherson and Steinhorn obtained a partial classification of dense C-minimal groups in [MS96], and Delon and Simonetta completely classified abelian valued C-minimal groups in [DS17]. However, there is still no complete classification of C-minimal groups. Moreover, until now, there was no description of C-groups in general (i.e. without the minimality assumption).

My results

Discovering the connection between compatible quasi-orders and C-groups gave me the intuition that quasi-orders could be very useful in the study of C-groups, and thus prompted me to push forward the study of C-groups with the help of another notion of quasi-orders. My idea came from the following remark: if (M, C) is any C-structure, then fixing one variable of the C-relation gives us a quasi-order on M. More precisely, for any $a \in M$, the binary relation $\neg C(.,.,a)$ is a quasi-order. In the case of groups, the natural candidate for a is the neutral element 1. A C-quasi-order on a group is then a quasi-order of the form $\neg C(.,.,1)$, where C is a compatible C-relation. Remarkably, a compatible C-relation is completely determined by the C-quasi-order it induces. This means that we can study compatible C-relations by just looking at C-quasi-orders. This makes the study of C-groups easier since quasi-orders are binary relations, whereas C-relations are ternary. Moreover, the fact that quasi-orders are similar to orders makes working with C-quasi-orders more intuitive than working with C-relations. When using quasi-orders, one can use techniques similar to those used for ordered structures. For example, the notion of convexity plays an essential role in my work on C-quasi-orders, as well as the idea that a C-quasi-order induces a C-quasi-order on a quotient by a convex subgroup (see Proposition 4.2.2). This idea was inspired by previous work on ordered groups (see for example [Fuc63]) which generalize well to C-quasi-orders.

My main result on C-groups is Theorem 4.3.33, which gives the structure of an arbitrary C-group. It essentially states that totally ordered groups and valued groups are the “building blocks” of C-groups. This result was obtained not by working directly with C-relations but with C-quasi-orders. The ideas used were very similar to the one I used for compatible quasi-orders. I separated the elements of the groups into two categories, the o-type and the v-type elements. I showed that the C-q.o. is “valuational-like” around v-type elements and “order-type-like” around o-type elements. This allowed me to partition the group into a collection of strictly convex subsets, on each of which the C-q.o. is either “order-type-like” or “valuational-like”.

I applied Theorem 4.3.33 to describe C-minimal groups. The main result concerning C-minimal groups is Theorem 4.4.37, which states that any welding-free (see Definition 4.3.2) abelian C-minimal group is a finite direct product of o-minimal groups and C-minimal valued groups. I also showed a “Feferman-Vaught” theorem for C-groups. More precisely, I defined a notion of product for the category of C-q.o. groups called the valuational product (see Definition 2.7.14) and showed that any such product with finitely many factors preserves elementary equivalence (see Theorem 4.4.13).

Introducing C-quasi-orders also allowed me to state a group analog of the Baer-Krull theorem (Theorem 4.2.11), which I had failed to do when considering compatible quasi-orders. This is due to the fact that, unlike compatible quasi-ordered groups, the class of C-quasi-ordered groups is stable under lifting. As corollaries of Theorem 4.2.11, I obtained a Baer-Krull theorem for ordered groups (Corollary 4.2.13) and another one for valued groups (Corollary 4.2.12). I then showed how we can recover the classical Baer-Krull theorem from my Baer-Krull theorem on ordered groups (see Section 4.2.3).

The differential rank

Background

The rank (respectively, the principal rank) of a valued field (K, v) is an important characteristic of the valued field. It is defined as the order type of the set of all coarsenings of v (respectively, all principal coarsenings of v), ordered by inclusion of their valuation rings. It has three equivalent characterizations: one at the level of the valued field (K, v)

itself, another one at the level of the value group $G := v(K^\times)$ (as the order type of the set of all convex subgroups of $(G, \leq)$, ordered by inclusion) and a third one at the level of the value chain $\Gamma := v_G(G^{\neq 0})$ of the value group (as the order type of the set of final segments of Γ, ordered by inclusion). Recently, notions of ranks have been developed for valued fields endowed with an operator. Examples of this are the exponential rank of an ordered exponential field (see [Kuh00]) and the difference rank of a valued difference field (see [KMP17]). In [Kuh00], Kuhlmann defined a notion of compatibility between the exponential and a valuation of the field, and defined the (principal) exponential rank of the field as the set of (principal) convex valuations which are compatible with the exponential. She then showed that the logarithm of the field naturally induces a contraction map χ on the value group, and that the exponential rank is equal to the order type of the set of convex subgroups of the value group $(G, \leq)$ which are closed under χ. Similarly, the map χ induces a map ζ on the value chain Γ of G, and the exponential rank of the field is also equal to the order type of the set of final segments of Γ which are closed under ζ. This shows that, as happens with the classical rank, the exponential rank is characterized at three levels. Kuhlmann also showed that any totally ordered set can be realized as the principal exponential rank of some exponential ordered field. In [KMP17], a very similar work was done for difference fields. The authors of [KMP17] defined a notion of compatibility between a valuation and an automorphism σ of the field, and defined the (principal) difference rank of a difference valued field as the set of (principal) coarsenings of v compatible with σ. They then showed that the difference rank can also be characterized at the level of the value group via a map σ_G induced by σ on G and at the level of the value chain Γ via a map σ_Γ induced by σ_G on Γ. They also showed that the difference rank can be characterized via a certain equivalence relation induced by σ on K (see [KMP17, Theorem 5.3, Corollary 5.4, Corollary 5.5]). Finally, they showed that any totally ordered set can be realized as the principal difference rank of some difference field.

It is then natural to ask whether we can develop a similar theory for differential fields. More precisely, we are interested in the class of differential-valued fields introduced by Rosenlicht in [Ros80]. They are generalizations of Hardy fields. Differential-valued fields play a central role in the work of Aschanbrenner, van den Dries and van der Hoeven on the model theory of transseries (see [MAv17]).

A natural way of constructing a valued field of given principal rank is to consider fields of power series. We know that this is also possible in the case of difference fields, i.e. any totally ordered set is the difference rank of some field of power series (see the proof of [KMP17, Corollary 4.13]). However, we also know that this is not possible in the exponential case, because no field of power series can be endowed with an exponential (see [Kuh00]). It is then natural to wonder if it is possible in the case of differential fields. This requires solving the following problem:

Question 1: Given an arbitrary field of power series K, how can we endow K with a derivation D which makes (K, v, D) a differential-valued field?

This problem was addressed in [KM12] and [KM11].

My results

I defined a notion of ϕ-rank of a field (K, v) endowed with an arbitrary operator ϕ. This notion generalizes the notions of exponential and difference rank. I showed that, with some reasonable assumptions on ϕ, the ϕ-rank is characterized at three levels (see Proposition 5.1.8). I then applied my definition of ϕ-rank to the special case of differential-valued fields, where ϕ is set to be the logarithmic derivative. I then immediately got the result that the differential rank is characterized at three levels (Theorem 5.3.3). I then interpreted the differential rank via a certain quasi-order induced by ϕ (Theorem 5.3.5). This is an analog of [KMP17, Theorem 5.3, Corollary 5.4, Corollary 5.5]. I then established that, unlike what happens in the difference and exponential case, one cannot characterize the compatibility of a valuation with ϕ by looking at the map induced by ϕ on the residue field. However, I get another characterization of compatibility (Theorem 5.3.11).

This notion of differential rank is a bit naive as it only consists in imitating the definitions of rank for difference and exponential rank. It turned out that the differential rank is not really a satisfying notion of rank, as it is too coarse. Indeed, the differential rank does not say anything on the behavior of ϕ for elements of the field whose valuation is close to zero. In the context of Hardy fields for example, any convex subring containing the identity function is ϕ-compatible. This means that the ϕ-rank does not see the behavior of ϕ on slowly growing functions, i.e functions that grow more slowly than polynomials. This prompted me to define a new notion of rank, which I call the unfolded differential rank. The unfolded differential rank is defined via the asymptotic couple (G, ψ) of the field. The idea is to consider a family of translates of ψ which allows us to "unfold" the map ψ on a neighborhood of 0. I then defined the unfolded differential rank as the union of all the ψ_a-ranks of the group for all these translates ψ_a of ψ. I then established a connection between the unfolded differential rank and the exponential rank: If K happens to be an exponential field and a differential-valued field, then the exponential rank of K coincides with the unfolded differential rank of K (see Corollary 5.3.22).

Finally, I showed that, for any pair (P, Q) of totally ordered sets such that P is either Q or a principal final segment of Q, there exists a field of power series K and a derivation D on K such that (K, v, D) is a differential-valued field, P is the principal differential rank of K and Q is the principal unfolded differential rank of K (see Theorem 5.4.28). In doing so, I answered a variant of Question 1 above:

Question 2: Given an asymptotic couple (G, ψ) and a field k, is there a strongly linear derivation D on $k((G))$ making (K, v, D) a differential-valued field?

I answered question 2 by giving a necessary and sufficient condition on (G, ψ) for the existence of D (see Theorem 5.4.12). It then allowed me to answer Question 1 in the special case where we require D to be a strongly linear Hardy-type derivation (see Theorem 5.4.22).

I now briefly describe the structure of this thesis. Chapter 2 gives preliminaries on orders, valuations, C-relations and quasi-orders. Except for Section 2.7, which introduces quasi-orders, every definition and result in chapter 2 can be found in the literature. Chapter 3 introduces and studies compatible quasi-ordered abelian groups. I start by describing the structure of a compatible quasi-ordered abelian group in Section 3.1. I then give a characterization of compatibility between a quasi-order and a valuation in Section 3.2. In Section 3.3, I introduce a notion of product of compatible q.o.a.g.'s and show a quasi-order analog of Hahn's embedding theorem. Finally, in Section 3.4, I introduce a notion of quasi-order-minimality and show that it is equivalent to C-minimality.

Chapter 4 is dedicated to the study of C-groups via C-quasi-orders. I start by introducing and axiomatizing C-quasi-orders in Section 4.1. In the same section, I also describe the "order-type" C-q.o.'s (i.e., C-q.o.'s induced by a group order) and explain the connection between C-q.o.'s and compatible q.o.'s. In Section 4.2, I establish a Baer-Krull Theorem for C-q.o.'s. In Section 4.3, I describe the structure of an arbitrary C-q.o. group. Finally, in Section 4.4, I apply the results of Section 4.3 to describe C-minimal groups.

Chapter 5 is dedicated to differential-valued fields. I develop the notion of ϕ-rank of a valued field endowed with an operator ϕ in Section 5.1. In Section 5.3, I introduce the notion of differential rank of a differential-valued field and give several characterizations of it. I then introduce the notion of unfolded differential rank. Finally, in Section 5.4, I describe how to define a derivation on a field of power series, and then use these results to construct a differential-valued field of given principal differential rank and given principal unfolded differential rank.

Chapter 2

Preliminaries: orders, valuations, C-groups and quasi-orders

2.1 Conventions

When working with arbitrary groups which are not necessarily abelian, we shall denote the group operation multiplicatively. However, the group operation will be denoted additively in most instances where the group is assumed to be abelian. For any group G and $g, z \in G$, g^z denotes zgz^{-1}, and $\mathrm{ord}(g)$ denotes the order of g. If $(G, 1, .)$ is a group and $A \subseteq G$ is a subset of G, then $A.A$ denotes the set $\{a.b \mid a, b \in A\}$ and A^{-1} denotes $\{g^{-1} \mid g \in A\}$. If H is a normal subgroup of G, we will denote by π_H the canonical projection $\pi_H : G \to G/H$. If $(G_i)_{i\in I}$ is a family of groups, then $\prod_{i\in I} G_i$ denotes the direct product of this family. If $g \in \prod_{i\in I} G_i$, we will usually denote by g_i the coefficient of g at $i \in I$, and $\mathrm{supp}(g)$ will denote the support of g, i.e $\mathrm{supp}(g) = \{i \in I \mid g_i \neq 1\}$. If G, H are two groups and $\alpha : G \to \mathrm{Aut}(H)$, then $G \ltimes_\alpha H$ denotes the corresponding semi-direct product. We will denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \dots\}$ without zero. The set $\mathbb{N} \cup \{0\}$ is denoted by $\mathbb{N}_0$. In Sections dealing with model theory (Sections 3.3.3, 3.4 and 4.4), the languages we consider always contain equality, which is denoted by "$=$". In contrast, equality between formulas will be denoted by "$\equiv$".

2.2 Ordered and valued groups

We recall here some basic facts about the theory of ordered groups. Most of these results can be found in [Fuc63] and [Gla99]. Let G be a group. A total order $\leq$ on G is called a **group order** if it satisfies the following condition:

$$\forall x, y, z \in G, x \leq y \Rightarrow xz \leq yz \wedge zx \leq zy. \tag{OG}$$

Throughout this thesis, an **ordered group** is a pair $(G, \leq)$ consisting of a group G with a *total* group order $\leq$ on G. Note that the author of [Fuc63] considers orders which are not necessarily total, but we want to insist on the fact that all ordered groups considered in this thesis are totally ordered.

It follows easily from axiom (OG) that any ordered group is torsion-free. If $(G, \leq)$ is an ordered group, we denote by G^+ the **positive cone** of $(G, \leq)$, i.e $G^+ := \{g \in G \mid 1 \leq g\}$. Note that G^+ entirely characterizes the order $\leq$, since $g \leq h$ is equivalent to $hg^{-1} \in G^+$. The following result can be found in [Fuc63]:

Theorem 2.2.1
A subset P of G is the positive cone of some total group order on G if and only if the following holds:

(i) $P \cap P^{-1} = \{1\}$

(ii) $P.P \subseteq P$

(iii) $zPz^{-1} = P$ for any $z \in G$

(iv) $P \cup P^{-1} = G$

The order is then unique and can be defined from P as follows: $g \leq h$ if and only if $hg^{-1} \in P$.

Note that condition (iv) of Theorem 2.2.1 is not part of [Fuc63, Theorem 2, page 13], because the author of [Fuc63] considers partial orders. However, we need condition (iv) if we just want to consider total orders. If P is any subset of G satisfying conditions (i)-(iv) of Theorem 2.2.1, we will say that P is a **positive cone** of G.

If G is a torsion-free abelian group, then we denote by $\widehat{G}$ the divisible abelian group $G \bigotimes_{\mathbb{Z}} \mathbb{Q}$. We call $\widehat{G}$ the **divisible hull** of G, and we view G as a subgroup of $\widehat{G}$ via the embedding $g \mapsto g \otimes 1$. Note that $\widehat{G}$ is the smallest divisible abelian group containing G, in the sense that $\widehat{G}$ is embeddable into any divisible abelian group containing G. Now assume that $(G, \leq)$ is an ordered group with positive cone P. Then there is a unique group order on $\widehat{G}$ extending $\leq$. Indeed, if we set $\widehat{P} := \{g \otimes r \mid g \in P \wedge 0 \leq r\}$, then one can check that $\widehat{P}$ is the unique positive cone on $\widehat{G}$ which extends P.

An important notion related to ordered groups is the notion of archimedeanity. Let us fix an ordered group $(G, \leq)$. If $g \in G$, we denote by $|g|$ the absolute value of g, i.e $|g| = g$ if $1 \leq g$ and $|g| = g^{-1}$ otherwise. We say that two elements $g, h \in G$ are **archimedean-equivalent**, and write $g \sim_{\text{arch}} h$, if there are $n, m \in \mathbb{N}$ such that $|g| \leq |h|^n$ and $|h| \leq |g|^m$. The relation $\sim_{\text{arch}}$ is an equivalence relation on G, and we denote by $\text{cl}_{\text{arch}}(g)$ the class of g for the relation $\sim_{\text{arch}}$. An ordered group is called archimedean if it has exactly one archimedean class apart from $\{1\}$. The set of archimedean classes of G is totally ordered by $\text{cl}_{\text{arch}}(g) \leq \text{cl}_{\text{arch}}(h) \Leftrightarrow \exists n, m \in \mathbb{N}, |g^n| \leq |h^m|$. We recall the Hölder theorem on archimedean ordered groups:

Theorem 2.2.2 ([Fuc63, chapter IV, section 1, Theorem 1])
Let $(G, \leq)$ be an ordered group. Then $(G, \leq)$ is archimedean if and only if it is isomorphic as an ordered group to a subgroup of $(\mathbb{R}, +, \leq)$, where $\leq$ denotes the usual order of $\mathbb{R}$.

If H is a convex normal subgroup of G, then $\leq$ naturally induces an order on the quotient group G/H by $g_1H \leq g_2H \Leftrightarrow \exists h_1, h_2 \in H, g_1h_1 \leq g_2h_2$ (see [Fuc63, chapter

II, section 4]). The category of ordered groups is naturally endowed with a notion of product called the lexicographic product, whose definition we recall here. If $(B_\gamma)_{\gamma\in\Gamma}$ is a family of groups indexed by a totally ordered set $(\Gamma,\leq)$, we define the **Hahn product** of the family $(B_\gamma)_{\gamma\in\Gamma}$, denoted $\underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma$, as the subgroup of $\prod_\gamma B_\gamma$ consisting of all elements of $\prod_\gamma B_\gamma$ whose support is well-ordered. If each B_γ is endowed with a group order $\leq_\gamma$, then we can define the **lexicographic product** of the family $(B_\gamma,\leq_\gamma)_{\gamma\in\Gamma}$ as the ordered group $(G := \underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma,\leq)$, where $\leq$ is defined as follows: we say that $g\leq h$ if $g_\gamma\leq h_\gamma$, where $\gamma=\min\operatorname{supp}(gh^{-1})$ (see [Fuc63, chapter II, section 7]). We will sometimes denote elements of $\underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma$ as formal sums $\sum_{\gamma\in\Gamma} g_\gamma$, with $g_\gamma\in B_\gamma$ for all $\gamma\in\Gamma$.

We now want to recall the basic facts of valuation theory. We insist on the fact that the valued groups which we will consider are not necessarily abelian. This is unusual in the literature, as most works on valued groups seem to focus on abelian groups. However, several notions of valuations for arbitrary groups have been introduced, for example in [PC83] and [Sim03]. We chose to use the definition given in [Sim03], as it is particularly useful for the study of C-groups.

Definition 2.2.3
A **valuation** on a group G is a map $v : G\to\Gamma\cup\{\infty\}$ such that:

(i) $\Gamma\cup\{\infty\}$ is a totally ordered set with maximum ∞.

(ii) For any $g\in G$, $v(g)=\infty\Leftrightarrow g=1$

(iii) For any $g,h\in G$, $v(gh^{-1})\geq\min(v(g),v(h))$.

(iv) For any $g,h,z\in G$, $v(g)\leq v(h)\Leftrightarrow v(g^z)\leq v(h^z)$

Remark 2.2.4: In [PC83], Priess-Crampe only requires (i),(ii) and (iii) in the definition of valuation. However, Simonetta in [Sim03] also requires (iv). We choose here the definition of valuation given in [Sim03] because it is a more convenient definition when working with C-groups as we do in chapter 4. In particular, a map v satisfying (i),(ii) and (iii) but not (iv) would not induce a compatible C-relation on G.

Notation
If $v : G\to\Gamma\cup\{\infty\}$ is a valuation, then for any $\gamma\in\Gamma$, G^γ and G_γ respectively denote $\{g\in G\mid v(g)\geq\gamma\}$ and $\{g\in G\mid v(g)>\gamma\}$. We also set $G^\infty := G_\infty := \{1\}$.

Remark 2.2.5: Here are a few remarks concerning Definition 2.2.3. Note that due to the fact that $(g^z)^{z^{-1}}=g$, we can replace "$\Leftrightarrow$" by "$\Rightarrow$" in (iv). Also, assuming that (ii) holds, one easily sees that (iii) holds if and only if for any $g,h\in G$, $v(g)=v(g^{-1})\wedge v(gh)\geq\min(v(g),v(h))$ holds. Moreover, we can easily show that following facts are true for any valued group (G,v):

(a) For any $g,h\in G$, $v(g)<v(h)\Rightarrow v(g^z)<v(h^z)$ and $v(g)=v(h)\Rightarrow v(g^z)=v(h^z)$ (it follows from (iv)).

(b) If $v(g) < v(h)$ then $v(gh) = v(g) = v(hg)$. If $v(gh) > v(g)$, then $v(g) = v(h)$.

(c) For any $\gamma \in \Gamma$, G^γ and G_γ are subgroups of G, and G_γ is a normal subgroup of G^γ. Note however that it can happen that $v(g) \neq v(g^z)$, and in particular G^γ and G_γ are not always normal in G. This is showed by Example 4.3.36.

(d) Thanks to axiom (iv), conjugation by an element $z \in G$ induces an automorphism of Γ defined by $v(g) \mapsto v(g^z)$ (note that this map is onto since $v(g^{z^{-1}})$ is a pre-image of $v(g)$). If $\gamma = v(g)$ then we denote $v(g^z)$ by γ^z. Conjugation by z also induces a group homomorphism $G^\gamma \to G^{\gamma^z}$ and another one from G^γ/G_γ to $G^{\gamma^z}/G_{\gamma^z}$. The latter is defined by $gG_\gamma \mapsto g^z G_{\gamma^z}$.

We will often use the notation B_γ for the quotient group G^γ/G_γ. To simplify notation, we will denote by π_γ instead of π_{G_γ} the canonical projection $\pi_\gamma : G^\gamma \to B_\gamma$. The groups B_γ are called the **components** of the valued group (G, v), and the pair $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$ is called the **skeleton** of the valued group (G, v). There is a natural way of constructing valued groups of given skeleton. Let an ordered family $(B_\gamma)_{\gamma\in\Gamma}$ of groups be given. Then the Hahn product G of this family is naturally endowed with a valuation defined by $v(g) := \min \operatorname{supp}(g)$. This makes (G, v) a valued group with skeleton $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$.

A particularly interesting example of valuations are the $\mathbb{Z}$-module valuations on abelian groups. If $(G, 0, +)$ is an abelian group and $v : G \to \Gamma \cup \{\infty\}$ a valuation, we say that v is a **$\mathbb{Z}$-module valuation** if for any $g \in G$ and $n \in \mathbb{Z}\backslash\{0\}$, $v(ng) = g$. Note that the existence of a $\mathbb{Z}$-module valuation v on G implies that G as well as each component B_γ of (G, v) is torsion-free. It is then easy to see that v extends uniquely to a $\mathbb{Z}$-module valuation on $\widehat{G}$ by setting $v(g \otimes r) := v(g)$ for any $g \in G$ and $r \in \mathbb{Q}$ with $r \neq 0$. The skeleton of $(\widehat{G}, v)$ is then $(\Gamma, (\widehat{B}_\gamma)_{\gamma\in\Gamma})$. We refer to [Kuh00, chapter 0] for more information about $\mathbb{Z}$-module valuations.

Valuations are naturally related to orders. In particular, if $(G, \leq)$ is an ordered abelian group, then $\leq$ canonically induces a valuation on G. Indeed, let Γ denote the set of all archimedean classes of G apart from $\{1\}$. We saw earlier that $\leq$ naturally induces an order on Γ. Now define $v_{\text{arch}}(g) := \text{cl}_{\text{arch}}(g)$ for $g \neq 1$ and $v_{\text{arch}}(1) := \infty$. This defines a $\mathbb{Z}$-module valuation $v_{\text{arch}} : G \to (\Gamma \cup \{\infty\}, \leq^*)$, where $\leq^*$ is the reverse order of $\leq$. The valuation v_{arch} is called the **natural valuation**, or **archimedean valuation**, associated to the order $\leq$. We then define the value chain, the components and the skeleton of the ordered group $(G, \leq)$ as the value chain, the components and the skeleton of the valued group (G, v_{arch}). For any $\gamma \in \Gamma$, the subgroups G^γ and G_γ are convex in G (so in particular $\leq$ induces an order on G^γ/G_γ). Conversely, any convex subgroup of $(G, \leq)$ is of the form $\bigcup_{\gamma\in\Delta} G^\gamma$ for a certain $\Delta \subseteq \Gamma$. The set of non-trivial convex subgroups of $(G, \leq)$ is totally ordered by inclusion, its order-type is called the **rank** of the ordered group $(G, \leq)$. The rank of $(G, \leq)$ is order-isomorphic to the set of final segments of Γ via the map $H \mapsto v_{\text{arch}}(H\backslash\{0\})$. We now recall Hahn's embedding theorem for ordered groups. If $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$ is the skeleton of (G, v_{arch}), then for all $\gamma \in \Gamma$, $\leq$ induces an order $\leq_\gamma$ on B_γ. We then have the following:

Theorem 2.2.6 (Hahn's embedding theorem, see [Gla99, Theorem 4.C])
Let $(G, \leq)$ be an ordered abelian group. Then $(G, \leq)$ is embeddable into a lexicographic product of archimedean ordered groups. More precisely, $(G, \leq)$ is embeddable as an ordered group into the lexicographic product of the family $(\widehat{B}_\gamma, \leq_\gamma)_{\gamma\in\Gamma}$.

Remark 2.2.7: [Gla99, Example 4.5.2] shows that Theorem 2.2.6 fails if we replace $\widehat{B}_\gamma$ by B_γ.

In [Kuh00], a variant of Theorem 2.2.6 was stated for $\mathbb{Z}$-module valuations:

Theorem 2.2.8 ([Kuh00, chapter 0, Theorem 0.27])
Let G be a divisible abelian group and v a $\mathbb{Z}$-module valuation on G with skeleton $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$. There is a group embedding $\phi : G \hookrightarrow \underset{\gamma\in\Gamma}{\mathrm{H}} \widehat{B}_\gamma$ and an automorphism of ordered sets $\psi : \Gamma \to \Gamma$ such that $\psi(v(g)) = \min \mathrm{supp}(\phi(g))$ for all $g \in G$ (in other words, ϕ is an embedding of valued groups).

We can even improve Theorem 2.2.8 into the following, stronger statement, which will be particularly useful in Section 3.3.2:

Theorem 2.2.9
Let (G, v) be a group endowed with a $\mathbb{Z}$-module valuation, $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$ the skeleton of (G, v), $H := \underset{\gamma\in\Gamma}{\mathrm{H}} \widehat{B}_\gamma$ and w the usual valuation on H, i.e $w(h) = \min \mathrm{supp}(h)$. There exists a group embedding $\phi : G \hookrightarrow H$ such that the following holds:

(1) For any $g \in G$, $w(\phi(g)) = v(g)$.

(2) For any $\gamma \in \Gamma$ and $g \in G$ with $v(g) = \gamma$, the coefficient of $\phi(g)$ at γ is $g + G_\gamma$.

Proof. Assume first that G is divisible. Then for every $\gamma \in \Gamma$, B_γ is also divisible, so $H = \underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma$. By Theorem 2.2.8, there exists a group embedding $\psi : G \to H$ and an isomorphism of ordered set $\lambda : \Gamma \to \Gamma$ such that $w(\psi(g)) = \lambda(v(g))$ for every $g \in G$. Now consider $\chi : H \to H, (h_\gamma)_{\gamma\in\Gamma} \mapsto (h_{\lambda(\gamma)})_{\gamma\in\Gamma}$. One can easily check that χ is a group isomorphism and that $\chi \circ \psi : G \to H$ satisfies condition (1) of the theorem. Therefore, we can assume that ψ satisfies (1). For every $\gamma \in \Gamma$, consider now $\epsilon_\gamma : B_\gamma \to B_\gamma, g + G_\gamma \mapsto \phi(g)_\gamma$, where $\phi(g)_\gamma$ denotes the coefficient of $\phi(g)$ at γ. For each γ, ϵ_γ is well-defined: indeed, if $g, h \in G^\gamma$ are such that $v(g-h) > \gamma$, then since ϕ satisfies condition (1) we have $w(\phi(g) - \phi(h)) = v(g-h) > \gamma$ so $\phi(g)_\gamma = \phi(h)_\gamma$. One easily sees that ϵ_γ is a group isomorphism. Define $\zeta : H \to H, (h_\gamma)_{\gamma\in\Gamma} \mapsto (\epsilon_\gamma^{-1}(h_\gamma))_{\gamma\in\Gamma}$. ζ is a group isomorphism, and it is easy to see that $\zeta \circ \psi : G \to H$ satisfies all the conditions of the theorem.

Now let us go back to the general case. We know that v extends uniquely to a $\mathbb{Z}$-module valuation on $\widehat{G}$ and that the skeleton of $(\widehat{G}, v)$ is $(\Gamma, (\widehat{B}_\gamma)_{\gamma\in\Gamma})$. We already proved that there exists a map $\phi : \widehat{G} \to H$ such that $w(\phi(g)) = v(g)$ and $(\phi(g))_\gamma = g + \widehat{G}_\gamma$ for any $g \in \widehat{G}$ and $\gamma = v(g)$. It follows that ϕ restricted to G satisfies the conditions we want. □

Remark 2.2.10: It follows from Remark 2.2.7 that Theorem 2.2.9 would fail if we replaced $\widehat{B}_\gamma$ by B_γ.

2.3 Ordered and valued fields

We now recall basic facts about ordered and valued fields, which will be the basis for Chapter 5. *All fields considered in this thesis are commutative.* Most of the results presented here can be found in [EP05] and [PC83]. Let $(K, 0, 1, +, .)$ be a field. If $\leq$ is a total order on K, we say that $\leq$ is a **field order** if the following holds:

(i) $\leq$ is a group order on $(K, 0, +)$.

(ii) for any $x, y, z \in K$, $x \leq y \wedge 0 \leq z \Rightarrow xz \leq yz$.

An **ordered field** is a pair $(K, \leq)$ consisting of a field K endowed with a total field order $\leq$. An ordered field $(K, \leq)$ is called archimedean if $(K, 0, +, \leq)$ is an archimedean ordered group.

A **valued field** is a a field K endowed with a map $v : K \to (G, \leq) \cup \{\infty\}$ such that the following holds:

(i) $(G, 0, + \leq)$ is an ordered abelian group.

(ii) the order $\leq$ is extended to $G \cup \{\infty\}$ by declaring $G < \infty$.

(iii) for any $a \in K$, $v(a) = \infty \Leftrightarrow a = 0$.

(iv) for any $a, b \in K$, $v(a + b) \geq \min(v(a), v(b))$.

(v) $v(ab) = v(a) + v(b)$ for all $a, b \neq 0$.

G is then called the **value group** of (K, v). Note that if (K, v) is a valued field then in particular $(K, 0, +, v)$ is a valued group. If $(K, \leq)$ is an ordered field, then we define the **natural valuation** associated to $(K, \leq)$ as the natural valuation associated to the ordered group $(K, 0, +, \leq)$. We use the following notation: $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ to mean that (K, v) is a valued field with value group G, v_G is the natural valuation of the ordered group G and Γ is the value chain of (G, v_G). For a given valuation v on a field K, we denote by $\mathcal{O}_v := \{x \in K \mid v(x) \geq 0\}$ its **valuation ring**, by $\mathcal{M}_v := \{x \in K \mid v(x) > 0\}$ the maximal ideal of $\mathcal{O}_v$ and by $\mathcal{U}_v := \{x \in K \mid v(x) = 0\}$ the group of units of $\mathcal{O}_v$. The residue field of (K, v) is the field $\mathcal{O}_v/\mathcal{M}_v$ and will be denoted by Kv. We recall that a field valuation is entirely determined by its valuation ring, in the sense that for any $a, b \in K$, $v(a) \leq v(b)$ if and only if $ba^{-1} \in \mathcal{O}_v$. We will sometimes assimilate a valuation with its valuation ring.

If $(K, \leq)$ is an ordered field and v is a valuation on K, then v is called a **$\leq$-convex** valuation if for any $a, b \in K$, $0 \leq a \leq b \Rightarrow v(a) \geq v(b)$. In this case we also say that v is **compatible** with the order $\leq$. The theory of convex valuations is studied in chapter 2 of [EP05]. An important theorem for convex valuations is the Baer-Krull theorem:

Theorem 2.3.1 (Baer-Krull theorem, see [EP05, theorem 2.2.5])
Let $v : K \to G \cup \{\infty\}$ be a valuation and let $(\pi_i)_{i \in I}$ be a family of elements of K such that $(v(\pi_i) + 2G)_{i \in I}$ is an $\mathbb{F}_2$-basis of $G/2G$. Let $\mathcal{O}(K)$ be the set of of field orders on K with

which v is compatible and $\mathcal{O}(Kv)$ the set of field orders of the residue field Kv. There is a bijection: $\phi : \mathcal{O}(K) \longleftrightarrow \{-1,1\}^I \times \mathcal{O}(Kv)$ defined as follows: $\phi(\leq) = (\eta, \leq_{Kv})$, where $\eta(i) = 1$ if and only if $0 \leq \pi_i$ and $\leq_{Kv}$ is the order induced by $\leq$ on the residue field.

An important characteristic of a valued field is its rank, whose definition we recall here. If v and w are two valuations on a field K, we say that w is a **coarsening** of v if $\mathcal{O}_v \subseteq \mathcal{O}_w$. We say that w is a **strict coarsening** of v if moreover $v \neq w$. We recall that if w is a coarsening of v, then v induces a valuation on the residue field Kw which we denote by $\frac{v}{w}$. It is the valuation whose valuation ring is the image of $\mathcal{O}_v$ under the canonical homomorphism $\mathcal{O}_w \to Kw$. We also recall that, if $(K, \leq)$ is an ordered field and w a coarsening of its natural valuation, then $\leq$ induces a field order on Kw which we denote by $\leq_w$ and defined by $0 + \mathcal{M}_w <_w a + \mathcal{M}_w \Leftrightarrow (0 < a \wedge a \notin \mathcal{M}_w)$. The set of all strict coarsenings of a given valuation v is totally ordered by inclusion, and its order type is what we call the **rank of the valued field** (K, v). If w is a coarsening of v and $a \in K$, we say that w is **the principal coarsening of** v **generated by** a if $\mathcal{O}_w$ is the smallest overring of $\mathcal{O}_v$ containing a. We say that w is a **principal coarsening** of v if there exists an $a \in K$ such that w is the principal coarsening of v generated by a. We then define the **principal rank** of the valued field (G, v) as the order type of the set of all principal coarsenings of v.

The rank and the principal rank of a valued field can be characterized at three levels: at the level of the field itself, at the level of its value group and at the level of the value chain of the value group. Indeed, if $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ is a valued field, then the map $\mathcal{O}_w \mapsto G_w := v(\mathcal{U}_w)$ defines a bijection from the set of strict coarsenings of v to the set of non-trivial convex subgroups of $(G, \leq)$. Similarly, the map $G_w \mapsto \Gamma_w := v_G(G_w^{\neq 0})$ defines a bijection from the set of non-trivial convex subgroups of G to the set of final segments of Γ. From this we get the following result:

Proposition 2.3.2 (see [EP05, Section 2.3])
Let $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ be a valued field. Then the rank of (K, v) (respectively, the principal rank) is equal to the rank (respectively, the principal rank) of the ordered group $(G, \leq)$ and it is equal to the order type of the set of final segments (respectively, of principal final segments) of $(\Gamma, \leq)$.

Ordered fields can be seen as valued fields; indeed, we can equip an ordered field $(K, \leq)$ with its natural valuation v_{arch}, which makes (K, v_{arch}) a valued field. We then define the rank of an ordered field $(K, \leq)$ as the rank of the valued field (K, v_{arch}).

In [Kuh00], Kuhlmann developed a notion of rank for exponential ordered fields, called the exponential rank. The exponential rank is a subset of the rank of the ordered field $(K, \leq)$, and takes the exponential structure into account. More precisely, the author of [Kuh00] defines a notion of compatibility between coarsenings of v_{arch} and the exponential map, and then defines the exponential rank of the field as the order type of the set of all coarsenings of v_{arch} which are compatible with exp. She also showed that, under some additional assumptions on the exponential, the logarithm log naturally induces a contraction map χ defined on G by $\chi(v(a)) := v(\log(a))$, and that χ itself induces a map ζ on Γ by $\zeta(v_G(g)) := v_G(\chi(g))$. We recall from [Kuh94] that a **contraction map** on

G is a surjective precontraction, and that a **precontraction** on G is a map $\chi : G \to G$ satisfying the following properties:

(i) $\chi(g) = 0 \Leftrightarrow g = 0$.

(ii) χ preserves $\leq$.

(iii) $\chi(g) = -\chi(-g)$.

(iv) If $v_G(g) = v_G(h)$ and g, h have the same sign, then $\chi(g) = \chi(h)$.

If moreover χ satisfies $|\chi(g)| < |g|$ for all g, then χ is called a **centripetal precontraction**. Kuhlmann showed in [Kuh00] that the map χ induced on G by the logarithm of K satisfies these conditions. She also showed the following result:

Theorem 2.3.3 (see [Kuh00, Theorem 3.25])
The exponential rank of $(K, \leq)$ is equal to the order type of the set of all convex subgroups of G which are closed under χ, and it is also equal to the order type of the set of final segments of Γ which are closed under ζ.

Similarly, a notion of difference rank for quasi-ordered fields endowed with an automorphism was developed by Kuhlmann, Matusinski and Point in [KMP17]. Since our work on differential-valued fields in chapter 5 is inspired by their work, we recall now the main ideas and results of [KMP17]. Given a valued field $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ and a field automorphism σ of K, the authors of [KMP17] define σ to be compatible with v if $v(a) \leq v(b) \Leftrightarrow v(\sigma(a)) \leq v(\sigma(b))$ holds for every $a, b \in K$. They gave several characterizations of compatibility:

Theorem 2.3.4 ([KMP17, Theorem 4.2])
Let σ be compatible with v and w be a coarsening of v. The following conditions are equivalent:

(1) σ is compatible with w.

(2) $\sigma(\mathcal{O}_w) = \mathcal{O}_w$

(3) $\sigma(\mathcal{M}_w) = \mathcal{M}_w$

(4) $\sigma(\mathcal{U}_w) = \mathcal{U}_w$

(5) The map $Kw \to Kw, a + \mathcal{M}_w \mapsto \sigma(a) + \mathcal{M}_w$ is a well-defined $\frac{v}{w}$-compatible automorphism of Kw.

If σ is compatible with v, then σ naturally induces a map σ_G on G and a map σ_Γ on Γ, respectively defined by $\sigma_G(v(a)) := v(\sigma(a))$ and $\sigma_\Gamma(v_G(g)) := v_G(\sigma_G(g))$. If σ is compatible with v, then the authors of [KMP17] defined the σ-rank of the valued difference field (K, v, σ) as the the set of coarsenings w of v such that σ is compatible with w. They also defined the σ_G-rank of G as the set of convex subgroups of G which are σ_G-invariant. Finally, they defined the σ_Γ-rank of Γ as the set of final segments of Γ which are σ_Γ-invariant. They then showed the following:

Theorem 2.3.5 ([KMP17, Lemma 4.5 and Theorem 4.7])
The σ-rank of (K, v, σ) is in bijection with the σ_G-rank of G and with the σ_Γ-rank of Γ.

The authors of [KMP17] also defined the notion of principal σ-rank. A coarsening w of v is called σ-principal generated by a if $\mathcal{O}_w$ is the smallest σ-compatible overring of $\mathcal{O}_v$ containing a. The principal σ-rank of (K, v, σ) is the set of σ-principal coarsenings of v. Now assume that σ satisfies the condition $v(\sigma(a)) < v(a^2)$ for all $a \in K$ with $v(a) < 0$. Now define $\sim_\sigma$ on K by $a \sim_\sigma b$ if and only if there is $n \in \mathbb{N}_0$ with $v(\sigma^n(a)) \leq v(b)$ and $v(\sigma^n(b)) \leq v(a)$. One can check that $\sim_\sigma$ defines an equivalence relation on K. We denote by $\mathrm{cl}_\sigma(a)$ the $\sim_\sigma$-class of a. The set $[K/\sim_\sigma]$ is then naturally ordered by $\mathrm{cl}_\sigma(a) < \mathrm{cl}_\sigma(b) \Leftrightarrow \mathrm{cl}_\sigma(a) \neq \mathrm{cl}_\sigma(b) \wedge v(a) > v(b)$. Similarly, σ_G induces an equivalence relation on G, and σ_Γ induces an equivalence relation on Γ. The authors of [KMP17] showed the following:

Theorem 2.3.6 ([KMP17, Theorem 5.3 and Corollaries 5.4 and 5.5])
The σ-rank of (K, v, σ) is isomorphic to the set of initial segments of $K/\sim_\sigma$ and to the set of final segments of $\Gamma/\sim_{\sigma_\Gamma}$. The σ-principal rank of (K, v, σ) is isomorphic to the set of initial segments of $K/\sim_\sigma$ which have a maximum, and to the set of final segments of $\Gamma/\sim_{\sigma_\Gamma}$ which have a minimum.

The main goal of chapter 5 below will be to do the analog of the work done in [KMP17], this time for differential-valued fields.

2.4 Differential-valued fields

In chapter 5, we will develop a notion of rank for differential fields, in the spirit of what was done for exponential fields in [Kuh00] and for difference fields in [KMP17]. The differential fields which we are interested in are all generalizations of Hardy fields. This encompasses several classes of fields, whose definitions we want to recall here. We recall that a derivation on a a field K is a map $D : K \to K$ such that for any $a, b \in K$, $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$. If D is a derivation on K, then (K, D) is called a differential field, and the set $\{a \in K \mid D(a) = 0\}$ is a subfield of K called the field of constants of (K, D). A **differential-valued field** is a triple (K, v, D), where v is a field valuation on K and D a derivation such that the following is satisfied:

(DV1) $\mathcal{O}_v = \mathcal{M}_v + \mathcal{C}$, where $\mathcal{C}$ is the field of constants of (K, D).

(DV2) If $a \in \mathcal{O}_v, b \in \mathcal{M}_v$ and $b \neq 0$, then $v(D(a)) > v(\frac{D(b)}{b})$.

This is the definition given by Rosenlicht in [Ros80]. In [AvdD02a], Aschenbrenner and v.d.Dries defined the slightly more general notion of **pre-differential-valued field**: these are the valued fields with a derivation D satisfying (DV2). A pre-differential valued field does not necessarily satisfy (DV1). Note however that (DV2) implies $\mathcal{C} \subseteq \mathcal{U}_v$. In [AvdD02a], Aschenbrenner and v.d.Dries showed that any pre-differential-valued field can be embedded into a differential-valued field. If (K, v, D) is a pre-differential valued field, then we will denote by ϕ its logarithmic derivative restricted to elements of non-trivial valuation, i.e $\phi : K\backslash(\mathcal{U}_v \cup \{0\}) \to K$, $a \mapsto \frac{D(a)}{a}$. Note that if (K, v, D) is a pre-differential-valued field and $a \in K$, $a \neq 0$, then (K, v, aD) is also a pre-differential-valued field, where aD denotes the derivation $b \mapsto aD(b)$. A pre-differential-valued field (K, v, D) is said to have **asymptotic integration** if for any $a \in K$, there exists $b \in K$ such that $v(D(b) - a) > v(a)$.

The authors of [AvdD02a] also introduced a class of differential-valued fields called H-fields which are particularly significant for the theory of transseries and the model-theoretic study of Hardy fields (see in particular [MAv17]). They also introduced the weaker notion of pre-H-field and showed that any pre-H-field can be embedded into a H-field. We recall their definition: A **pre-H-field** is a valued ordered differential field $(K, v, \leq, D)$ such that :

(PH1) (K, v, D) is a pre-differential-valued field

(PH2) $\mathcal{O}_v$ is $\leq$-convex.

(PH3) for all $a \in K, a > \mathcal{O}_v \Rightarrow D(a) > 0$.

A **H-field** is an ordered differential field $(K, \leq, D)$ such that

(H1) $(K, v, \leq, D)$ is a pre-H-field, where $\mathcal{O}_v := \{a \in K \mid \exists c \in \mathcal{C}, |a| \leq c\}$.

(H2) $\mathcal{O}_v = \mathcal{C} + \mathcal{M}_v$

An important consequence of axiom (DV2) is that the map ϕ naturally induces a map $\psi : G\backslash\{0\} \to G$ defined by $\psi(v(a)) := v(\phi(a))$. Rosenlicht showed in [Ros80] that this map satisfies the following properties:

(AC1) $\psi(g + h) \geq \min(\psi(g), \psi(h))$ for any $g, h \neq 0$ with $g + h \neq 0$.

(AC2) $\psi(ng) = \psi(g)$ for any $g \neq 0$, $n \in \mathbb{Z}\backslash\{0\}$.

(AC3) $\psi(g) < \psi(h) + |h|$ for any $g, h \neq 0$.

If moreover K is a H-field, then one can show that ψ also satisfies the condition:

(ACH) $\forall g, h \neq 0, g \leq h < 0 \Rightarrow \psi(g) \leq \psi(h)$.

This remark lead the authors of [AvdD02a] to introduce the notion of asymptotic couple: an **asymptotic couple** is a pair (G, ψ) satisfying axioms (AC1),(AC2) and (AC3) above. If moreover (ACH) is satisfied then we say that (G, ψ) is a **H-type asymptotic couple**. If (G, ψ) and (H, ψ') are two asymptotic couples, we say that $\iota : (G, \psi) \hookrightarrow (H, \psi')$ is an **embedding of asymptotic couples** is ι is an embedding of ordered groups such that $\iota(\psi(g)) = \psi'(\iota(g))$ for any $g \in G\backslash\{0\}$. The class of H-type asymptotic couples is particularly simple to study, because in this case the map ψ is constant on archimedean classes of G, which is not the case for asymptotic couples in general. This will become important in Section 5.4 when we define a derivation on power series. If (G, ψ) is H-type, we then say that D is a **H-derivation**. If $(K, v, \leq, D)$ is a pre-H-field, then D is a H-derivation. Note that if (G, ψ) is an asymptotic couple, then (G, ψ') is still an asymptotic couple for any translate $\psi' := \psi + h$ of ψ (where $h \in G$). Note also that if (K, v, D) is a pre-differential-valued field with asymptotic couple (G, ψ) and if $a \in K$, then the pre-differential-valued field (K, v, aD) has asymptotic couple (G, ψ'), where ψ' is the map $g \mapsto \psi(g) + v(a)$. Given an asymptotic couple (G, ψ), we denote by D_G the map $D_G(g) := \psi(g) + g$ from $G\backslash\{0\}$ to G. We denote by Ψ the image of ψ. We know from [AvdD02a, Proposition 2.3(4)] that $D_G : G\backslash\{0\} \to G$ is strictly increasing, so in particular it is injective. An asymptotic couple (G, ψ) is said to have **asymptotic integration** if the map D_G is surjective. Obviously, a pre-differential-valued field has asymptotic integration if and only if its asymptotic couple has it.

2.5 Generalized power series

We have seen above that one way of constructing a valued group of given skeleton is taking a Hahn product. Similarly, it is possible to construct a valued field of given value group and given residue field by taking a field of generalized power series. Let k be a field and let $(G, \leq)$ be an ordered abelian group. Define $k((G)) := \{a = (a_g)_{g \in G} \in k^G \mid \operatorname{supp}(a) \text{ is well-ordered}\}$. We endow $k((G))$ with component-wise addition and a multiplication defined as follows: $a.b = (c_g)_{g \in G}$, where $c_g = \sum_{h \in G} a_h b_{g-h}$. One can show that this operation is well-defined and makes $(k((G)), 0, 1, +, .)$ a field, which we call **the field of generalized power series with coefficients in k and exponents in G**. The field $K := k((G))$ is naturally endowed with a valuation defined by $v(a) := \min \operatorname{supp}(a)$. The valued field (K, v) then has value group G and residue field k. It is usual to denote an element $(a_g)_{g \in G}$ of K as a formal sum $\sum_{g \in G} a_g t^g$. If $g = v(a)$, we say that a_g is the **leading coefficient** and $a_g t^g$ the **leading term** of a. Note that, as a group, $(K, 0, +)$ is the Hahn product of the family indexed by $(G, \leq)$ whose every member is the group $(k, 0, +)$. If $(k, \leq)$ is an ordered field, then there is a natural way of extending $\leq$ to an order on K, namely by taking the lexicographic product of the family $(\leq)_{g \in G}$ (one can easily show that this is indeed a field order). In other words, we define $\sum_{g \in G} a_g t^g \leq \sum_{g \in G} b_g t^g$ if and only if $a_h \leq b_h$, where $h = v(a - b)$.

In chapter 5, we will have to deal with infinite sums in fields of generalized power series. However, an infinite sum is *a priori* not well-defined, which is why we need to introduce the notion of summable family. Let $(a_i)_{i \in I}$ be a family of elements of K with $a_i = \sum_{g \in G} a_{i,g} t^g$. We say that the family $(a_i)_{i \in I}$ is **summable** if the following two conditions are satisfied:

(i) The set $A := \bigcup_{i \in I} \operatorname{supp}(a_i)$ is well-ordered

(ii) for any $g \in A$, the set $A_g := \{i \in I \mid a_{i,g} \neq 0\}$ is finite.

If the family $(a_i)_{i \in I}$ is summable, we define its sum as $\sum_{g \in G} b_g t^g \in K$, where $b_g = 0$ if $g \notin A$ and $b_g = \sum_{i \in A_g} a_{i,g}$ if $g \in A$.

The "classical" field of power series $\mathbb{C}((\mathbb{Z}))$ admits a natural derivation defined by $D(\sum_{n\in\mathbb{Z}} a_n t^n) = \sum_{n\in\mathbb{Z}} n a_n t^{n-1}$. However, this definition makes no sense for an arbitrary group of exponents G, since it is not clear what "na_n" and "$n-1$" should mean. Therefore, there is no "obvious" way of defining a derivation on $k((G))$. If k itself is already endowed with a derivation D, then we can extend D to $k((G))$ by setting $D(\sum_{g\in G} a_g t^g) := \sum_{g\in G} D(a_g)t^g$. This idea was developed by Scanlon in [Sca00, Section 6]. However, such a derivation does not make $(k((G)), v, D)$ a pre-differential-valued field. Indeed, if D is a derivation as in [Sca00], then for all $g \in G$ we have $D(t^g) = 0$, which contradicts axiom (DV2). Scanlon's derivation also has another drawback, which is the fact that it is not strongly linear. A derivation D on K is said to be **strongly linear** if $D(\sum_{g\in G} a_g t^g) = \sum_{g\in G} a_g D(t^g)$ for any $\sum_{g\in G} a_g t^g \in K$. Strong linearity is a natural condition to require when working with generalized power series, because this condition is satisfied by the usual derivation of $\mathbb{C}((\mathbb{Z}))$. The problem of defining a derivation making (K, v, D) a differential-valued field was tackled by Kuhlmann and Matusinski in [KM12] and [KM11]. We will also address this problem in Section 5.4.

2.6 C-relations

Chapter 4 of this thesis deals with C-groups, whose definition we recall here. The notion of C-structure was first introduced by Adeleke and Neumann in [AN96] and [AN98]. Macpherson and Steinhorn then developed the notion of C-minimality and C-minimal groups in [MS96]. Delon then gave a slightly more general definition of C-structures in [Del11], which is the one we give here.

A **C-relation** on a set M (see [Del11]) is a ternary relation C satisfying the universal closure of the following axioms:

(C_1) $C(x,y,z) \Rightarrow C(x,z,y)$

(C_2) $C(x,y,z) \Rightarrow \neg C(y,x,z)$

(C_3) $C(x,y,z) \Rightarrow C(w,y,z) \vee C(x,w,z)$

(C_4) $x \neq y \Rightarrow C(x,y,y)$

Note that (C_2) implies $\neg C(x,x,x)$ for all x. Note also that In [MS96], Macpherson and Steinhorn consider a more restrictive notion of C-relations, were axiom (C_4) is replaced with the axioms $\exists x, y, x \neq y$ and $x \neq y \Rightarrow (\exists z, (y \neq z \wedge C(x,y,z)))$. The notion of C-relation studied in [MS96] is now called a **dense C-relation**.

If G is a group and C a C-relation on G, then we say that C is **compatible** (with the group operation) if $C(x,y,z)$ implies $C(vxu, vyu, vzu)$ for any $x, y, z, u, v \in G$. A **C-group** is a pair (G, C) consisting of a group G with a compatible C-relation C.

Example 2.6.1
There are two fundamental examples of C-groups:

(a) If $(G, \leq)$ is a totally ordered group, then $\leq$ induces a compatible C-relation defined by $C(x, y, z) \Leftrightarrow (y < x \wedge z < x) \vee (y = z \neq x)$. Such a C-relation is called an **order-type** C-relation. If P is the positive cone of $(G, \leq)$, then we can express C with P by the formula $C(x, y, z) \Leftrightarrow (yx^{-1} \notin P \wedge zx^{-1} \notin P) \vee (y = z \neq x)$.

(b) If (G, v) is a valued group, then v induces a compatible C-relation by $C(x, y, z) \Leftrightarrow v(yz^{-1}) > v(xz^{-1})$. Such a C-relation is called a **valuational** C-relation.

If (G, C) is a C-group, then we say that C is a **fundamental C-relation** if it is either order-type or valuational.

We say that a structure $\mathcal{M} = (M, C, \dots)$ endowed with a C-relation is **C-minimal** if, for every $\mathcal{N} = (N, C, \dots)$ such that $\mathcal{N} \equiv \mathcal{M}$, every subset of N definable with parameters is quantifier-free definable in the language $\{C\}$.

C-relations are connected to meet-semilattice trees. We recall that a meet-semilattice tree is a partially ordered set $\mathfrak{T}$ such that:

(i) For any $x \in \mathfrak{T}$, $\{y \in \mathfrak{T} \mid y \leq x\}$ is totally ordered.

(ii) Any two elements of $\mathfrak{T}$ has a greatest lower bound.

If $\mathfrak{T}$ is a meet-semilattice tree and M a set of maximal branches of $\mathfrak{T}$, then we can define a C-relation on M as follows: $C(x, y, z)$ holds if and only if the meet of x and z lies strictly below the meet of y and z. Conversely, if (M, C) is an arbitrary C-structure then we can canonically associate a meet-semilattice tree $\mathfrak{T}$, called the canonical tree of (M, C), so that (M, C) is isomorphic to a set of maximal branches of $\mathfrak{T}$ endowed with the C-relation given above. To study C-minimal structures, it might be practical to consider their canonical trees: in [MS96], Macpherson and Steinhorn described C-minimal groups by looking at the action induced by the group on its canonical tree. We will do the same in Section 4.4.1.

2.7 Quasi-orders

We now introduce the central notion of this thesis, the notion of quasi-order. Our use of quasi-orders is inspired by Fakhruddin's work on quasi-ordered fields in [Fak87] (see further below).

Definition 2.7.1
A **quasi-order** (q.o.) is a binary relation which is reflexive, transitive and total.

If $\precsim$ is a quasi-order on a set A, then $\precsim$ induces an equivalence relation on A defined as follows: we say that $a \sim b$ if and only if $a \precsim b \precsim a$.

Notation
The symbol $\precsim$ will always denote a quasi-order, whereas $\leq$ will always denote an order.

The symbol $\sim$ will always denote the equivalence relation induced by the quasi-order $\precsim$ and $\mathrm{cl}(a)$ will denote the class of a for this equivalence relation. The notation $a \precnsim b$ means $a \precsim b \wedge a \nsim b$. If S, T are two subsets of a quasi-ordered set $(A, \precsim)$, the notation $S \precsim T$ (respectively $S \precnsim T$) means that $s \precsim t$ (respectively $s \precnsim t$) for any $(s,t) \in S \times T$. If $a \in A$, we write $S \precsim a$ instead of $S \precsim \{a\}$.

The q.o $\precsim$ induces a total order on the quotient $A/\sim$ by $\mathrm{cl}(a) \leq \mathrm{cl}(b)$ if and only if $a \precsim b$. There are two particularly important examples of q.o.'s:

Example 2.7.2 (a) A total order is in particular a q.o.

(b) Let (G, v) be a valued group. Then $a \precsim b \Leftrightarrow v(b) \leq v(a)$ defines a quasi-order on G, called the quasi-order induced by v. We say that a q.o. on a group is **valuational** if it is induced by a valuation. A q.o. induced by a $\mathbb{Z}$-module valuation is called **$\mathbb{Z}$-module valuational**. We will say that $(G, \precsim)$ is a **valuationally quasi-ordered group** if $\precsim$ is valuational.

Remark 2.7.3: If $\precsim$ is a q.o. on a group G, then $\precsim$ is valuational if and only if the following conditions hold:

(i) For all $g \in G$, $1 \neq g \Rightarrow 1 \precnsim g$.

(ii) For all $g, h \in G$, $g \precsim h \Rightarrow gh \precsim h$.

(iii) For all $g \in G$, $g \sim g^{-1}$.

(iv) For all $g, h, z \in G$, $g \precsim h \Rightarrow g^z \precsim h^z$.

If $(A, \precsim_A)$ and $(B, \precsim_B)$ are two quasi-ordered sets and $\phi : A \to B$ a map, then we say that ϕ is **increasing** if for any $a, b \in A$, $a \precsim_A b \Rightarrow \phi(a) \precsim_B \phi(b)$. We say that ϕ is **quasi-order-preserving** if the stronger condition $a \precsim_A b \Leftrightarrow \phi(a) \precsim_B \phi(b)$ holds. We say that ϕ is **quasi-order-reversing** if $a \precsim_A b \Leftrightarrow \phi(b) \precsim_B \phi(a)$. A **coarsening** of a q.o. $\precsim$ is a q.o. $\precsim^*$ such that $a \precsim b \Rightarrow a \precsim^* b$ for any $a, b \in A$. We also say that $\precsim$ is a **refinement** of $\precsim^*$. The **trivial q.o.** on A is the q.o which only has one equivalence class, i.e $a \precsim b$ for every $a, b \in A$; we usually denote it by $\precsim_t$. Many notions pertaining to orders have a useful analog for quasi-orders, which we will define now. If $a, c, b \in A$, then we say that c is **between a and b** if $a \precsim c \precsim b$ or $b \precsim c \precsim a$ holds. If the stronger condition $a \precnsim c \precnsim b \vee b \precnsim c \precnsim a$ holds, then we then say that c is **strictly between** a and b. If S is a subset of A, we define the **maximum** (respectively **minimum**) of S as the set of all elements s of S such that $t \precsim s$ (respectively $s \precsim t$) for every $t \in S$; we denote it by $\max(S)$ (respectively $\min(S)$). Note that the maximum of S is always defined but can be empty. We say that S is:

- an **initial segment** (respectively, a final segment) of A if for every $s \in S$ and $a \in A$, $a \precsim s$ (respectively, $s \precsim a$) implies $a \in S$
- **convex** in A if for every $s, t \in S$ and every $a \in A$, $s \precsim a \precsim t$ implies $a \in S$.

- **strictly convex** in A if for every $s, t \in S$ and every $a \in A$, $s \lnsim a \lnsim t$ implies $a \in S$.
- **left-convex** (respectively, **right-convex**) if for every $s, t \in S$ and every $a \in A$, $s \lesssim a \lnsim t$ (respectively $s \lnsim a \lesssim t$) implies $a \in S$.

Remark 2.7.4: In the case of orders, the notions of convex and strictly convex coincide, but they are different in general for quasi-orders.

If S is strictly convex, we define the **convexity complement** of S as the smallest subset T of $A \backslash S$ such that $S \cup T$ is convex. Note that being left-convex or right-convex implies being strictly convex. We can characterize strict convexity by the following lemma:

Lemma 2.7.5
For any $S \subseteq A$, S is strictly convex if and only if one of the following conditions holds:

(i) S is convex. In that case the convexity complement of S is $\varnothing$.

(ii) $\min(S) \neq \varnothing$ and $S \cup \mathrm{cl}(m)$ is convex for any $m \in \min(S)$. In that case S is right-convex and its convexity complement is $\mathrm{cl}(m) \backslash S$.

(iii) $\max(S) \neq \varnothing$ and $S \cup \mathrm{cl}(M)$ is convex for any $M \in \max(S)$. In that case S is left-convex and its convexity complement is $\mathrm{cl}(M) \backslash S$.

(iv) $\min(S), \max(S)$ are both non-empty and $S \cup \mathrm{cl}(m) \cup \mathrm{cl}(M)$ is convex for any $m \in \min(S)$ and $M \in \max(S)$. In that case the convexity complement of S is $(\mathrm{cl}(m) \cup \mathrm{cl}(M)) \backslash S$.

Proof. It is easy to check that if one of these conditions hold then S is strictly convex. Let us prove the converse. Assume that S is strictly convex but not convex. Then there exists $m, t \in S$ and $a \notin S$ such that $m \lesssim a \lesssim t$. However, since S is strictly convex, we cannot have $m \lnsim a \lnsim t$. Without loss of generality, we can thus assume that $m \sim a$. Assume that $m \notin \min(S)$ and $m \notin \max(S)$. Then there are $s, M \in S$ with $s \lnsim a \sim m \lnsim M$. Since S is strictly convex it follows that $a \in S$, which is a contradiction. Thus, we either have $m \in \min(S)$ or $m \in \max(S)$. If $S \cup \mathrm{cl}(m)$ is convex we are in case (ii) or (iii). Assume then that it is not convex. Without loss of generality, we may assume $m \in \min(S)$. Take $b \notin S \cup \mathrm{cl}(m)$ and $M \in S \cup \mathrm{cl}(m)$ with $m \lnsim b \lesssim M$. Since $M \notin \mathrm{cl}(m)$ we have $M \in S$. By strict convexity of S we must have $b \sim M$. If $M \notin \max(S)$ then we would have $m \lnsim b \lnsim M'$ for a certain $M' \in S$ which would imply $b \in S$, so we must have $M \in \max(S)$. Now let us proves that $S \cup \mathrm{cl}(m) \cup \mathrm{cl}(M)$ is convex, so that we are in case (iv). Let $c \in A$ such that there is $s, t \in S \cup \mathrm{cl}(m) \cup \mathrm{cl}(M)$ with $s \lesssim c \lesssim t$. Since m, M are respectively minimal and maximal in S we have $m \lesssim c \lesssim M$. If $c \notin \mathrm{cl}(m) \cup \mathrm{cl}(M)$ then we even have $m \lnsim c \lnsim M$, which by strict convexity of S implies $c \in S$. The statements about the convexity complement are clear. $\square$

Our original motivation for considering quasi-orders was to find a good generalization of total orders and valuations. Indeed, many results from the theory of ordered fields are similar to some results from the theory of valued fields, but are still stated as separate statements. In [Fak87], Fakhruddin introduced the notion of quasi-ordered field, in the hope of unifying the theory of ordered fields with the theory of valued fields. He gave the following definition of a quasi-ordered field:

Definition 2.7.6
A *quasi-ordered field* is a field K endowed with a quasi-order $\precsim$ satisfying the following axioms:

(Q_1) $\forall x(x \sim 0 \Rightarrow x = 0)$

(Q_2) $\forall x, y, z(x \precsim y \nsim z \Rightarrow x + z \precsim y + z)$

(Q_3) $\forall x, y, z, (x \precsim y \wedge 0 \precsim z) \Rightarrow xz \precsim yz$

From the definition of q.o. fields, we easily see the following:

Proposition 2.7.7
If $(K, \leq)$ is an ordered field, then it is in particular a quasi-ordered field (i.e $\leq$ satisfies the axioms above). If (K, v) is a valued field then $(K, \precsim_v)$ is a quasi-ordered field, where $\precsim_v$ denotes the q.o. induced by v.

Conversely, Fakhruddin showed the following:

Theorem 2.7.8 (Fakhruddin's dichotomy, see [Fak87, Theorem 2.1])
Let $(K, \precsim)$ be a quasi-ordered field. Then $\precsim$ is either a field order or the quasi-order induced by a field valuation.

Proposition 2.7.7 and Theorem 2.7.8 show that the axiomatization of Definition 2.7.6 is a good unification of orders and valuations on fields, by which we mean that some statements from the theory of ordered fields can be unified into a single statement with statements from the theory of valued fields. For example, in [KMP17], Kuhlmann, Point and Matusinski defined the notion of compatibility between a valuation and a q.o., which generalizes both the notion of v being a convex valuation (when the q.o. is an order) and the notion of v being a coarsening of a valuation (when the q.o. is valuational). They then showed the following, which generalizes a well-known statement about convex valuations:

Theorem 2.7.9 (see [KMP17, Theorem 2.2])
Let $(K, \precsim)$ be a quasi-ordered field and v a valuation on K. The following are equivalent:

(1) v is compatible with $\precsim$.

(2) $\mathcal{O}_v$ is convex in $(K, \precsim)$.

(3) $\mathcal{M}_v$ is convex in $(K, \precsim)$.

(4) $\mathcal{M}_v \precsim 1$.

(5) $\precsim$ induces a q.o on the residue field $\mathcal{O}_v/\mathcal{M}_v$.

In the special case where $\precsim$ is an order, Theorem 2.7.9 gives conditions for a valuation v to be $\leq$-convex. If $\precsim$ comes from a valuation w then Theorem 2.7.9 gives conditions for v to be a coarsening of w.

The main motivation behind the theory developed in chapter 3 is to unify the theory of totally ordered groups with the theory of valued groups, and in particular we would like to find analogs of Theorems 2.7.9 and 2.3.1 for quasi-ordered groups. Throughout this thesis, a **quasi-ordered group** (q.o. group) is just a group endowed with a quasi-order without any further assumption. The following definition will be important in Chapters 3 and 4:

Definition 2.7.10
Let $(G, \precsim)$ be a q.o. group and $g \in G$. We say that g is:

- **v-type** if $g \sim g^{-1}$.
- **o-type** if $g = 1 \vee g \nsim g^{-1}$.
- **o$^+$-type** if $g^{-1} \precnsim g$.
- **o$^-$-type** if $g \precnsim g^{-1}$.

Note that 1 is the only element which is both v-type and o-type. The reason for this terminology will become clear in chapters 3 and 4. Roughly speaking, in the setting of this thesis, the o-type elements will correspond to a part of the group where the q.o. is an order, and the v-type elements will correspond to a part of the group where the q.o. behaves like a valuation. The set of v-type (respectively o-type/ o$^+$-type/o$^-$-type) elements of a q.o. group will usually be denoted by $\mathcal{V}$ (respectively $\mathcal{O}$/ $\mathcal{O}^+$/$\mathcal{O}^-$).

A **homomorphism of q.o. groups** is a homomorphism of groups $\phi : (G, \precsim_G) \to (H, \precsim_H)$ which is also increasing. If moreover ϕ is injective, and if ϕ^{-1} is also increasing, then we will say that ϕ is an **embedding of q.o. groups**. If moreover ϕ is bijective, then we say that ϕ is an **isomorphism of q.o. groups**. Note that a bijective homomorphism of q.o. groups is not necessarily an isomorphism of q.o. groups. To see this, consider $\mathbb{Q}^2$ endowed with the lexicographic product $\leq$ of the usual order of $\mathbb{Q}$. Now let $\precsim$ denote the valuational q.o associated to the archimedean valuation of $(\mathbb{Q}^2, \leq)$. The identity map on $\mathbb{Q}^2$ is a homomorphism of q.o groups from $(\mathbb{Q}^2, \leq)$ to $(\mathbb{Q}^2, \precsim)$, but it is not an isomorphism.

In the theory of ordered groups, we know that there is a notion of induced order on a quotient G/H if H is a convex normal subgroup of $(G, \leq)$. We will want to develop a similar notion for q.o.'s in Chapters 3 and 4. Note that, if $(G, \precsim)$ is any q.o. group and H a normal subgroup of G, then $\precsim$ naturally induces a binary relation on G/H, which we will also denote by $\precsim$, and which is defined as follows:

$$g_1H \precsim g_2H \Leftrightarrow (\exists h_1, h_2 \in H, g_1h_1 \precsim g_2h_2)$$

It is easy to see that $\lesssim$ thus defined on G/H is always reflexive and total, but not always transitive, which is why we introduce the following definition:

Definition 2.7.11
Let $(G, \lesssim)$ be a q.o. group and H a normal subgroup of G. We say that $\lesssim$ **induces a q.o. on** G/H if the following relation defined on G/H is transitive:

$$g_1H \lesssim g_2H \Leftrightarrow (\exists h_1, h_2 \in H, g_1h_1 \lesssim g_2h_2)$$

In that case, the relation $\lesssim$ thus defined on G/H is a total q.o which we call **the q.o induced by** $\lesssim$ **on** G/H.

In the case where $\lesssim$ is an order, the definition of the q.o. induced on the quotient given in Definition 2.7.11 coincides with the definition given in [Fuc63].

We also need to introduce a notion of lifting of q.o.'s in order to obtain an analog of theorem 2.3.1. The process of lifting should be thought of as the opposite process of quotienting. For example, if H is a normal subgroup of G and if the quotient G/H is endowed with a q.o. $\lesssim_{G/H}$, then a lifting of $\lesssim_{G/H}$ to G should be a q.o. whose quotient on G/H is $\lesssim_{G/H}$. We actually define a more general notion of lifting which holds not just for a quotient G/H but for a (possibly infinite) family of quotients. Let G be a group and $v : G \to \Gamma \cup \{\infty\}$ a valuation. Assume that for each $\gamma \in \Gamma$, the quotient G^γ/G_γ is endowed with a q.o $\lesssim_\gamma$.

Definition 2.7.12
A **lifting** of $(\lesssim_\gamma)_{\gamma\in\Gamma}$ to G is a q.o. $\lesssim$ on G such that for every $\gamma \in \Gamma$, $\lesssim$ induces the q.o. $\lesssim_\gamma$ on G^γ/G_γ.

Note that a lifting does not always exist, and if it exists it might not be unique.

In the theory of ordered groups, we have a natural notion of product which is the lexicographic product. It seems natural to define a notion of product for quasi-orders as well. Several notions of product are possible. The notion of lexicographic product of ordered sets can easily be generalized to q.o. sets: given two q.o. sets $(A, \lesssim_A)$ and $(B, \lesssim_B)$, we define their lexicographic product as the q.o. set $(A \times B, \lesssim)$, where $\lesssim$ is defined as follows: $(a_1, b_1) \lesssim (a_2, b_2) \Leftrightarrow (a_1 \precnsim a_2) \vee (a_1 \sim a_2 \wedge b_1 \lesssim b_2)$. However, this notion of product is not quite adapted to the study of q.o. groups, because the product of two valuational q.o.'s is not in general valuational:

Example 2.7.13
Take $B_1 = B_2 = \mathbb{Z}$ and let $\lesssim_1 = \lesssim_2$ be the q.o induced by the trivial valuation on $\mathbb{Z}$. Let $(G, \lesssim)$ be the lexicographic of $(B_1, \lesssim_1)$ and $(B_2, \lesssim_2)$. Now let $g := (1, 0)$ and $h := (0, 1)$. Then we have $h \precnsim g$, but $g \precnsim g + h$, which contradicts the ultrametric inequality.

For this reason, we introduce another notion of product. Let $(B_\gamma, \lesssim_\gamma)_{\gamma\in\Gamma}$ be a family of q.o. groups indexed by an ordered set Γ. Set $G := \mathrm{H}_{\gamma\in\Gamma} B_\gamma$ and let v denote the usual valuation on G.

Definition 2.7.14
The **valuational product** of the family $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$ is the q.o. group $(G, \precsim_{\text{val}})$, where $\precsim_{\text{val}}$ is defined on G as follows: $(g_\gamma)_{\gamma\in\Gamma} \precsim_{\text{val}} (h_\gamma)_{\gamma\in\Gamma} \Leftrightarrow g_\delta \precsim h_\delta$ where $\delta = \min(v(g), v(h))$.

We then have the following:

Proposition 2.7.15
Let $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$ be an ordered family of valuationally quasi-ordered groups and let $(G, \precsim_{\text{val}})$ be the valuational product of the family $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$. Then $\precsim_{\text{val}}$ is valuational.

Proof. We use Remark 2.7.3 Let $g, h, z \in G$. Assume $h \neq 1$ and set $\delta := v(h)$. Since $\precsim_\delta$ is valuational, we have $1 \precnsim_\delta h_\delta$ and $h_\delta \sim_\delta h_\delta^{-1}$, which implies by definition of $\precsim_{\text{val}}$ that $1 \precnsim_{\text{val}} h$ and $h \sim_{\text{val}} h^{-1}$. Now assume $g \precsim_{\text{val}} h$ and set $\delta = \min(v(g), v(h))$. We have $g_\delta \precsim_\delta h_\delta$. Since $\precsim_\delta$ is valuational, it follows that $g_\delta^{z_\delta} \precsim_\delta h_\delta^{z_\delta}$ and $g_\delta h_\delta \precsim_\delta h_\delta$. Now note that $h^z = (h_\gamma^{z_\gamma})_{\gamma\in\Gamma}$ and $g^z = (g_\gamma^{z_\gamma})_{\gamma\in\Gamma}$. It follows that $\delta = \min(v(g^z), v(h^z))$, and since $g_\delta^{z_\delta} \precsim_\delta h_\delta^{z_\delta}$ it follows from the definition of $\precsim_{\text{val}}$ that $g^z \precsim_{\text{val}} h^z$. Moreover, we have $v(gh) \geq \min(v(h), v(g))$, so $\delta = \min(v(g), v(gh))$. Since $g_\delta h_\delta \precsim_\delta h_\delta$, it follows from the definition of $\precsim_{\text{val}}$ that $gh \precsim_{\text{val}} h$.

□

However, it is easy to see that the valuational product of ordered sets is not an ordered set:

Example 2.7.16
Let $B_1 = B_2 = \mathbb{Z}$ and let $\precsim_1 = \precsim_2$ be the usual order on $\mathbb{Z}$. Let $(G, \precsim)$ be the valuational product of the family $(B_1, \precsim_1), (B_2, \precsim_2)$. Then we have $(1, 0) \sim (1, 1)$, so $\precsim$ cannot be an order on G.

Note however that the set of positive elements in $(G, \precsim)$ remains the same as in the lexicographic product of B_1, B_2. In chapter 4, this will allow us to use the valuational product as a product which can be used both for valuations and orders.

In view of finding an analog of Theorem 2.7.9 for groups, we are now going to give a few lemmas which will allow us to prove the analog of 2.7.9 both for compatible q.o.'s (in chapter 3) and for C-q.o.'s (in chapter 4). We fix an arbitrary q.o. group $(G, \precsim)$ and a valuation $v : G \to \Gamma \cup \{\infty\}$.

Definition 2.7.17
We say that v is **compatible with** $\precsim$ if for every $g, h \in G$, $1 \precsim g \precsim h \Rightarrow v(g) \geq v(h)$.

Lemma 2.7.18
The following statements are equivalent:

(i) For all $\gamma \in \Gamma$, G^γ is convex in $(G, \precsim)$.

(ii) For all $\gamma \in \Gamma$, G_γ is convex in $(G, \precsim)$.

(iii) For all $\gamma \in \Gamma$, G_γ is convex in $(G^\gamma, \precsim)$.

Proof. Let us show (i)⇒(ii). Assume (i) holds and let $\gamma \in \Gamma$, $g, h \in G_\gamma$ and $f \in G$ with $h \precsim f \precsim g$. Set $\delta := \min(v(g), v(h))$. We have $\delta > \gamma$ so $G^\delta \subseteq G_\gamma$. Moreover, $g, h \in G^\delta$, which by $\precsim$-convexity of G^δ implies $f \in G^\delta \subseteq G_\gamma$. This proves (ii). The implication (ii)⇒(iii) is obvious, so let us show (iii)⇒(i). Assume (iii) holds and let $g, h \in G^\gamma$. If $f \notin G^\gamma$ then $v(f) < \gamma \leq v(g), v(h)$ hence $g, h \in G_{v(f)}$. Since $f \in G^{v(f)} \backslash G_{v(f)}$, it follows from assumption (iii) that we cannot have $h \precsim f \precsim g$. This proves (i).

□

Lemma 2.7.19
v is compatible with $\precsim$ if and only if $\{g \in G_\gamma \mid 1 \precsim g\}$ is convex in $(G, \precsim)$ for all $\gamma \in \Gamma$.

Proof. Assume v is compatible with $\precsim$. Take $f, h \in G_\gamma$ and $g \in G$ with $1 \precsim f \precsim g \precsim h$. Then by compatibility of v with $\precsim$, we have $v(g) \geq v(h) > \gamma$, hence $g \in G_\gamma$. This proves that $\{g \in G_\gamma \mid 1 \precsim g\}$ is convex in $(G, \precsim)$. Conversely, assume that each $\{g \in G_\gamma \mid 1 \precsim g\}$ is convex in $(G, \precsim)$. Let $g, h \in G$ with $v(h) > v(g)$. By definition of $G_{v(g)}$, we then have $h \in G_{v(g)}$. By convexity of $\{f \in G_{v(g)} \mid 1 \precsim f\}$, it follows that we cannot have $1 \precsim g \precsim h$. Thus, we have $1 \precsim g \precsim h \Rightarrow v(g) \geq v(h)$, so v is compatible. □

Lemma 2.7.20
Let $\precsim$ be a q.o. such that:

$$(*) \begin{cases} \mathrm{cl}(1) = \{1\}. \\ g \precsim h \nsim f^{-1} \Rightarrow gf \precsim hf \wedge fg \precsim fh \text{ for any } g, h, f \in G. \end{cases}$$

Let H be a normal subgroup of G. Then H is convex in $(G, \precsim)$ if and only if $\precsim$ induces a q.o. on G/H such that $\mathrm{cl}(1.H) = \{1.H\}$. Moreover, if H is convex in $(G, \precsim)$, then for every $g \in G \backslash H$, $g \precsim g^{-1}$ if and only if $gH \precsim g^{-1}H$. In particular, g is v-type (resp. o$^+$-type, o$^-$-type) if and only if gH is v-type (resp. o$^+$-type, o$^-$-type).

Proof. Assume H is convex and take $g_1, g_2, g_3 \in G$ with $g_1H \precsim g_2H$ and $g_2H \precsim g_3H$. We want to show that $g_1H \precsim g_3H$. If $g_1 \precsim g_3$, then this is immediate, so assume $g_3 \precsim g_1$. There is $h_1, h_2, h_2', h_3 \in H$ with $g_1h_1 \precsim g_2h_2$ and $g_2h_2' \precsim g_3h_3$. Assume first that $g_2, g_3 \in H$. If $g_1 \notin H$, then by convexity we have $g_1 \nsim h_1^{-1}$. By $(*)$, the inequality $g_3 \precsim g_1$ then implies $g_3h_1 \precsim g_1h_1$. We thus have $g_3h_1 \precsim g_1h_1 \precsim g_2h_2$, which by convexity of H implies $g_1 \in H$, which is a contradiction. Therefore, we must have $g_1 \in H$, and it follows that $g_1H \precsim g_3H$. Now assume $g_3 \in H$ and $g_2 \notin H$. By convexity of H, we have $g_2h_2 \nsim (h_2')^{-1}h_2$, so by $(*)$ the inequality $g_1h_1 \precsim g_2h_2$ implies $g_1h_1h_2^{-1}h_2' \precsim g_2h_2'$ hence $g_1h_1h_2^{-1}h_2' \precsim g_3h_3$, hence $g_1H \precsim g_3H$. Now assume that $g_3 \notin H$. Then by convexity of H, we have $g_3h_3 \nsim h_2^{-1}h_2'$, so by $(*)$ the inequality $g_2h_2' \precsim g_3h_3$ implies $g_2h_2 \precsim g_3h_3(h_2')^{-1}h_2$, hence $g_1h_1 \precsim g_3h_3(h_2')^{-1}h_2$, hence $g_1H \precsim g_3H$. This proves that $\precsim$ induces a q.o. on H. If $gH \sim H$, then there are $h_1, h_2, h_3, h_4 \in H$ with $gh_1 \precsim h_2$ and $h_3 \precsim gh_4$. If $g \notin H$, then by convexity $gh_4 \nsim h_1^{-1}h_4$, so by $(*)$ the inequality $h_3 \precsim gh_4$ implies $h_3h_4^{-1}h_1 \precsim gh_1$, so we have $h_3h_4^{-1}h_1 \precsim gh_1 \precsim h_2$, which by convexity of H implies $g \in H$: contradiction. Thus, $g \in H$. This proves that $\mathrm{cl}(1.H) = \{1.H\}$. Now assume that $\precsim$ induces a q.o. on

G/H such that $\mathrm{cl}(1.H) = \{1.H\}$, and let us show that H is convex in G. Take $f, g, h \in G$ with $f \precsim g \precsim h$ and $f, h \in H$. By definition of the induced q.o., we have $fH \precsim gH \precsim hH$, hence $gH \sim 1.H$, hence $g \in H$ by assumption. This proves that H is convex in G.

For the second statement, note that $g \precsim g^{-1}$ obviously implies $gH \precsim g^{-1}H$, so we just have to prove the converse. Assume then without loss of generality that $g^{-1} \precnsim g$. If $g \precnsim 1$, then by $(*)$ we have $1 \precsim g^{-1}$, which is a contradiction, so we must have $1 \precsim g \nsim g^{-1}$. By $(*)$, this implies $1 \precsim g \precsim g^2$. Now assume for a contradiction that $gH \precsim g^{-1}H$. Then there are $h_1, h_2 \in H$ with $gh_1 \precsim g^{-1}h_2$. By convexity of H, we have $g^{-1}h_2 \nsim h_1$, hence by $(*)$: $g \precsim g^{-1}h_2h_1^{-1}$. Since $g^{-1} \precnsim g$, we cannot have $g^{-1}h_2h_1^{-1} \sim g^{-1}$, so by $(*)$ we have $g^2 \precsim h_2h_1^{-1}$. We thus have $1 \precsim g \precsim g^2 \precsim h_2h_1^{-1}$, which by convexity of H implies $g \in H$: contradiction. Thus, we must have $g^{-1}H \precnsim gH$. $\square$

Chapter 3

Compatible quasi-ordered abelian groups

Introduction

The object of this chapter is the study of a class of quasi-ordered groups, which we call *compatible quasi-ordered abelian groups* (compatible q.o.a.g.'s), whose axiomatization was inspired by the axiomatization of quasi-ordered fields given by Fakhruddin in [Fak87] (see Definition 2.7.6). The original purpose behind the theory of quasi-ordered fields developed in [Fak87] was to develop a theory which would unify the theory of ordered fields with the theory of valued fields. Proposition 2.7.7 and Theorem 2.7.8 show that the axiomatization of Definition 2.7.6 is very suitable for this purpose. It is thus natural to try to do the same for groups. More precisely, we want to know if the group analog of Definition 2.7.6 (which is what we call a compatible q.o.a.g.) gives us an interesting class of quasi-ordered groups, i.e a good generalization of ordered and valued abelian groups. This raises the following question: Is there a group analog of Theorem 2.7.8? If not, what is the structure of a compatible q.o.a.g.? We answer these questions in Section 3.1, where we show that a compatible q.o.a.g. is an extension of a valued group by an ordered group (see Theorems 3.1.26, 3.1.28 and 3.1.29). In Section 3.2, we use compatible q.o.a.g.'s to show a group analog of Theorem 2.7.9 (Theorem 3.2.2). In Section 3.3, we define a notion of product for the category of compatible q.o.a.g.'s, called the compatible product. We use this notion of product to show a "Hahn's embedding theorem" for compatible quasi-ordered groups which generalizes Theorem 2.2.6 (see Theorem 3.3.8). We also show that the compatible product of an ordered group by a valued group preserves elementary equivalence in Section 3.3.3 (Theorem 3.3.13). Finally, in Section 3.4, we develop a notion of model-theoretic minimality for compatible q.o.a.g.'s which generalizes o-minimality. We show that this notion of minimality is equivalent to C-minimality, thus establishing a connection between compatible q.o.a.g.'s and C-groups (see Proposition 3.4.4).

Every group appearing in this chapter is abelian, which is why we will adopt the additive notation.

3.1 The structure of compatible q.o.a.g.'s

The definition of compatible q.o.a.g.'s is an adaptation of definition 2.7.6 for abelian groups:

Definition 3.1.1
Let G be an abelian group and $\precsim$ a q.o. on G. We say that $\precsim$ is **compatible (with $+$)** if it satisfies the following axioms:

(Q_1) $\forall x(x \sim 0 \Rightarrow x = 0)$.

(Q_2) $\forall x, y, z(x \precsim y \nsim z \Rightarrow x + z \precsim y + z)$.

We also say that the pair $(G, \precsim)$ is a **compatible q.o.a.g.** (quasi-ordered abelian group).

As in the case of fields, it is easy to check that, if $(G, \precsim)$ is actually an ordered abelian group or if $\precsim$ is a valuational q.o., then $\precsim$ is compatible with $+$. However, we have no analog of Fakhruddin's dichotomy, i.e there are some compatible q.o.'s which are not an order and do not come from a valuation. We will show this now by giving three different examples where the q.o. is neither an order nor valuational. One could directly check that these q.o.'s satisfy axioms (Q_1) and (Q_2), but this will actually be a consequence of Theorem 3.1.26.

Example 3.1.2 (a) Consider the group $G := \mathbb{Z}^2$ endowed with the following quasi-order: $(a, b) \precsim (c, d) \Leftrightarrow (c \neq 0) \vee (c = a = 0 \wedge b \leq d)$, where $\leq$ is the usual order of $\mathbb{Z}$. The q.o. is an order on $G^o := 0 \times \mathbb{Z}$ (it coincides with $\leq$) so it cannot be valuational. However, it cannot be an order on $\mathbb{Z}^2$ since we have $(a, b) \sim (c, d)$ for any $a, c \in \mathbb{Z}\backslash\{0\}$ and any b, d.

(b) Set $G := \mathbb{Z}$ and $G^o := 5\mathbb{Z}$. Endow G^o with its usual order $\leq$, and extend $\leq$ to a q.o. $\precsim$ on G by declaring that $f \precsim g \sim h$ for any $f \in G^o$ and $g, h \notin G^o$. Then $(G, +, \precsim)$ is a compatible q.o.a.g.

(c) Let $(K, \leq, \sigma)$ be an ordered difference field with the assumptions of [KMP17, Section 5]. In [KMP17], the authors defined an equivalence relation $\sim_\sigma$ on $P_K := K^{\geq 0}\backslash\mathcal{O}_v$. This equivalence relation is related to the difference rank of $(K, \leq, \sigma)$ (see Theorem 2.3.6 above). They also showed that the $\sim_\sigma$-classes are naturally ordered. This gives rise to a q.o. on P_K defined as $a \precsim_\sigma b \Leftrightarrow \mathrm{cl}_\sigma(a) \leq \mathrm{cl}_\sigma(b)$ (cl_σ denotes the $\sim_\sigma$-class of a). This q.o. can easily be extended to $K\backslash\mathcal{O}_v$ by declaring that $-a \sim_\sigma a$ for every a. Note that $\precsim_\sigma$ satisfies the ultrametric inequality on $K\backslash\mathcal{O}_v$. Now define a q.o. $\precsim$ on K as follows: if $a, b \in \mathcal{O}_v$, then $a \precsim b \Leftrightarrow a \leq b$; if $a, b \notin \mathcal{O}_v$ then $a \precsim b \Leftrightarrow a \precsim_\sigma b$; finally, declare $a \precnsim b$ whenever $a \in \mathcal{O}_v$ and $b \notin \mathcal{O}_v$. This makes $(K, +, \precsim)$ a compatible q.o.a.g. The q.o. $\precsim$ contains both the information about the order $\leq$ of K and some information about the σ-rank of K. Note that we can do a similar construction with H-fields if we replace $\precsim_\sigma$ by the q.o. $\precsim_\phi$ defined in Section 5.3.1.

Remark 3.1.3: (i) In the case where $\precsim$ is actually an order, note that (Q_2) is technically weaker than axiom (OG) (see Section 2.2) because of the condition "$y \nsim z$". However, the only ordered group which satisfies (Q_2) but not (OG) is $\mathbb{Z}/2\mathbb{Z}$ with the order $0 < 1$ (see Proposition 3.1.8 below), so (Q_2) and (OG) are essentially equivalent for orders.

(ii) The condition "$y \nsim z$" in (Q_2) is essential if we want to include valuational q.o.'s. Indeed, if $\precsim$ is a valuational q.o., and if we take $x \neq y = -z$ such that $x \precsim y$, then we have $x + z \neq 0$ and $y + z = 0$ which implies $y + z \precnsim x + z$.

3.1.1 o-type and v-type elements

We now fix a compatible q.o.a.g. $(G, \precsim)$ and investigate its structure. As mentioned in the introduction, we want to show that $(G, \precsim)$ is a mix of ordered and valued groups. The idea is to use the distinction between o-type and v-type elements (see Definition 2.7.10). The set of o-type elements of G will be denoted by $\mathcal{O}$, whereas the set of v-type elements of G will be denoted by $\mathcal{V}$. We also set $\mathcal{V}^* := \mathcal{V}\backslash\{0\}$. We recall that $\mathrm{ord}(g)$ denotes the order of g.

Proposition 3.1.4
Let $g \in G$. The following conditions are equivalent:

(1) $g \in \mathcal{O}$.

(2) $\mathrm{cl}(g) = \{g\} \wedge \mathrm{ord}(g) \neq 2$.

(3) $g \precnsim 0 \vee -g \precnsim 0$.

Proof. (2)⇒(1) is immediate. Now assume that $g \in \mathcal{O}$ and $0 \precsim -g$. Since $g \in \mathcal{O}$, we have $-g \nsim g$. It then follows from $0 \precsim -g$ and (Q_2) that $g \precsim -g + g = 0$. This shows (1)⇒(3). Now assume that (3) holds and let us show (2). Without loss of generality, we may assume $g \precsim 0$ and $g \neq 0$. By (Q_1), we have $g \precsim 0 \nsim -g$, which by (Q_2) implies $0 \precsim -g$. If $g \sim -g$ were true, then we would have $g \sim 0$, which is a contradiction to (Q_1). Thus, we have $g \nsim -g$. This implies in particular $\mathrm{ord}(g) \neq 2$. Now let us show $\mathrm{cl}(g) = \{g\}$. Let $h \in G$ with $h \sim g$. We have $h \precsim g$ and $g \precsim h$. Since $-g \nsim g \sim h$, we can apply (Q_2) to both inequalities and we get $h - g \precsim 0$ and $0 \precsim h - g$ which implies $g - h \sim 0$, which by (Q_1) means $h = g$. This proves $\mathrm{cl}(g) = \{g\}$. □

Remark 3.1.5: Proposition 3.1.4 already motivates the terminology "o-type". Indeed, the fact that $cl(g) = \{g\}$ holds for o-type elements implies in particular that $\precsim$ is an order on $\mathcal{O}$.

As mentioned in Remark 3.1.3, (Q_2) is not the same as axiom (OG) of ordered abelian groups, and it can in fact happen that a compatible quasi-order is an order but does not satisfy (OG):

Example 3.1.6
If we order $\mathbb{Z}/2\mathbb{Z}$ by $0 < 1$ then $(\mathbb{Z}/2\mathbb{Z}, \leq)$ does not satisfy (OG) but it is a compatible q.o.a.g. More precisely, $\leq$ is the q.o induced by the trivial valuation on $\mathbb{Z}/2\mathbb{Z}$.

Remarkably, this is the only pathological case. To show this, we need the following lemma:

Lemma 3.1.7
Assume $\precsim$ is an order and assume that G has an element of order 2. Then $G = \mathbb{Z}/2\mathbb{Z}$.

Proof. Let g be an element of order 2. By Proposition 3.1.4, we then have $0 \precsim g$. Let $h \neq g$. Since $\precsim$ is an order, we have $h \nsim g$. Therefore, we can apply (Q_2) to $0 \precsim g$, which yields $h \precsim g + h$. If $h \neq 0$, then $g \nsim g + h$, so we can apply (Q_2) to the previous inequality and get $g + h \precsim g + g + h = h$, hence $h \sim g + h$. But since $\precsim$ is an order, this implies $h = g + h$ hence $g = 0$, which is a contradiction. This proves $h \neq g \Rightarrow h = 0$. We thus have $G = \{0, g\} \cong \mathbb{Z}/2\mathbb{Z}$. □

Proposition 3.1.8
Let $(G, \precsim)$ be a compatible q.o.a.g. If $\precsim$ is an order and if $G \neq \mathbb{Z}/2\mathbb{Z}$, then $(G, \precsim)$ is an ordered abelian group, i.e. (OG) is satisfied.

Proof. We want to prove: $\forall x, y, z \in G, x \precsim y \Rightarrow x + z \precsim y + z$. Since $\precsim$ is an order, we have $y \sim z \Rightarrow y = z$ for any $y, z \in G$. Thus, we only have to consider the case where $y = z$, since the other cases are given by axiom (Q_2). Assume then that $x \precsim y$. Since $G \neq \mathbb{Z}/2\mathbb{Z}$, Lemma 3.1.7 ensures that $y \neq -y$, so $y \nsim -y$. We can then apply (Q_2) to $x \precsim y$ and we get $x - y \precsim 0$. Since $2y \neq 0$, we can apply (Q_2) to this inequality and obtain $x + y \precsim y + y$, which is what we wanted. □

Remark 3.1.9: Since the case $\mathbb{Z}/2\mathbb{Z}$ is somewhat degenerate, it would be tempting to exclude this case from the definition of compatible q.o.a.g's. However, this seems rather unreasonable in view of Proposition 3.1.20 below. Indeed, we want the class of compatible q.o.a.g's to be stable under quotient by convex subgroups, which would not be the case if $\mathbb{Z}/2\mathbb{Z}$ were excluded.

An immediate consequence of Proposition 3.1.8 is the following:

Proposition 3.1.10
The compatible q.o.a.g. $(G, \precsim)$ is an ordered group if and only if every element of G is o-type.

Proof. If $(G, \precsim)$ is an ordered group, then in particular we must have $\mathrm{cl}(g) = \{g\}$ for all $g \in G$. It then follows from Proposition 3.1.4 that $\mathcal{O} = G$. Conversely, assume $\mathcal{O} = G$. Then it follows from Proposition 3.1.4 that $\precsim$ is an order on G and that G has no element of order 2. By Proposition 3.1.8, it follows that $(G, \precsim)$ is an ordered group. □

We are now going to investigate $\mathcal{O}$ and $\mathcal{V}$ in more details and show that they have remarkable properties.

3.1.2 Properties of $\mathcal{O}$ and $\mathcal{V}$

Set $\Gamma = G/\sim$ and denote by $\leq$ the order induced by $\precsim$ on Γ. For any $\gamma \in \Gamma$, set $G^\gamma := \{g \in G \mid \mathrm{cl}(g) \leq \gamma\}$ and $G_\gamma := \{g \in G \mid \mathrm{cl}(g) < \gamma\}$

Remark 3.1.11: If $\precsim$ is the q.o. induced by a valuation v, then Γ with the reverse order of $\leq$ is isomorphic to $v(G)$. In that case, our definition of G^γ, G_γ coincides with the definition given in Section 2.2 for valued groups, i.e $G^\gamma = \{g \in G \mid v(g) \geq \gamma\}$ and $G_\gamma = \{g \in G \mid v(g) > \gamma\}$. If $(G, \leq)$ is an ordered abelian group then Γ and G are isomorphic as ordered sets.

The following two lemmas will have important consequences on $\mathcal{O}$ and $\mathcal{V}$:

Lemma 3.1.12
If $h \in \mathcal{V}$ and $g \not\sim h$ then $g - h \sim g + h$.

Proof. Since h is v-type, we have $h \precsim -h \precsim h$. Since $g \not\sim h$, we can apply (Q_2) to these inequalities and we get $g + h \precsim g - h \precsim g + h$. □

Lemma 3.1.13
Let H be an initial segment of G containing $\mathcal{O}$. Then H is a subgroup of G.

Proof. Since 0 is o-type, $0 \in \mathcal{O} \subseteq H$. Let $h \in H$. If $h \in \mathcal{V}$, then in particular $-h \precsim h$, so $-h \in H$ because H is an initial segment of G. If $h \in \mathcal{O}$, then $-h \in \mathcal{O} \subseteq H$. This shows that H is closed under taking the inverse, and we are now going to show that it is closed under addition. Let $g, h \in H$. We can assume $h \neq 0$. If $g \precsim 0$ then (Q_2) implies $g + h \precsim h \in H$ hence $g + h \in H$ because H is an initial segment of G. Now assume $0 \precsim g$. If $-(g + h) \notin H$, then in particular $g \not\sim -(g + h)$ (because H is an initial segment of G). Thus, we can apply (Q_2) to the inequality $0 \precsim g$ and we get $-(g + h) \precsim -h \in H$, so $-(g + h) \in H$, which is a contradiction. Thus, $-(g + h) \in H$, and since H is closed under taking the inverse this implies $g + h \in H$. □

We can now give the main properties of $\mathcal{O}$:

Proposition 3.1.14
$\mathcal{O}$ is an initial segment and a subgroup of G. In particular, $(\mathcal{O}, \precsim)$ is an ordered abelian group.

Proof. Let $h \in \mathcal{V}^*$ and $g \in \mathcal{O}$. We want to show that $g \precnsim h$. If $h = g$ were true, then we would have $h \in \mathcal{O} \cap \mathcal{V} = \{0\}$ which is excluded. Therefore, $h \neq g$. Since $g \in \mathcal{O}$, it follows from Proposition 3.1.4(2) that $g \not\sim h$. By Lemma 3.1.12, we then have $g + h \sim g - h$. Note that we also have $-h \in \mathcal{V}^*$, so by the same argument we have $g \not\sim -h$. Moreover, Proposition 3.1.4(3) implies $0 \precsim -h$. By (Q_2), it follows that $g \precsim g - h$. Assume $h \not\sim g - h$. We can apply (Q_2) to the previous inequality and get $g + h \precsim g$, so we have $g + h \precsim g \precsim g - h \sim g + h$, which means $g \sim g + h$. Since $g \in \mathcal{O}$, this contradicts Proposition 3.1.4(2). Thus, we have $g \precsim g - h \sim h$, which is what we wanted. This shows that $\mathcal{O}$ is an initial segment, and by Lemma 3.1.13 it follows that $\mathcal{O}$ is a subgroup of G. By Proposition 3.1.10 $(\mathcal{O}, \precsim)$ is then an ordered abelian group. □

The main property of $\mathcal{V}^*$ is given by the ultrametric inequality satisfied by v-type elements:

Proposition 3.1.15 (ultrametric inequality for v-type elements)
$\mathcal{V}^*$ is a final segment of G. Moreover, for any $g \in \mathcal{V}^*$ and $h \in G$ we have $\mathrm{cl}(g+h) \leq \max(\mathrm{cl}(g), \mathrm{cl}(h))$. If $h \precnsim g$, then $g \sim g+h$.

Proof. The fact that $\mathcal{V}^*$ is a final segment follows directly from Proposition 3.1.14. Take $g \in \mathcal{V}^*$ and $h \in G$. We can assume that $h \precsim g$: otherwise, we have $h \in \mathcal{V}^*$ so we can exchange the roles of g and h. By Proposition 3.1.14, $G^{\mathrm{cl}(g)}, G_{\mathrm{cl}(g)}$ contain $\mathcal{O}$. By Lemma 3.1.13 it follows that they are subgroups of G. In particular, since $g, h \in G^{\mathrm{cl}(g)}$, we have $g+h \in G^{\mathrm{cl}(g)}$, hence $g+h \precsim g$. If $h \precnsim g$, then we even have $h \in G_{\mathrm{cl}(g)}$ but $g \in G^{\mathrm{cl}(g)} \backslash G_{\mathrm{cl}(g)}$, hence $g+h \in G^{\mathrm{cl}(g)} \backslash G_{\mathrm{cl}(g)}$, which means $g \sim g+h$. □

We can reformulate Proposition 3.1.15 by saying that $\precsim$ behaves like a valuation on $\mathcal{V}^*$:

Proposition 3.1.16
Set $\Gamma^v := \mathrm{cl}(\mathcal{V}^*)$ and take $\gamma_0 \in \Gamma$ with $\gamma_0 < \Gamma^v$. Let $\leq^*$ be the reverse order of $\leq$ on $\Gamma_v \cup \{\gamma_0\}$. Define v on G by:

$$v(g) = \begin{cases} \mathrm{cl}(g) & \text{if } g \in \mathcal{V}^* \\ \gamma_0 & \text{if } 0 \neq g \in \mathcal{O} \\ \infty & \text{if } g = 0 \end{cases}$$

Then $v : G \to (\Gamma_v \cup \{\gamma_0, \infty\}, \leq^*)$ is a valuation and we have $g \precsim h \Leftrightarrow v(g) \geq v(h)$ for any $g, h \in \mathcal{V}^*$.

Proof. It suffices to show that $v(g+h) \geq \min(v(g), v(h))$ for any $g, h \in G$. If g or h is in $\mathcal{V}^*$, then this is given by Proposition 3.1.15. If g and h are in $\mathcal{O}$, then this is given by Proposition 3.1.14. □

As a special case of Proposition 3.1.16 we have a v-type analog of Proposition 3.1.10:

Proposition 3.1.17
The compatible q.o. $\precsim$ is valuational if and only if every element of G is v-type. In that case, the map $\mathrm{cl} : G \to \Gamma$ with reverse order on Γ and with $\infty := \mathrm{cl}(0)$ is a valuation, and $\precsim$ is the q.o. induced by this valuation.

3.1.3 Quasi-order induced on a quotient

It is known that if $(G, \leq)$ is an ordered abelian group and if H is a convex subgroup, then $\leq$ naturally induces an order on the quotient G/H (see [Fuc63]). We now show that the same is true in the case of compatible q.o.a.g.'s, which will allow us to give a more elegant formulation of Proposition 3.1.16. We start by describing convex subgroups:

Proposition 3.1.18
Let H be a convex subgroup of $(G, \precsim)$. Then either $H \subseteq \mathcal{O}$ or $\mathcal{O} \subseteq H$. If the latter holds, then H is an initial segment of G.

Proof. Assume $\mathcal{O}$ is not contained in H, so there exists some o-type element g with $g \notin H$. Without loss of generality, we can assume $0 \precsim g$. By convexity of H, we have $H \precsim g$, which by Proposition 3.1.14 implies that every element of H is o-type. Now assume $\mathcal{O} \subseteq H$ and let us show that H is an initial segment of G. Take $h \in H$ and $g \in G$ with $g \precsim h$. If $g \in \mathcal{O}$ then $g \in H$ by assumption; if $g \notin \mathcal{O}$, then by Lemma 3.1.4 we have $0 \precsim g \precsim h$, hence $g \in H$ by convexity of H. This shows that H is an initial segment. □

We will need the following lemma to define a q.o. on quotients:

Proposition 3.1.19
For any $g, h, f \in G$, $g \precsim h \nsim -f \Rightarrow g + f \precsim h + f$.

Proof. Assume $g \precsim h \nsim -f$ holds. If $f \in \mathcal{V}$, the claim follows directly from axiom (Q_2). Assume $f \in \mathcal{O}$. If $h \in \mathcal{O}$, then by Proposition 3.1.14, $g \in \mathcal{O}$. By Proposition 3.1.14, $(\mathcal{O}, \precsim)$ satisfies (OG), hence $g + f \precsim h + f$. Assume that $h \in \mathcal{V}^*$. By Proposition 3.1.15, we have $h + f \sim h$. If $g \in \mathcal{O}$, then it follows from Proposition 3.1.14 that $g + f \precsim h \sim h + f$. If $g \in \mathcal{V}^*$, then Proposition 3.1.15 implies $g + f \sim g$, hence $g + f \sim g \precsim h \sim h + f$. In all cases, we have $g + f \precsim h + f$. □

Proposition 3.1.20
Let H be a subgroup of G. Then H is convex in $(G, \precsim)$ if and only if $\precsim$ induces a compatible q.o. on the quotient G/H. Moreover, the canonical projection π from G to G/H is a homomorphism of q.o. groups, and for all $g \in G \backslash H$, g is o-type if and only if $\pi(g)$ is o-type.

Proof. Proposition 3.1.19 and (Q_1) show that $\precsim$ satisfies the condition $(*)$ of Lemma 2.7.20. If $\precsim$ induces a compatible q.o. on the quotient G/H, Lemma 2.7.20 immediately implies that H is convex in G. Conversely, if H is convex, then by Lemma 2.7.20, $\precsim$ induces a q.o. on the quotient G/H which satisfies (Q_1). We just have to check that this q.o. satisfies (Q_2). Let $g_1, g_2, g_3 \in G$ with $g_1 + H \precsim g_2 + H \nsim g_3 + H$. There are $h_1, h_2 \in H$ with $g_1 + h_1 \precsim g_2 + h_2$. Since $g_2 + H \nsim g_3 + H$, we cannot have $g_2 + h_2 \sim g_3$. Therefore, we can apply (Q_2) to the relation $g_1 + h_1 \precsim g_2 + h_2$, which yields $g_1 + g_3 + h_1 \precsim g_2 + g_3 + h_2$. By definition of the induced q.o., this means $g_1 + g_3 + H \precsim g_2 + g_3 + H$. This proves that (Q_2) is satisfied on G/H. It is clear from the definition of the induced q.o. on G/H that π is a homomorphism of q.o. groups. The last sentence follows from Lemma 2.7.20. □

We can give a formula for the induced q.o.:

Proposition 3.1.21
Let H be a convex subgroup of $(G, \precsim)$. Then the q.o. induced by $\precsim$ on G/H is given by the following formula: $g + H \precsim h + H \Leftrightarrow (g - h \in H) \vee (g - h \notin H \wedge g \precsim h)$.

Proof. If $g-h \in H$ then $g+H = h+H$, so clearly $g+H \precsim h+H$. If $g \precsim h$, then clearly by definition of the q.o. induced on G/H we have $g+H \precsim h+H$. This shows $g+H \precsim h+H \Leftarrow (g-h \in H) \vee (g-h \notin H \wedge g \precsim h)$. Let us show the converse. Assume $g_1+H \precsim g_2+H$ and $g_1-g_2 \notin H$ holds and let us show $g_1 \precsim g_2$. Since $g_1+H \precsim g_2+H$, there are $h_1, h_2 \in H$ with $g_1+h_1 \precsim g_2+h_2$. Assume first that $g_2 \in H$. Since $g_1-g_2 \notin H$, we then have $g_1 \notin H$. By convexity of H, it follows that $h_1 \nsim g_1$. Assume for a contradiction that $g_2 \precsim g_1$. We then have $g_2 \precsim g_1 \nsim h_1$. By (Q_2), this implies $g_2+h_1 \precsim g_1+h_1$. By convexity of H, it follows from $g_2+h_1 \precsim g_1+h_1 \precsim g_2+h_2$ that $g_1 \in H$: contradiction. Therefore, we must have $g_1 \precsim g_2$. Now assume $g_2 \notin H$. By convexity of H, we have $g_2+h_2 \nsim -h_1$. By (Q_2), $g_1+h_1 \precsim g_2+h_2$ then implies $g_1 \precsim g_2+h_2-h_1$. If $g_2+h_2-h_1 \sim g_2$, it immediately follows that $g_1 \precsim g_2$. Assume that $g_2+h_2-h_1 \nsim g_2$. Then it follows from Proposition 3.1.19 that $g_1-g_2 \precsim h_2-h_1$. But since $g_1-g_2 \notin H$, the convexity of H implies $g_1-g_2 \precsim 0$. Since $g_2 \neq 0$, (CQ_2) then implies $g_1 \precsim g_2$.

□

Remark 3.1.22: Proposition 3.1.21 implies in particular that, for any $h \notin H$ and any $g \in G$, $g \precsim h \Leftrightarrow g+H \precsim h+H$.

As we noted in Section 2.7, a bijective homomorphism of q.o. groups is in general not an isomorphism. A consequence of this is that there is no equivalent of the fundamental homomorphism theorem of groups, i.e a homomorphism of q.o. groups is not always the product of a projection by an embedding. However, we can show that every homomorphism is a projection followed by a coarsening:

Proposition 3.1.23
Let $(G, \precsim_G)$ and $(H, \precsim_H)$ be two compatible q.o. groups, $\phi : G \to H$ a homomorphism of q.o. groups and π the canonical projection from G to $G/\ker\phi$. Then $\ker\phi$ is convex in $(G, \precsim_G)$, so $\precsim_G$ induces a compatible q.o. on $G/\ker\phi$. Moreover, the map:
$G/\ker\phi \to H, g+\ker\phi \mapsto \phi(g)$ is an injective homomorphism of q.o. groups. If moreover ϕ is quasi-order-preserving, then $g+\ker\phi \mapsto \phi(g)$ is an embedding of q.o. groups.

Proof. Let $g, f \in \ker\phi$ and $h \in G$ with $f \precsim_G h \precsim_G g$. Then $\phi(f) = 0 \precsim_H \phi(h) \precsim_H \phi(g) = 0$. By (Q_1), it follows that $h \in \ker\phi$. This proves that $\ker\phi$ is convex, and so by Proposition 3.1.20 $\precsim_G$ induces a compatible q.o. on $G/\ker\phi$. We know from general group theory that the map given by the formula $\psi(g+\ker\phi) := \phi(g)$ is a well-defined injective group homomorphism from $G/\ker\phi$ to H. Now let $g_1, g_2 \in G$ such that $g_1+\ker\phi \precsim_G g_2+\ker\phi$. There are $h_1, h_2 \in \ker\phi$ with $g_1+h_1 \precsim_G g_2+h_2$. Since ϕ is a homomorphism of q.o. groups, we then have $\phi(g_1+h_1) \precsim_H \phi(g_2+h_2)$. Since $h_1, h_2 \in \ker\phi$, it follows that $\phi(g_1) \precsim_H \phi(g_2)$, hence $\psi(g_1+H) \precsim_H \psi(g_2+H)$. Now assume that ϕ is quasi-order-preserving and assume that $\psi(g_1+H) \precsim_H \psi(g_2+H)$ holds. By definition of ψ, we have $\phi(g_1) \precsim_H \phi(g_2)$. Since ϕ is q.o.-preserving, it follows that $g_1 \precsim_G g_2$, which by definition of the induced q.o. on the quotient implies $g_1+\ker\phi \precsim_G g_2+\ker\phi$. □

Proposition 3.1.20 allows us to reformulate Proposition 3.1.16:

Proposition 3.1.24
Let H be a convex subgroup of $(G, \precsim)$. The induced q.o on G/H is valuational if and only if $\mathcal{O} \subseteq H$. In particular, $\mathcal{O}$ is the smallest convex subgroup of G such that the induced q.o on $G/\mathcal{O}$ is valuational.

Proof. If $\mathcal{O} \subseteq H$, then it follows from Proposition 3.1.20 that every element of G/H is v-type. It then follows from Proposition 3.1.17 that $\precsim$ is valuational on G/H. If $H \subsetneq \mathcal{O}$, then by Proposition 3.1.20 G/H contains a non-zero o-type element. It then follows from Proposition 3.1.17 that $\precsim$ is not valuational on G/H. $\square$

Propositions 3.1.14 and 3.1.24 show that $(G, \precsim)$ is an extension of a valued group by an ordered group. We now define the **ordered part** of $(G, \precsim)$ as the ordered group $(G^o, \precsim)$ where $G^o := \mathcal{O}$, and we define the **valued part** of $(G, \precsim)$ as the valued group (G^v, v), where $G^v := G/\mathcal{O}$ and v is the valuation corresponding to the q.o induced by $\precsim$ on G^v. We will now express $\precsim$ with a formula in which the order of its ordered part and the valuation of its valued part explicitly appear.

Proposition 3.1.25
Let $\leq_o$ denote the restriction of $\precsim$ to G^o. Then the q.o. $\precsim$ is given by the following formula for all $g, h \in G$:
$g \precsim h \Leftrightarrow (g, h \in G^o \wedge g \leq_o h) \vee (h \notin G^o \wedge v(g + G^o) \geq v(h + G^o))$

Proof. Denote by $\precsim^*$ the q.o given by the formula $g \precsim^* h \Leftrightarrow (g, h \in G^o \wedge g \leq_o h) \vee (h \notin G^o \wedge v(g + G^o) \geq v(h + G^o))$. We show that $\precsim^*$ coincides with $\precsim$. Assume $g \precsim h$. Then $g + G^o \precsim h + G^o$ holds by definition of the induced q.o on G/G^o. If $h \notin G^o$, then it directly follows from the definition of $\precsim^*$ that $g \precsim^* h$. If $h \in G^o$, then by Proposition 3.1.14 we must have $g \in G^o$. By definition of $\leq_o$, $g \precsim h$ then implies $g \leq_o h$. By definition of $\precsim^*$, this implies $g \precsim^* h$. Conversely, assume $g \precsim^* h$. If $g, h \in G^o$, then $g \leq_o h$ holds. By definition of $\leq_o$, this implies $g \precsim h$. Assume then that $(h \notin G^o \wedge v(g + G^o) \geq v(h + G^o))$ holds. Then it follows from Remark 3.1.22 that $g \precsim h$. $\square$

3.1.4 Structure theorems

We can summarize previous results into the following theorem which gives the structure of a compatible q.o.a.g.:

Theorem 3.1.26
Let $(G, \precsim)$ be a quasi-ordered abelian group. Then $\precsim$ is compatible with $+$ if and only if G admits a subgroup H satisfying the following properties:

(1) H is an initial segment of G.

(2) $(H, \precsim)$ is an ordered group.

(3) There exists a valuation v on G such that $v(H) > v(G \backslash H)$ and $g \precsim h \Leftrightarrow v(g) \geq v(h)$ for every g, h in $G \backslash H$.

Proof. We have already showed that, if $\precsim$ is compatible with $+$, then (1), (2) and (3) are satisfied with $H := \mathcal{O}$ (Propositions 3.1.14 and 3.1.16). Conversely, assume there exists H satisfying (1), (2) and (3). (Q_1) is clearly satisfied, so let us prove (Q_2). Let $x \precsim y \nsim z$. Assume $y \in H$. Since H is an initial segment, this implies $x \in H$. If $z \in H$, then since H is an ordered group we have $x \precsim y \Rightarrow x + z \precsim y + z$. If $z \notin H$, then $v(x), v(y) > v(z)$ so $v(z) = v(x+z) = v(y+z)$, and since v and $\precsim$ coincide outside of H this means $z \sim x + z \sim y + z$ so in particular $x + z \precsim y + z$. Assume $y \notin H$. Then $x \precsim y$ implies $v(x) \geq v(y)$ and $z \nsim y$ implies $v(z) \neq v(y)$. If $v(z) < v(y)$, then $z \notin H$ and $v(x+z) = v(y+z)$, hence $x + z \sim y + z$. If $v(y) < v(z)$, then $v(y+z) = v(y) \leq v(z+x)$, hence $z + x \precsim z + y$. In any case, we have $x + z \precsim y + z$. ◻

Remark 3.1.27: As we have seen in Proposition 3.1.24, (1)+(2)+(3) implies:

($3'$) The q.o. $\precsim$ induces a valuational q.o. on G/H.

It is tempting to replace (3) by ($3'$) in Theorem 3.1.26, as ($3'$) seems to be a more elegant reformulation of (3). However, condition (3) is in general stronger than ($3'$), so that Theorem 3.1.26 becomes false if we replace (3) by ($3'$). We can construct an example of a q.o. group satisfying (1), (2) and ($3'$) but which is not compatible: Take $G := (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ with the following q.o: $(a, b) \precsim (c, d) \Leftrightarrow (a = c \wedge b \leq d) \vee (a < c)$, where $\leq$ is the usual order of $\mathbb{Z}$. We have: $(0 \times \mathbb{Z}, \leq) \precnsim \dots (1, -n) \precnsim \dots \precnsim (1, -1) \precnsim (1, 0) \precnsim \dots \precnsim (1, n) \precnsim \dots$. Setting $H = 0 \times \mathbb{Z}$, H satisfies (1) and (2). It is also easy to see that $\precsim$ induces a q.o. on G/H, which is the q.o. $0 \precnsim 1$, so it is valuational. Therefore, $(G, \precsim)$ satisfies ($3'$). However, $\precsim$ cannot be compatible: the set of o-type elements is $G \backslash \{(1, 0)\}$ which is clearly not a group, thus contradicting Proposition 3.1.14. We can also give an explicit example of axiom (Q_2) failing: take $x := (0, 0), y := (1, 0)$ and $z := (1, 1)$. We have $x \precsim y \nsim z$ but $y + z = (0, 1) \precnsim x + z = (1, 1)$.

However, we can replace (3) by ($3'$) plus an extra condition, which gives us a second version of the structure theorem:

Theorem 3.1.28
Let $(G, \precsim)$ be a q.o.a.g. Then $\precsim$ is compatible with $+$ if and only if G admits a subgroup H satisfying the following properties:

(1) H is an initial segment of G.

(2) $(H, \precsim)$ is an ordered abelian group.

($3'$) The q.o. $\precsim$ induces a valuational q.o. on G/H.

(4) For any $g, h \in G$, $g \notin H$ and $g - h \in H$ implies $g \sim h$.

Proof. Assume that $\precsim$ is compatible and set $H := \mathcal{O}$. (1) and (2) follow from Theorem 3.1.26. ($3'$) follows from Proposition 3.1.24. Let us prove (4). Let $g \in \mathcal{V}^*$ and $h \in G$ with $g - h \in H$. Then by Proposition 3.1.14, we have $h - g \precnsim g$. It then follows from Proposition 3.1.15 that $g \sim g + h - g = h$, which proves (4).

Conversely, assume that (1), (2), (3′) and (4) hold. We just have to show that (3) of Theorem 3.1.26 holds. By (3′), we know that the q.o $\precsim$ on G/H is valuational, and we denote by $v : G/H \to \Gamma \cup \{\infty\}$ the corresponding valuation. We lift v to a valuation w on G as follows: add a point γ_0 to Γ such that $\Gamma < \gamma_0 < \infty$. For any $g \in G$ define $w(g)$ as follows:

$$w(g) = \begin{cases} v(g+H) & \text{if } g \notin H \\ \gamma_0 & \text{if } 0 \neq g \in H \\ \infty & \text{if } g = 0 \end{cases}$$

It is clear from its definition that w is a valuation (because v is a valuation and H is a subgroup of G). Take $g, h \notin H$. If $g \precsim h$, then $g + H \precsim h + H$ hence $v(g + H) \geq v(h + H)$ hence $w(g) \geq w(h)$. Conversely, if $w(g) \geq w(h)$, then by definition of w we must have $h+H \precsim g+H$. By definition of the induced q.o., there are $a, b \in H$ with $h+a \precsim g+b$.Since $g, h \notin H$, it follows from (4) that $h + a \sim h$ and $g + b \sim g$, hence $g \precsim h$. This shows that (3) of Theorem 3.1.26 holds. □

We can reformulate Theorem 3.1.28 into the language of exact sequences: a compatible q.o.a.g. is an extension of a valued group by an ordered group:

Theorem 3.1.29
Let $(G, \precsim)$ be an abelian quasi-ordered group. Then $\precsim$ is compatible with $+$ if and only if there exists an exact sequence $0 \to G^o \xrightarrow{\iota} G \xrightarrow{\pi} F \to 0$ such that there exists a group order $\leq_o$ on G^o and a valuation v on F such that for any $g, h \in G$
(†) $\quad g \precsim h \Leftrightarrow (g, h \in \iota(G^o) \wedge \iota^{-1}(g) \leq_o \iota^{-1}(h)) \vee (h \notin \iota(G^o) \wedge v(\pi(g)) \geq v(\pi(h)))$.

Proof. One direction is given by Proposition 3.1.25. For the other direction, assume that such an exact sequence as above exists. We use Theorem 3.1.28. Set $H := \iota(G^o)$. It follows directly from formula (†) that (1) and (2) are satisfied. If $g \notin H$ and $h \in G$ are such that $g - h \in H$, Then we have $v(\pi(g-h)) > v(\pi(g))$. This implies $v(\pi(g)) = v(\pi(h))$. By formula (†), this implies $g \sim h$. This shows (4). Now let us show (3′). By assumption, if we set $\pi(g+H) := \pi(g)$, then π is a group isomorphism from G/H to F. It follows that $v(g+H) := v(\pi(g))$ defines a valuation on G/H. Denote by $\precsim$ the relation defined on G/H by $g+H \precsim h+H \Leftrightarrow \exists a, b \in H(g+a \precsim h+b)$. To prove (3′), we just have to show that $\precsim$ is the q.o. induced by v. Assume that $g+a \precsim h+b$. If $h+b \notin H$, then it directly follows from formula (†) that $v(h + H) \leq v(g + H)$. If $h + b \in H$, then formula (†) implies $g + a \in H$, hence $v(h + H) = v(g + H) = \infty$, hence $v(h + H) \leq v(g + H)$. Conversely, assume $v(h + H) \leq v(g + H)$. If $h \notin H$, then it immediately follows from formula (†) that $g \precsim h$. If $h \in H$, then $v(h+H) = \infty$, hence $v(g+H) = \infty$, hence $g \in H$. Choose $a := h - g$ and we have $g + a \precsim h$. This proves that $v(h+H) \leq v(g+H) \Leftrightarrow \exists a, b \in H(g+a \precsim h+b)$. □

3.2 Compatible valuations and Baer-Krull

A motivation for the study of quasi-ordered groups was to find q.o. group analogs of Theorems 2.7.9 and 2.3.1. We now tackle this question. It is easy to obtain a group analog of Theorem 2.7.9:

Lemma 3.2.1
Assume $(G, \precsim)$ is a compatible q.o.a.g. and let $H \subseteq G$. Then H is convex in $(G, \precsim)$ if and only if $\{g \in H \mid 0 \precsim g\}$ is convex in $(G, \precsim)$.

Proof. It is clear that, if H is convex, then $\{g \in H \mid 0 \precsim g\}$ is also convex. Conversely, assume that $\{g \in H \mid 0 \precsim g\}$ is convex and let $g, h \in H$ and $f \in G$ with $g \precsim f \precsim h$. If $0 \precsim f \precsim h$ then by assumption we have $f \in H$. Assume that $g \precsim f \precsim 0$. Then by Proposition 3.1.4, we have $f, g \in \mathcal{O}$, and since $(\mathcal{O}, \leq)$ is an ordered group it follows that $0 \precsim -f \precsim -g$, hence by assumption $-f \in H$. $\square$

Theorem 3.2.2
Let $(G, \precsim)$ be a compatible q.o.a.g. and $v : G \to \Gamma \cup \{\infty\}$ a valuation. The following statements are equivalent:

(1) For all $\gamma \in \Gamma$, G^γ is convex in $(G, \precsim)$.

(2) For all $\gamma \in \Gamma$, G_γ is convex in $(G, \precsim)$.

(3) For all $\gamma \in \Gamma$, $\precsim$ induces a compatible q.o. $\precsim_\gamma$ on G^γ / G_γ.

(4) v is compatible with $\precsim$.

Moreover, $\precsim$ is valuational (respectively, an order) if and only if for all $\gamma \in \Gamma$, $\precsim_\gamma$ is valuational (respectively, an order).

Proof. (1)$\Leftrightarrow$(2) is given by Lemma 2.7.18. (2)$\Leftrightarrow$(4) follows from Lemma 3.2.1 and from Lemma 2.7.19. (2)$\Rightarrow$(3) is given by Proposition 3.1.20. If (3) holds, then by Proposition 3.1.20, G_γ is convex in $(G^\gamma, \precsim)$ for all $\gamma \in \Gamma$. It then follows from Lemma 2.7.18 that G_γ is convex in $(G, \precsim)$ for all $\gamma \in \Gamma$, hence (2). Finally, the last statement follows from the fact that the map π in Proposition 3.1.20 preserves the type of elements, and by then applying Propositions 3.1.10 and 3.1.17. $\square$

Theorem 3.2.2 has two immediate corollaries:

Corollary 3.2.3
Let $(G, \leq)$ be an ordered abelian group and $v : G \to \Gamma \cup \{\infty\}$ a valuation. The following statements are equivalent:

(1) For all $\gamma \in \Gamma$, G^γ is convex in $(G, \leq)$.

(2) For all $\gamma \in \Gamma$, G_γ is convex in $(G, \leq)$.

(3) For all $\gamma \in \Gamma$, $\leq$ induces a group order on G^γ / G_γ.

(4) v is compatible with $\leq$.

Corollary 3.2.4
Let (G, w) be a valued group and $v : G \to \Gamma \cup \{\infty\}$ another valuation. Denote by $\precsim_w$ the q.o. induced by w. The following statements are equivalent:

(1) For all $\gamma \in \Gamma$, G^γ is convex in $(G, \precsim_w)$.

(2) For all $\gamma \in \Gamma$, G_γ is convex in $(G, \precsim_w)$.

(3) For all $\gamma \in \Gamma$, w induces a valuation on G^γ/G_γ.

(4) v is a coarsening of w.

However, the Baer-Krull Theorem cannot hold for compatible q.o.a.g.'s. The reason for this is that the classical Baer-Krull theorem (Theorem 2.3.1) uses the fact that any order on the residue field can be lifted to an order of the original field. This situation is not transposable to the case of compatible q.o.a.g.'s. Indeed, Proposition 3.1.14 implies in particular that the class of compatible q.o.'s is not stable under lifting. This is illustrated by the following example:

Example 3.2.5
Take $G := \mathbb{Z} \times \mathbb{Z}$. Define $v : G \to \{1, 2, \infty\}$ by $v(g) = 1$ if $g \notin \{0\} \times \mathbb{Z}$, $v(g) = 2$ if $g \in \{0\} \times \mathbb{Z}\backslash\{(0, 0)\}$ and $v((0, 0)) = \infty$. We have $B_1 := G^1/G_1 \cong \mathbb{Z} \cong G^2/G_2 =: B_2$. Now let $\precsim_1, \precsim_2$ denote the same q.o., namely the q.o. on $\mathbb{Z}$ of example 3.1.2(b). Then no lifting of $(\precsim_1, \precsim_2)$ to G can be a compatible q.o. Indeed, assume that $(\precsim_1, \precsim_2)$ has a lifting $\precsim$ and that $\precsim$ is compatible. In particular, $\precsim$ induces the q.o. $\precsim_2$ on G^2/G_2. It then follows from Theorem 3.2.2 that G^2 is convex in $(G, \precsim)$. Moreover, because the map π from Proposition 3.1.20 preserves the type of elements, both $(5, 0)$ and $(0, 5)$ are o-type and $(0, 1)$ is v-type. By Proposition 3.1.14, we must have $(5, 0) \precnsim (0, 1)$. Because $\precsim_1$ is the q.o. induced by $\precsim$ on B_1, this implies $5 \precsim_1 0$. This contradicts the definition of $\precsim_1$.

This shows that compatible q.o.'s are not appropriate for a Baer-Krull theorem. However, we will see in Chapter 4 that C-q.o's are suitable; see Section 4.2.2.

We finish this section by giving a consequence of Theorem 3.2.2 which will be useful for asymptotic couples in Section 5.4.

Proposition 3.2.6
Let $(G, \leq)$ be an ordered group and v a $\mathbb{Z}$-module valuation compatible with $\leq$. Let $(\Psi, (C_\lambda)_{\lambda\in\Psi})$ be the skeleton of (G, v). The following holds:

(1) For any $\lambda \in \Psi$, $\leq$ induces an order $\leq_\lambda$ on C_λ.

(2) Denote by $(H, \leq_H)$ the lexicographic product of the family $(\widehat{C}_\lambda, \leq_\lambda)_{\lambda\in\Psi}$, and let w be the valuation defined on H by $w(g) = \min \operatorname{supp}(g)$. Then any embedding of valued groups $\phi : (G, v) \hookrightarrow (H, w)$ as in Theorem 2.2.9 is also an embedding of ordered groups $(G, \leq) \hookrightarrow (H, \leq_H)$.

Proof. It follows from Corollary 3.2.3 that $\leq$ induces an order on C_λ. Now let us prove (2). Denote by P the positive cone of $(G,\leq)$, by P_λ the positive cone of $(C_\lambda,\leq_\lambda)$ and by P_H the positive cone of $\leq_H$. We just have to prove that $\phi(P)\subseteq P_H$. Take $g\in G$. Set $\lambda := v(g)$. By condition (1) of Theorem 2.2.9, we have $w(\phi(g))=\lambda$. Because of property (2) of 2.2.9, we have $\phi(g)+H_\lambda = g+G_\lambda$. Because P_λ is the order induced by P on C_λ, we have $g\in P\Leftrightarrow g+G_\lambda\in P_\lambda$. Moreover, by definition of P_H, we have $\phi(g)\in P_H\Leftrightarrow \phi(g)+H_\lambda\in P_\lambda$. It follows that $g\in P\Leftrightarrow\phi(g)\in P_H$, which is what we wanted. □

3.3 Products of compatible q.o.'s

In the theory of ordered abelian groups, there is a natural notion of product, namely the lexicographic product (see Section 2.2). The goal of this section is to develop a similar notion for compatible q.o.a.g.'s. We first introduce the notion of compatible product, and we then use this notion to prove a generalization of Hahn's embedding theorem for compatible q.o.a.g.'s. We then show that the compatible product of an ordered group by a valued group preserves elementary equivalence. For any compatible q.o.a.g. $(G,\precsim)$, $(G^o,\precsim)$ and $(G^v,\precsim)$ respectively denote the ordered part and the valued part of $(G,\precsim)$.

3.3.1 The compatible product

In Chapter 2, we recalled the definition of the lexicographic product of a family of ordered groups, and we introduced the notion of valuational product of q.o. groups. This allows us to define a notion of product for compatible q.o.a.g.'s. Let $(B_\gamma,\precsim_\gamma)_{\gamma\in\Gamma}$ be an ordered family of compatible q.o.a.g.'s. Let $G := \mathrm{H}_{\gamma\in\Gamma}B_\gamma$, let $(G^o,\leq_o)$ be the lexicographic product of the family $(B^o_\gamma,\precsim_\gamma)_{\gamma\in\Gamma}$ and set $\mathcal{V}^* := G\backslash G^o$. For each $\gamma\in\Gamma$, the q.o. $\precsim_\gamma$ induces a valuational q.o. on $B^v_\gamma = B^\gamma/B^o_\gamma$ by Proposition 3.1.24. Let $(G^v,\precsim_{\mathrm{val}})$ be the valuational product of the family $(B^v_\gamma,\precsim_\gamma)_{\gamma\in\Gamma}$. Note that G/G^o is canonically isomorphic to G^v via the isomorphism $\psi : (g_\gamma)_\gamma + G^o \mapsto (g_\gamma + B^o_\gamma)_\gamma$.

Definition 3.3.1
The **compatible product** of the family $(B_\gamma,\precsim_\gamma)_{\gamma\in\Gamma}$ is the compatible q.o.a.g. $(G,\precsim)$, where $\precsim$ is defined by the following formula:
$g\precsim h\Leftrightarrow (g,h\in G^o\wedge g\leq_o h)\vee(h\in\mathcal{V}^*\wedge\psi(g+G^o)\precsim_{\mathrm{val}}\psi(h+G^o))$.

The fact that $\precsim$ is compatible follows directly from Theorem 3.1.29. We denote the compatible product of the family $(B_\gamma,\precsim_\gamma)_{\gamma\in\Gamma}$ by $\mathrm{H}_{\gamma\in\Gamma}(B_\gamma,\precsim_\gamma)$. If Γ is finite with elements $\gamma_1<\gamma_2<\cdots<\gamma_n$, then we denote it by $(B_{\gamma_1},\precsim_{\gamma_1})\times(B_{\gamma_2},\precsim_{\gamma_2})\times\cdots\times(B_{\gamma_n},\precsim_{\gamma_n})$.

One particular case of compatible product is the case of the product of an ordered group by a valued group. This gives us a way of constructing compatible q.o.a.g.'s from ordered and valued groups. In particular, we have the following:

Proposition 3.3.2
Let $(G,\leq)$ be an ordered abelian group and (F,v) a valued group. Then there exists a compatible q.o.a.g. whose ordered part is $(G,\leq)$ and whose valued part is (F,v).

Proof. Just take the compatible product $(F, \precsim_{\mathrm{val}}) \times (G, \leq)$, where $\precsim_{\mathrm{val}}$ is the q.o. induced by v. □

In view of Theorem 3.1.28, it is natural to ask whether every compatible q.o.a.g. can be obtained as the product of an ordered group by a valued group. However, Example 3.1.2(b) shows that it is not the case: $G^o = 5\mathbb{Z}$ is not a direct factor of $G = \mathbb{Z}$, so $(G, \precsim)$ is not the compatible product of G^o by G/G^o. Fortunately, we have the following:

Proposition 3.3.3
Let $(G, \precsim)$ be a compatible q.o.a.g. If G^o is a direct factor of G with complement F, then $(F, \precsim)$ is canonically isomorphic to the valued part of $(G, \precsim)$ and we have $(G, \precsim) = (F, \precsim) \times (G^o, \precsim)$. In other words, $(G, \precsim)$ is the compatible product of its ordered part by its valued part.

Proof. It follows directly from Proposition 3.1.25 and from the definition of the compatible product. □

Proposition 3.3.2 shows that if $(G, \precsim)$ is a compatible q.o.a.g, then the valuation appearing in the valued part of G can a priori be any valuation, in particular it does not have to be a $\mathbb{Z}$-module valuation. However, it can be interesting to restrict our attention to such valuations. Consider the following family of axioms indexed by $n \in \mathbb{N}$: (VM_n) $\forall g, -g \precsim g \Rightarrow g \precsim ng$ ("VM" stands for "valued module"). This family of axioms gives an axiomatization of the class of compatible q.o.a.g.'s whose valuation on its valuational part is a $\mathbb{Z}$-module valuation. If $(G, \precsim)$ is such a compatible q.o.a.g., then G^o is pure in G, from which we get the following result:

Proposition 3.3.4
Let $(G, \precsim)$ be a compatible q.o.a.g. satisfying the axiom (VM_n) for every $n \in \mathbb{N}$. Assume that G is divisible. Then $(G, \precsim)$ is the compatible product of its ordered part by its valued part.

Proof. Because of (VM_n), G^o is pure in G. Since G is divisible, G^o is then a direct factor of G. The result then follows from Proposition 3.3.3. □

3.3.2 Hahn's embedding theorem

We now want to generalize Hahn's embedding theorem (Theorem 2.2.6) to compatible q.o.a.g.'s. This implies defining a notion of archimedeanity for q.o. groups. To do this, we will associate a valuational q.o. $\precsim_{\mathrm{arch}}$ to each compatible q.o. $\precsim$, which we will call the archimedean q.o. associated to $\precsim$.

Let $(G, \precsim)$ be a compatible q.o.a.g. Consider the relation $\lessdot$ defined as follows: we say that $g \lessdot h$ if and only if there is $n, m \in \mathbb{Z}\backslash\{0\}$ such that $0 \precsim ng \precsim mh$. We have the following:

Lemma 3.3.5
The relation $\lessdot$ defines a valuational q.o. on G^o.

Proof. $\lessdot$ is clearly reflexive, let us show transitivity. Assume $f \lessdot g \lessdot h$. There are n, m, k, l with $0 \lesssim nf \lesssim mg$ and $0 \lesssim kg \lesssim lh$. It follows that $0 \lesssim |n||f| \lesssim |m||g|$ and $0 \lesssim |k||g| \lesssim |l||h|$. Since $\lesssim$ is an order on G^o these relations imply $0 \lesssim |n||f| \lesssim |m||g| \lesssim |km||g| \lesssim |ml||h|$, so either $0 \lesssim nf \lesssim mlh$ or $0 \lesssim nf \lesssim -mlh$ holds, hence $f \lessdot h$. This proves that $\lessdot$ is a q.o. on G^o. Now let us prove that $\lessdot$ is valuational. If $g \lessdot 0$ then $0 \lesssim ng \lesssim 0$ holds for some $n \neq 0$ which implies $ng = 0$ by (Q_1), and since G^o is an ordered abelian group it is torsion-free, hence $g = 0$. Thus, we have $0 \lnsim g$ for every $g \in G^o$ with $g \neq 0$. Clearly, $g \lessdot -g \lessdot g$ holds for every g. Now assume $g \lessdot h$ and take n, m with $0 \lesssim ng \lesssim mh$. We have $0 \lesssim |n||g| \lesssim |m||h|$, which by compatibility implies $|n||g+h| \lesssim |n+m||h|$ hence $g + h \lessdot h$. This proves that the ultrametric inequality holds. □

However, the relation $\lessdot$ is not transitive in general. Indeed, consider the following example: set $G := \mathbb{Z}^2$ and let $v : G \to \{1, 2, 3, \infty\}$ be the valuation defined as follows:

$$v(n,m) = \begin{cases} 1 \text{ if } p \nmid m \\ 2 \text{ if } 0 \neq n \wedge p \mid m \\ 3 \text{ if } n = 0 \wedge p \mid m \neq 0 \\ \infty \text{ if } n = m = 0 \end{cases}$$

Now let $\lesssim$ be the q.o induced by v. Let $f := (0, p)$, $g := (1, p)$ and $h := (0, 1)$. We have $p.h \sim f$, hence $h \lessdot f$. Moreover, $g \lesssim h$, so $g \lessdot h$. However, for every $n, m \in \mathbb{Z}\backslash\{0\}$ we have $mf \sim f \lnsim g \sim ng$, so $g \lessdot f$ does not hold. To make $\lessdot$ transitive, we define the relation $\lesssim_{\text{arch}}$ as follows: we say that $g \lesssim_{\text{arch}} h$ if there exists $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in G$ such that $g \lessdot x_1 \lessdot x_2 \lessdot \ldots \lessdot x_r \lessdot h$. Note that $\lesssim_{\text{arch}}$ is the same as $\lessdot$ for ordered groups. In order to prove that $\lesssim_{\text{arch}}$ is a valuational q.o, we need the following lemma:

Lemma 3.3.6
Let $\lesssim$ be a compatible q.o., $g \in \mathcal{V}^*$ and let $\lesssim^*$ be a coarsening of $\lesssim$. Then for any $h \in G$, $h \lesssim^* g$ implies $g + h \lesssim^* g$.

Proof. Assume $h \lesssim^* g$ and $g \lnsim^* g+h$. Since $\lesssim^*$ is a coarsening of $\lesssim$, this implies $g \lnsim g+h$. Since g is v-type, $g+h$ must also be v-type by Proposition 3.1.14, and it then follows from Proposition 3.1.15 that $g + h \sim h$. Since $\lesssim^*$ is a coarsening of $\lesssim$, this implies $h \sim^* g + h$. We thus have $h \lesssim^* g \lnsim^* g + h \sim^* h$, which is a contradiction. □

Proposition 3.3.7
Let $(G, \lesssim)$ be a torsion-free compatible q.o.a.g. The relation $\lesssim_{\text{arch}}$ is a q.o on G. Moreover, it is the finest $\mathbb{Z}$-module-valuational coarsening of $\lesssim$.

Proof. The fact that $\lesssim_{\text{arch}}$ is transitive and is a coarsening of $\lesssim$ is clear from its definition. It is also clear that $g \sim_{\text{arch}} ng$ for all $g \in G$ and $n \in \mathbb{Z}\backslash\{0\}$. Now let us show that $\lesssim_{\text{arch}}$ is valuational. Note that for any $g \in G$, $g \lessdot 0$ implies $g = 0$: indeed, if $g \lessdot 0$ then there exists $n \neq 0$ with $0 \lesssim ng \lesssim 0$, which by (Q_1) implies $ng = 0$ and since G is torsion-free it follows that $g = 0$. By definition of $\precsim_{\text{arch}}$, it then follows that $g \precsim_{\text{arch}} 0$ implies $g = 0$, so we have $0 \precnsim_{\text{arch}} g$ whenever $g \neq 0$. Now let us show that $\precsim_{\text{arch}}$ satisfies the ultrametric inequality. Let $g, h \in G$ with $g \precsim_{\text{arch}} h$. If $h \in \mathcal{V}^*$, it follows from Lemma 3.3.6 that $g + h \precsim_{\text{arch}} h$, so assume $h \in G^o$. If $g \in G^o$, it follows from Lemma 3.3.5 that $g + h \precsim_{\text{arch}} h$. Assume then that $g \in \mathcal{V}^*$. By Proposition 3.1.14 we then have $h \precnsim g$, hence by Proposition 3.1.15 $g + h \sim g$, and since $\precsim_{\text{arch}}$ is a coarsening of $\precsim$ this implies $g + h \sim_{\text{arch}} g$, hence $g + h \precsim_{\text{arch}} h$. This proves that $\precsim_{\text{arch}}$ is valuational.

Now let $\precsim^*$ be another coarsening of $\precsim$ such that $\precsim^*$ is $\mathbb{Z}$-module valuational. We show that $\precsim^*$ is a coarsening of $\precsim_{\text{arch}}$. Let $g, h \in G$ with $g \precsim_{\text{arch}} h$. We first assume that $g \lessdot h$. There is n, m with $0 \precsim ng \precsim mh$. Since $\precsim^*$ is a coarsening of $\precsim$, this implies $ng \precsim^* mh$. Since $\precsim^*$ is $\mathbb{Z}$-module-valuational, we have $ng \sim^* g$ and $mh \sim^* h$, hence $g \precsim^* h$. Now for the general case, we know that there are $x_1, \ldots, x_r$ with $g \lessdot x_1 \lessdot \ldots \lessdot x_r \lessdot h$. By what we just proved this implies $g \precsim^* x_1 \precsim^* \ldots \precsim^* x_r \precsim^* h$, which by transitivity of $\precsim^*$ implies $g \precsim^* h$.This shows that $\precsim^*$ is a coarsening of $\precsim_{\text{arch}}$. □

The valuation v_{arch} corresponding to $\precsim_{\text{arch}}$ is called **the archimedean valuation associated to** $\precsim$. $(G, \precsim)$ is an **archimedean compatible q.o.a.g.** if v_{arch} is the trivial valuation on G. Because $\precsim_{\text{arch}}$ is a coarsening of $\precsim$, v_{arch} is compatible with $\precsim$. It follows from Theorem 3.2.2 that, for every $\gamma \in \Gamma$, $\precsim$ induces a compatible q.o. $\precsim_\gamma$ on B_γ. Note that $(B_\gamma, \precsim_\gamma)$ is archimedean.

We can now state a Hahn's embedding theorem for compatible q.o.a.g's:

Theorem 3.3.8
Let $(G, \precsim)$ be a torsion-free divisible compatible q.o.a.g. and let v_{arch} be the archimedean valuation associated to $\precsim$. Let $(\Gamma, (B_\gamma)_{\gamma \in \Gamma})$ be the skeleton of (G, v_{arch}) and let $\precsim_\gamma$ be the q.o induced by $\precsim$ on B_γ. Then there exists an embedding of quasi-ordered groups from $(G, \precsim)$ into the compatible Hahn product $\mathrm{H}_{\gamma \in \Gamma}(B_\gamma, \precsim_\gamma)_\gamma$.

Proof. Let $(H, \precsim^*)$ denote the compatible product of the family $(B_\gamma, \precsim_\gamma)_{\gamma \in \Gamma}$ and let v denote the usual valuation on H (i.e $v(h) = \min \operatorname{supp}(h)$). We denote by $(F, \precsim_{\text{val}})$ the valuational product of the family $(B_\gamma / B_\gamma^o)_{\gamma \in \Gamma}$. We take a group embedding $\phi : G \to H$ as given by Theorem 2.2.9 (note that, since G is divisible, $\widehat{B}_\gamma = B_\gamma$). For $g \in G$ we denote by g_γ the coefficient of $\phi(g)$ at γ. We need the following claims:

Claim 1: For any $\gamma \in \Gamma$ and $g \in G$ with $v_{\text{arch}}(g) = \gamma$, g is v-type if and only if $g + G_\gamma$ is v-type.

Proof. It follows from Proposition 3.1.20. □

Claim 2: If $h \in G$ is o-type and $\delta = v_{\text{arch}}(h)$, then for every $\gamma > \delta$, $B_\gamma^o = B_\gamma$.

Proof. Let $g \in G$ with $v_{\text{arch}}(g) = \gamma > \delta$. Since $v_{\text{arch}}(g) > v_{\text{arch}}(h)$ and since $\precsim_{\text{arch}}$ is a coarsening of $\precsim$, we must have $g \precsim h$. By Proposition 3.1.14, it follows that g is o-type. By Claim 1, $g + G_\gamma$ is then o-type. This shows that every element of B_γ is o-type. □

Claim 3: For any $g \in G$ and $\delta := v_{\text{arch}}(g)$, we have $\min \operatorname{supp}((g_\gamma + B_\gamma^o)_{\gamma \in \Gamma}) \geq \delta$. If moreover $g \notin G^o$, then $\min \operatorname{supp}((g_\gamma + B_\gamma^o)_{\gamma \in \Gamma}) = \delta$

Proof. By condition (1) of Theorem 2.2.9, we have $\delta = v(\phi(g))$. It then follows from the definition of v and δ that $g_\epsilon = 0$ for every $\epsilon < \delta$, hence $g_\epsilon + B^o_\epsilon = 0$ for every $\epsilon < \delta$ hence $\min\operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma}) \geq \delta$. Now if $g \notin G^o$, then by Claim 1 $g + G_\delta$ is v-type, and by condition (2) of Theorem 2.2.9 we have $g_\delta = g + G_\delta$, hence $g_\delta \notin B^o_\delta$ hence $\delta \in \operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma})$, which proves $\delta = \min\operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma})$. □

Now let us show the theorem. Take $g, h \in G$ and set $\alpha := v_{\text{arch}}(g) = v(\phi(g))$ and $\beta := v_{\text{arch}}(h) = v(\phi(h))$. We want to show that $g \precsim h \Leftrightarrow \phi(g) \precsim^* \phi(h)$. Without loss of generality, we assume $g \neq h$. We first assume that $g \precsim h$ and we show that $\phi(g) \precsim^* \phi(h)$. Note that $g \precsim h$ implies $\alpha \geq \beta$ by Proposition 3.3.7. We use the formula of Proposition 3.1.25. We first consider the case $h \in G^o$, which implies $g \in G^o$ by Proposition 3.1.14. Since g, h are o-type, it follows from Claim 1 and from condition (2) of Theorem 2.2.9 that g_α and h_β are o-type. Moreover, it follows from Claim 2 that $B^o_\epsilon = B_\epsilon$ for every $\epsilon > \beta$. It follows that $g_\epsilon, h_\epsilon \in B^o_\epsilon$ for every $\epsilon \geq \beta$. Since $v_{\text{arch}}(g), v_{\text{arch}}(h) \geq \beta$, this implies that $\phi(g)$ and $\phi(h)$ both lie in $H^o = \mathrm{H}_{\gamma\in\Gamma} B^o_\gamma$. Now set $\epsilon := v_{\text{arch}}(g - h) = v(\phi(g) - \phi(h))$. Since $g \precsim h \in G^o$, we have $0 \precsim h - g$, hence $0 \precsim_\epsilon (h - g) + G_\epsilon$. It then follows from condition (2) of 2.2.9 that $0 \precsim h_\epsilon - g_\epsilon$, and since $g_\epsilon, h_\epsilon \in B^o_\epsilon$ this implies $g_\epsilon \precsim_\epsilon h_\epsilon$. By definition of $\precsim^*$ on H^o, this implies $\phi(g) \precsim^* \phi(h)$. Now consider the case where $h \notin G^o \wedge g + G^o \precsim h + G^o$. It follows from Claim 1 that h_β is v-type, hence $\phi(h) \notin H^o$. Now note that $g_\beta \precsim h_\beta$. Indeed, if $\alpha > \beta$, then $g_\beta = 0$. Since h_β is v-type, we have $0 \precsim_\beta h_\beta$. If $\beta = \alpha$, then we have $g_\beta = g + G_\beta$ and $h_\beta = h + G_\beta$. Now $g \precsim h$ implies $g + G_\beta \precsim_\beta h + G_\beta$, hence $g_\beta \precsim h_\beta$. This in turn implies $g_\beta + B^o_\beta \precsim h_\beta + B^o_\beta$. Moreover, Claim 3 implies that $\beta = \min(\min\operatorname{supp}((h_\gamma + B^o_\gamma)_{\gamma\in\Gamma}), \min\operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma}))$. It then follows from the definition of $\precsim_{\text{val}}$ that $((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma}) \precsim_{\text{val}} ((h_\gamma + B^o_\gamma)_{\gamma\in\Gamma})$, which by definition of $\precsim^*$ implies $\phi(g) \precsim^* \phi(h)$.

This shows $g \precsim h \Rightarrow \phi(g) \precsim^* \phi(h)$, let us show the converse. Assume that $\phi(g) \precsim^* \phi(h)$ holds and let us show that $g \precsim h$. Note that if $\beta < \alpha$, then since v_{arch} is a coarsening of $\precsim$ we have $g \precsim h$, so we can assume $\beta \geq \alpha$. We can also assume $g \neq h$. We first consider the case $\phi(h) \in H^o$, which implies $\phi(g) \in H^o$ by Proposition 3.1.14. Since $\precsim^*$ is an order on H^o and since $g \neq h$, we have $\phi(g) \precnsim^* \phi(h)$. By definition of $\precsim^*$ on H^o, we have $g_\gamma \precnsim_\gamma h_\gamma$ for $\gamma := v(\phi(g) - \phi(h))$. Since $h_\gamma, g_\gamma \in B^o_\gamma$, this implies $0 \precnsim_\gamma (h - g)_\gamma$. Since $\gamma = v(\phi(g) - \phi(h))$, we have $(h - g)_\gamma = (h - g) + G_\gamma$ by condition (2) of Theorem 2.2.9, hence $0 \precnsim_\gamma (h - g) + G_\gamma$. By Remark 3.1.22, this implies $0 \precsim h - g$. By Claim 1, $h - g$ is o-type, so the previous inequality implies $g \precsim h$. Now assume that $\phi(h) \notin H^o$. By definition of $\precsim^*$, this implies $(g_\gamma + B^o_\gamma)_{\gamma\in\Gamma} \precsim_{\text{val}} (h_\gamma + B^o_\gamma)_{\gamma\in\Gamma}$. By definition of $\precsim_{\text{val}}$, this implies $\min(\operatorname{supp}((h_\gamma + B^o_\gamma)_{\gamma\in\Gamma})) \leq \min(\operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma}))$. By Claim 3, this implies $\beta \leq \alpha$, hence $\beta = \alpha = \min(\min(\operatorname{supp}((h_\gamma + B^o_\gamma)_{\gamma\in\Gamma})), \min(\operatorname{supp}((g_\gamma + B^o_\gamma)_{\gamma\in\Gamma}))))$. The definition of $\precsim_{\text{val}}$ then also implies $g_\beta + B^o_\beta \precsim_\beta h_\beta + B^o_\beta$. Because $\phi(h) \notin H^o$, it follows from Claim 1 that $h_\beta \notin B^o_\beta$. By Remark 3.1.22, $g_\beta + B^o_\beta \precsim_\beta h_\beta + B^o_\beta$ then implies $g_\beta \precsim h_\beta$. Since $\beta = v_{\text{arch}}(g) = v_{\text{arch}}(h)$, condition (2) of Theorem 2.2.9 implies $g_\beta = g + G_\beta$ and $h_\beta = h + G_\beta$, hence $g + G_\beta \precsim_\beta h + G_\beta$. This implies $g \precsim h$ by Remark 3.1.22. This finishes the proof.

□

3.3.3 Elementary equivalence and products

In view of Theorem 3.1.28, it is natural to ask whether elementary equivalence of two compatible q.o.a.g.'s is equivalent to the elementary equivalence of their respective ordered parts and the elementary equivalence of their valued parts. This is the object of this subsection. We first show that one implication is always true : if two compatible q.o.a.g.'s are elementarily equivalent, then so are their ordered parts and so are their valued parts (Proposition 3.3.10). The converse fails in general (see example 3.3.11), but we show then that it holds for groups which are obtained as the compatible product of their ordered parts by their valued parts. In other words, we show that the compatible product of ordered groups by valued groups preserves elementary equivalence (Theorem 3.3.13).

We let $\mathcal{L}$ denote the language of quasi-ordered groups: $\mathcal{L} = \{0, +, -, \precsim\}$ ($-$ is interpreted as a unary relation). Note that the atomic formulas are all formulas of the form $P(\bar{x}) \precsim Q(\bar{x})$ or $P(\bar{x}) = 0$, where $P(\bar{x}), Q(\bar{x})$ are expressions of the form $\sum_{i=1}^{k} n_i x_i$ with $n_1, \ldots, n_k \in \mathbb{Z}$. Note also that for any compatible q.o.a.g. $(G, \precsim)$, G^o is definable in G by the formula $x = 0 \vee -x \not\sim x$, which we will thus abbreviate as the formula $x \in G^o$. Finally, note that for any term $P(\bar{x})$ and for every tuples $\bar{g}, \bar{h} \subseteq G$ we have $P(\bar{g} + \bar{h}) = P(\bar{g}) + P(\bar{h})$ and $P(\bar{g} + G^o) = P(\bar{g}) + G^o$.

Lemma 3.3.9
Let $\phi(\bar{x})$ be a formula of $\mathcal{L}$. Then there exists two formulas $\phi^o(\bar{x}), \phi^v(\bar{x})$ in $\mathcal{L}$, each of the same arity as ϕ, such that the following holds for any compatible q.o.a.g. $(G, \precsim)$:

(i) For any $\bar{g}_o \subseteq G^o$, $G^o \vDash \phi(\bar{g}_o)$ if and only if $G \vDash \phi^o(\bar{g}_o)$

(ii) For any $\bar{g}_v \subseteq G^v$, $G^v \vDash \phi(\bar{g}_v)$ if and only if for all $\bar{g} \subseteq G$, $\bar{g} + G^o = \bar{g}_v \Rightarrow G \vDash \phi^v(\bar{g})$ if and only if there exists $\bar{g} \subseteq G$ with $\bar{g} + G^o = \bar{g}_v$ and $G \vDash \phi^v(\bar{g})$.

Proof. For (i): write ϕ in prenex form: $\phi(\bar{x}) \equiv Q_1 y_1 \ldots Q_n y_n \psi(\bar{y}, \bar{x})$, where each Q_i is a quantifier and ψ is quantifier-free. Since G^o is definable in G (by the formula $x = 0 \vee -x \not\sim x$), we can define the formula $\phi^o(\bar{x}) \equiv Q_1 y_1 \in G^o \ldots Q_n y_n \in G^o \psi(\bar{y}, \bar{x})$, and it is then easy to see that ϕ^o has the desired property.

For (ii): We proceed by induction on ϕ. Assume first that ϕ is atomic. If ϕ has the form $P(\bar{x}) = 0$, then define $\phi^v(\bar{x}) \equiv P(\bar{x}) \in G^o$. Now assume that $\phi(\bar{x}) \equiv P(\bar{x}) \precsim Q(\bar{x})$ and define $\phi^v(\bar{x})$ as $(P(\bar{x}) \in G^o \wedge Q(\bar{x}) \in G^o) \vee (Q(\bar{x}) \notin G^o \wedge \phi(\bar{x}))$. Assume that $G^v \vDash \phi(\bar{g}_v)$ and take $\bar{g} \subseteq G$ with $\bar{g} + G^o = \bar{g}_v$. We have $G^v \vDash P(\bar{g}_v) \precsim Q(\bar{g}_v)$. If $Q(\bar{g}_v) = 0$, then since $\precsim$ is valuational on G^v we must have $P(\bar{g}_v) = 0$, hence $G \vDash P(\bar{g}) \in G^o \wedge Q(\bar{g}) \in G^o$. If $Q(\bar{g}_v) \neq 0$, then Remark 3.1.22 implies that $P(\bar{g}) \precsim Q(\bar{g})$. This shows that $G \vDash \phi^v(\bar{g})$. Conversely, assume that there exists a $\bar{g} \subseteq G$ such that $\bar{g} + G^o = \bar{g}_v$ and $G \vDash \phi^v(\bar{g})$. If $G \vDash (P(\bar{g}) \in G^o \wedge Q(\bar{g}) \in G^o)$ then $P(\bar{g}_v) = Q(\bar{g}_v) = 0$, so in particular $G^v \vDash P(\bar{g}_v) \precsim Q(\bar{g}_v)$. If $G \vDash Q(\bar{g}) \notin G^o \wedge \phi(\bar{g})$, then Remark 3.1.22 implies that $G^v \vDash P(\bar{g}_v) \precsim Q(\bar{g}_v)$. This shows that $G^v \vDash \phi(\bar{g}_v)$ and concludes the case where ϕ is atomic. Assume now that $\phi \equiv \neg\psi$ and set $\phi^v :\equiv \neg\psi^v$. If $G^v \vDash \phi(\bar{g}_v)$, then $G^v \nvDash \psi(\bar{g}_v)$, so by induction hypothesis we have $G \nvDash \psi^v(\bar{g})$ for all $\bar{g} \subseteq G$ with $\bar{g} + G^o = \bar{g}_v$, hence $G \vDash \phi^v(\bar{g})$. Conversely, if there is $\bar{g} \subseteq G$ with $\bar{g} + G^o = \bar{g}_v$ and $G \vDash \phi^v(\bar{g})$, then $G \nvDash \psi^v(\bar{g})$

which by induction hypothesis means $G^v \nvDash \psi(\bar{g}_v)$ hence $G^v \vDash \phi(\bar{g}_v)$. If $\phi \equiv \phi_1 \wedge \phi_2$, one can easily show that $\phi^v :\equiv \phi_1^v \wedge \phi_2^v$ satisfies the desired property and if $\phi \equiv \exists y \psi(y, \bar{x})$, it is also easy to see that $\phi^v \equiv \exists y \psi^v(y, \bar{x})$ is suitable. □

Proposition 3.3.10
Let $(G_1, \precsim_1)$ and $(G_2, \precsim_2)$ be two compatible q.o.a.g.'s such that $(G_1, 0, +, -, \precsim_1) \equiv (G_2, 0, +, -, \precsim_2)$. Then we have $(G_1^o, 0, +, -, \precsim_1) \equiv (G_2^o, 0, +, -, \precsim_2)$ and $(G_1^v, 0, +, -, \precsim_1) \equiv (G_2^v, 0, +, -, \precsim_2)$.

Proof. Assume that $G_1 \equiv G_2$ holds and let ϕ be a sentence of $\mathcal{L}$. Take ϕ^o, ϕ^v as in Lemma 3.3.9. If $G_1^o \vDash \phi$, then $G_1 \vDash \phi^o$, hence by assumption $G_2 \vDash \phi^o$, hence by choice of ϕ^o: $G_2^o \vDash \phi$. We could show similarly that $G_1^v \vDash \phi$ implies $G_2^v \vDash \phi$, hence $G_1^o \equiv G_2^o$ and $G_1^v \equiv G_2^v$. □

The next example shows that the converse of Proposition 3.3.10 is false in general:

Example 3.3.11
Take $G_1 := \mathbb{Z}$ with ordered part $G_1^o := 5\mathbb{Z}$ (with the usual order $\leq$) and valued part $H_1 := \mathbb{Z}/5\mathbb{Z}$ equipped with the trivial valuation $\precsim$. Now take $(G_2, \precsim_2) := (H_1, \precsim) \times (G_1^o, \leq)$ (compatible product). Since G_2 has torsion and G_1 does not, it is clear that G_1 and G_2 are not elementarily equivalent.

However, the next Lemma shows that the converse of Proposition 3.3.10 is true if we restrict ourselves to compatible q.o.a.g.'s which are obtained as the product of their ordered part by their valued part (which is not the case of G_1 in example 3.3.11):

Lemma 3.3.12
Let $\phi(\bar{x})$ be a formula of $\mathcal{L}$. Then there is $n \in \mathbb{N}$ such that there are $2n$ formulas $\phi_1^o(\bar{x}), \ldots, \phi_n^o(\bar{x}), \phi_1^v(\bar{x}), \ldots, \phi_n^v(\bar{x})$, each having the same arity as ϕ, such that the following holds:

For any ordered abelian group $(G^o, \precsim)$ and any valuationally quasi-ordered group $(G^v, \precsim)$, for any $\bar{g} = \bar{g}_o + \bar{g}_v$ in $(G, \precsim) := (G^v, \precsim) \times (G^o, \precsim)$, we have: $G \vDash \phi(\bar{g})$ if and only if there exists $i \in \{1, \ldots, n\}$ such that $G^o \vDash \phi_i^o(\bar{g}_o)$ and $G^v \vDash \phi_i^v(\bar{g}_v)$.

Proof. We identify G^v with G/G^o. Note that the q.o induced by $\precsim$ on G/G^o coincides with the q.o of G^v. We proceed by induction on ϕ. We first assume that ϕ is an atomic formula. If ϕ is of the form $P(\bar{x}) = 0$, set $n = 1$ and $\phi_1^o \equiv \phi_1^v \equiv \phi$. Assume that ϕ is of the form $P(\bar{x}) \precsim Q(\bar{x})$. Set $n = 2$ and define $\phi_1^o(\bar{x}) :\equiv (\bar{x} = \bar{x})$, $\phi_1^v(\bar{x}) :\equiv (Q(\bar{x}) \neq 0 \wedge \phi(\bar{x}))$, $\phi_2^o :\equiv \phi$ and $\phi_2^v(\bar{x}) :\equiv (Q(\bar{x}) = P(\bar{x}) = 0)$. We must check that these formulas satisfy the desired condition. Note that for any $G^o, G^v, \bar{g}$ as above, we have $P(\bar{g}) = P(\bar{g}_o) + P(\bar{g}_v)$ with $P(\bar{g}_o) \in G^o$ and $P(\bar{g}_v) \in G^v$, and in particular we have $P(\bar{g}) + G^o = P(\bar{g}_v)$ and $P(\bar{g}) \in G^o$ if and only if $P(\bar{g}_v) = 0$. With this remark in mind, it follows directly from Proposition 3.1.25 that the formulas $\phi_1^o, \phi_2^o, \phi_1^v, \phi_2^v$ satisfy the condition we want. This settles the case where ϕ is atomic. If $\phi \equiv \psi \vee \chi$, and if $\psi_1^o, \ldots, \psi_k^o, \psi_1^v, \ldots, \psi_k^v, \chi_1^o, \ldots, \chi_l^o, \chi_1^v, \ldots, \chi_l^v$ are the desired formulas for ψ and χ, we simply set $n := k + l$, $\phi_i^o :\equiv \psi_i^o, \phi_i^v :\equiv \psi_i^v$ for $1 \leq i \leq k$ and $\phi_i^o :\equiv \chi_i^o, \phi_i^v :\equiv \chi_i^v$ for $k < i \leq n$. Now assume that $\phi \equiv \exists y \psi(y, \bar{x})$ and let $\psi_1^o, \dots, \psi_k^o, \psi_1^v, \dots, \psi_k^v$ be the desired formulas for ψ. Define $n := k$, $\phi_i^o :\equiv \exists y \psi_i^o(y, \bar{x})$ and $\phi_i^v :\equiv \exists y \psi_i^v(y, \bar{x})$ for every $i \in \{1, \dots, n\}$. If $G \vDash \phi(\bar{g})$, then there is $h \in G$ with $G \vDash \psi(h, \bar{g})$, which implies by induction hypothesis that there is i with $G^o \vDash \psi_i^o(h_o, \bar{g}_o)$ and $G^v \vDash \psi_i^v(h_v, \bar{g}_v)$, hence $G^o \vDash \phi_i^o(\bar{g}_o)$ and $G^v \vDash \phi_i^v(\bar{g}_v)$. Conversely, if we assume that $G^o \vDash \phi_i^o(\bar{g}_o)$ and $G^v \vDash \phi_i^v(\bar{g}_v)$, then there is some $h_o \in G^o$ and $h_v \in G^v$ with $G^o \vDash \psi_i^o(h_o, \bar{g}_o)$ and $G^v \vDash \psi_i^v(h_v, \bar{g}_v)$, and by induction hypothesis we then have $G \vDash \psi(h_o + h_v, \bar{g})$ hence $G \vDash \phi(\bar{g})$. This shows that the formulas $\phi_1^o, \dots, \phi_n^o, \phi_1^v, \dots, \phi_n^v$ have the desired property.

Now we just have to consider the case $\phi \equiv \neg\psi$. Let $\psi_1^o, \dots, \psi_k^o, \psi_1^v, \dots, \psi_k^v$ be given. Let $P := \mathcal{P}(\{1, \dots, k\})$ denote the power set of $\{1, \dots, k\}$. For any $I \in P$, we define ϕ_I^o, ϕ_I^v as follows: $\phi_I^o \equiv \bigwedge_{i \in I} \neg\psi_i^o$ and $\phi_I^v \equiv \bigwedge_{i \notin I} \neg\psi_i^v$. Now let us check that the formulas $(\phi_I^o)_{I \in P}$ and $(\phi_I^v)_{I \in P}$ satisfy the desired property. Assume that $G \vDash \phi(\bar{g})$, so $G \nvDash \psi(\bar{g})$. By induction hypothesis, this means that for all $i \in \{1, \dots, k\}$, either $G^o \nvDash \psi_i^o(\bar{g}_o)$ or $G^v \nvDash \psi_i^v(\bar{g}_v)$. Choose $I \in P$ as the set of all i with $G^o \nvDash \psi_i^o(\bar{g}_o)$. Then $G^o \vDash \phi_I^o(\bar{g}_o)$ and $G^v \vDash \phi_I^v(\bar{g}_v)$. Conversely, assume there is $I \in P$ with $G^o \vDash \phi_I^o(\bar{g}_o)$ and $G^v \vDash \phi_I^v(\bar{g}_v)$. Then for any $i \in \{1, \dots, k\}$, we either have $G^o \nvDash \psi_i^o(\bar{g}_o)$ (when $i \in I$) or $G^v \nvDash \psi_i^v(\bar{g}_v)$ (when $i \notin I$). By induction hypothesis, this means that $G \nvDash \psi(\bar{g})$.

□

An immediate consequence of Lemma 3.3.12 is the following Theorem:

Theorem 3.3.13
Let $(G_1^o, \precsim_1), (G_2^o, \precsim_2)$ be two ordered abelian groups and $(G_1^v, \precsim_1), (G_2^v, \precsim_2)$ two valuationally quasi-ordered groups.
Set $(G_1, \precsim_1) := (G_1^v, \precsim_1) \times (G_1^o, \precsim_1)$ and $(G_2, \precsim_2) := (G_2^v, \precsim_2) \times (G_2^o, \precsim_2)$. Then $(G_1, 0, +, -, \precsim_1) \equiv (G_2, 0, +, -, \precsim_2)$ if and only if $(G_1^o, 0, +, -, \precsim_1) \equiv (G_2^o, 0, +, -, \precsim_2)$ and $(G_1^v, 0, +, -, \precsim_1) \equiv (G_2^v, 0, +, -, \precsim_2)$.

Proof. One direction of the theorem is given by Proposition 3.3.10, let us now prove the converse. Assume that $G_1^o \equiv G_2^o$ and $G_1^v \equiv G_2^v$ holds and let ϕ be a sentence of $\mathcal{L}$ with $G_1 \vDash \phi$. Take $\phi_1^o, \dots, \phi_n^o, \phi_1^v, \dots, \phi_n^v$ as in Lemma 3.3.12. Since $G_1 \vDash \phi$, there is $i \in \{1, \dots, n\}$ such that $G_1^o \vDash \phi_i^o$ and $G_1^v \vDash \phi_i^v$. By assumption, we then have $G_2^o \vDash \phi_i^o$ and $G_2^v \vDash \phi_i^v$, which by choice of ϕ_i^o, ϕ_i^v implies that $G_2 \vDash \phi$. We could show similarly that $G_2 \vDash \phi$ implies $G_1 \vDash \phi$, hence $G_2 \equiv G_1$. □

3.4 Quasi-order-minimality and C-relations

The goal of this section is to define a q.o. analog of o-minimal groups. More precisely, we want to find a notion of minimality for compatible quasi-ordered abelian groups. We want this notion of minimality to generalize o-minimality, but we also want it to give an interesting class of valued groups when applied to valuationally quasi-ordered groups. In this section, we will present our approach to answer this problem, which allows us to define a notion of q.o.-minimality satisfying the criteria that we want. We will then show that this notion of q.o.-minimality turns out to be equivalent to C-minimality.

An o-minimal group is defined as an ordered group $(G, \leq)$ such that any definable subset of G is a finite disjoint union of intervals. By analogy, we want to define a quasi-order-minimal group as a group such that every definable subset of the group is a finite disjoint union of "simple" definable sets. This requires first determining what the "simple definable sets" are in the case of quasi-ordered groups.

Jan Holly (see [Hol95]) already gave the shape of simple definable sets of valued fields: they are what she called swiss cheeses, i.e sets of the form $X\backslash\bigcup_{i=1}^n X_i$ where X and each X_i is an ultrametric ball. Following her idea, we define a **$\precsim$-ball** of a compatible q.o.a.g. $(G,\precsim)$ as a set which is either a singleton, the empty set or a set of the form $\{g\in G \mid g-a\precsim b\}$ (closed ball) or $\{g\in G\mid g-a\precnsim b\}$(open ball) for some parameters $a,b\in G$. We then define a **$\precsim$-swiss cheese** of G as a subset of G of the form $X\backslash\bigcup_{i=1}^n X_i$ where X is either G or a $\precsim$-ball and each X_i is a $\precsim$-ball. Now consider compatible q.o.a.g.'s as structures of the language $\{0,+,-,\precsim\}$. We say that a compatible q.o.a.g. $(G,0,+,-,\precsim)$ is **quasi-order-minimal** if the following condition holds: for every compatible q.o.a.g. $(H,0,+,-,\precsim)$ which is elementarily equivalent to $(G,0,+,-,\precsim)$, every definable subset of H is a finite disjoint union of $\precsim$-swiss cheeses. Note that if $(G,\precsim)$ happens to be an ordered abelian group then the $\precsim$-balls are just initial segments, and the class of finite unions of swiss cheeses is exactly the class of finite unions of intervals. Therefore, an ordered group is quasi-order minimal if and only if it is o-minimal.

We will now show that compatible quasi-orders naturally induce a C-relation and that quasi-order-minimality is equivalent to C-minimality. Since a compatible q.o. is a mix of order and valuation, we can define a C-relation from a compatible q.o. by mixing the definition of an order-type C-relation with the definition of a valuational C-relation.

Proposition 3.4.1
Let $(G,\precsim)$ be a compatible q.o.a.g. Consider the relation $C(x,y,z)$ defined by the following formula:

$$(x\neq y=z)\vee(x-z\in\mathcal{V}^*\wedge(y-z\precnsim x-z))\vee(y-z,x-z\in G^o\wedge(0\precnsim x-y\wedge 0\precnsim x-z))$$

Then C is a C-relation compatible with $+$. Moreover, $\precsim$ is the only compatible q.o inducing C, C is quantifier-free definable without parameters in the language $\{0,+,-,\precsim\}$ and $\precsim$ is quantifier-free definable without parameters in the language $\{0,+,-,C\}$.

Proof. Let v be the valuation defined in Proposition 3.1.16 and C^v the C-relation induced by v, i.e $C^v(x,y,z)\Leftrightarrow v(y-z)>v(x-z)$. Let us show that C satisfies the axioms of C-relations. It is clear from the definition of C that (C_4) holds. We prove $(C_1),(C_2),(C_3)$ simultaneously. Take $x,y,z,w\in G$ such that $C(x,y,z)$ and let us show that $C(x,z,y),\neg C(y,x,z)$ and $C(w,y,z)\vee C(x,w,z)$ hold. Assume first that $y=z$. Then clearly $C(x,z,y)$ holds. Because $y-y\in G^o$ and $y-y\precsim 0$, it follows from the definition of C that $C(y,x,y)$ cannot hold, so $\neg C(y,x,z)$ holds. Finally, if $w\neq y$, then by (C_4), $C(w,y,z)$ holds. If $w=y$, then $C(x,w,z)$ holds. Now assume $z\neq y$. Assume first that $x-z\in\mathcal{V}^*$. Then $C(x,y,z)$ implies $C^v(x,y,z)$. Because C^v is a C-relation, this implies $C^v(x,z,y)$, which means $v(z-y)>v(x-y)$. Since v takes its maximal non-infinite value on $G^o\backslash\{0\}$ and is constant on $G^o\backslash\{0\}$, it follows that $x-y\in\mathcal{V}^*$, so $C(x,z,y)$ holds. We also have $\neg C^v(y,x,z)$, and since $x-z\notin G^o$ this implies $\neg C(y,x,z)$.

If $x-z \precsim w-z$, then Proposition 3.1.14 implies $w-z \in \mathcal{V}^*$. Moreover, we then have $y-z \precnsim x-z \precsim w-z$, so $y-z \precnsim w-z$, so $C(w,y,z)$ holds. If $w-z \precnsim x-z$, then it directly follows from the definition of C that $C(x,w,z)$ holds. Assume now that $x-z, y-z \in G^o \wedge 0 \precnsim x-y \wedge 0 \precnsim x-z$ holds. We can obviously exchange z and y in this formula, hence $C(x,z,y)$. However, $0 \precnsim x-y \in G^o$ implies $y-x \precnsim 0$, so $C(y,x,z)$ does not hold. If $w-z \in \mathcal{V}^*$, then Proposition 3.1.14 implies $y-z \precnsim w-z$, hence $C(w,y,z)$. Assume $w-z \in G^o$, which also implies $w-x, w-y \in G^o$. If $C(w,y,z)$ does not hold, then either $w-z \precnsim 0$ or $w-y \precnsim 0$ must be true. If $w-z \precnsim 0$, then $0 \precnsim z-w$, hence $0 \precnsim x-w = x-z+z-w$, so $C(x,w,z)$ holds; the same reasoning holds if $w-y \precnsim 0$.

The fact that C is compatible with $+$ is obvious from its definition. Note that G^o, $\mathcal{V}^*$ are both quantifier-free definable in the language $\{0,+,-,\precsim\}$ since we have $G^o = \{g \in G \mid g \nsim -g \vee g = 0\}$ and $\mathcal{V}^* = \{g \in G \mid g \sim -g \neq 0\}$, so C is defined with a quantifier-free formula of that language. We want to show the converse. Set $G^+ := \{g \in G \mid 0 \precsim g\}$ and $G^- = \{g \in G \mid g \precnsim 0\}$. We want to find a formula defining $\precsim$ in the language $\{0,+,-,C\}$. Note first that we have

$$x \precsim y \Leftrightarrow (x \in G^- \wedge y \in G^+) \vee (x, y \in G^+ \wedge x \precsim y) \vee (x, y \in G^- \wedge -y \precsim -x)$$

It is easy to see from the definition of C that for any $x, y \in G^+$, $x \precnsim y \Leftrightarrow C(y,x,0)$. Moreover, G^- and G^+ are quantifier-free definable with C: $G^+ \cap G^o$ is given by the formula $C(x,-x,0) \vee (x=0)$. Indeed, by definition of C we have $C(x,-x,0) \Leftrightarrow (x \neq -x = 0) \vee (x \in \mathcal{V}^* \wedge -x \precnsim x) \vee (x, -x \in G^o \wedge 0 \precnsim x \wedge -x \precnsim x)$. Obviously $(x \neq -x = 0) \vee (x \in \mathcal{V}^* \wedge -x \precnsim x)$ is impossible (if x is in $\mathcal{V}^*$ then $x \sim -x$) so $C(x,-x,0) \Leftrightarrow (x, -x \in G^o \wedge 0 \precnsim x \wedge -x \precnsim x)$ which means $x \in G^o \cap G^+$. It follows that G^- is defined by the formula $C(-x,x,0)$ and that $\mathcal{V}^*$ is defined by $\neg C(x,-x,0) \wedge \neg C(-x,x,0)$. Thus, the formula

$$\phi(x,y) :\equiv (x \in G^- \wedge y \in G^+) \vee (x, y \in G^+ \wedge C(y,x,0)) \vee (x, y \in G^- \wedge C(-x,-y,0))$$

is a quantifier-free formula of the language $\{0,+,-,C\}$ and we have $x \precsim y \Leftrightarrow \neg\phi(y,x)$ for any $x, y \in G$. This proves that $\precsim$ is quantifier-free definable in $\{0,+,-,C\}$ and it also proves that $\precsim$ is the only compatible q.o inducing C since we can recover $\precsim$ from C. $\square$

Thus, compatible q.o.a.g.'s can be seen as C-groups. We now want to show that quasi-order-minimality is equivalent to C-minimality. In [Del11], Delon defined a notion of swiss cheeses for C-structures. Let $(M, C, \dots)$ be a C-structure. A **cone** is a subset of M of the form $\{x \mid C(a,x,b)\}$ for some parameters $a, b \in M$. A **thick cone** is a subset of M of the form $\{x \mid \neg C(x,a,b)\}$ for some parameters $a, b \in M$. A **C-swiss cheese** is a subset of M of the form $X \setminus \bigcup_{i=1}^n X_i$ where X is either M or a (possibly thick) cone and each X_i is a (possibly thick) cone. The author of [Del11] then gives a characterization of C-minimal structures in terms of C-swiss cheeses, which we reformulate here:

Proposition 3.4.2 (Proposition 3.3 of [Del11])
Let $\mathcal{M} = (M, C, \dots)$ be a C-structure. Then $(M, C, \dots)$ is C-minimal if and only if for

every $\mathcal{N} = (N, C, \dots)$ with $\mathcal{N} \equiv \mathcal{M}$, every definable subset of N is a finite disjoint union of C-swiss cheeses.

It turns out that the notion of C-swiss cheese coincides with the notion of $\precsim$-swiss cheese which we just defined:

Lemma 3.4.3
Let $(G, \precsim)$ be a compatible q.o.a.g. and C the C-relation defined in Proposition 3.4.1. Then any $\precsim$-ball in G is a cone or a thick cone, and any cone or thick cone is a $\precsim$-ball. In particular, for any $X \subseteq G$, X is a $\precsim$-swiss cheese if and only if X is a C-swiss cheese.

Proof. Let $X \subseteq G$. Assume first that X is a (possibly thick) cone. Takes $a, b \in G$ and assume X is defined by the formula $\neg C(x, a, b)$. Assume first that $a - b \in \mathcal{V}^*$. Then it follows from the definition of C that $\neg C(x, a, b)$ is equivalent to $(x = a \vee a \neq b) \wedge (x - b \precsim a - b)$. Since $a - b \in \mathcal{V}^*$, we have $a \neq b$, so X is defined by $x - b \precsim a - b$, which is a ball. Assume now that $a - b \in G^o$. Then it follows from the definition of C that $\neg C(x, a, b)$ is equivalent to $x - b \precsim 0 \vee x - a \precsim 0$. If $b \in G^o$, then also $a \in G^o$, and it then follows that $x - b \precsim 0 \vee x - a \precsim 0$ is equivalent to $x \precsim \max(a, b)$, which is a ball. Assume that $b \in \mathcal{V}^*$. Since $a - b \in G^o$, Proposition 3.1.4 implies that either $a - b \precsim 0$ or $b - a \precsim 0$ holds. Assume that $a - b \precsim 0$, and assume that $x - a \precsim 0$. We then have $x - b = x - a + a - b$. Since $x - a \precsim 0$ and $a - b \precsim 0$, this implies $x - b \precsim 0$. This proves that $x - a \precsim 0 \Rightarrow x - b \precsim 0$, so $x - b \precsim 0 \vee x - a \precsim 0$ is equivalent to $x - b \precsim 0$. If $b - a \precsim 0$, we can show similarly that $x - b \precsim 0 \vee x - a \precsim 0$ is equivalent to $x - a \precsim 0$. This shows that every thick cone is a $\precsim$-ball. Now assume that X is given by the formula $C(a, x, b)$. If $a - b \in \mathcal{V}^*$, then $C(a, x, b)$ is equivalent to $a - x \lnsim a - b$. If $a - b \in \mathcal{O}^-$, then $X = \varnothing$. If $a - b \in \mathcal{O}^+$, then $C(a, x, b)$ is equivalent to $0 \lnsim a - x \in G^o$, which is equivalent to $x - a \precsim 0$. This proves that every cone is a $\precsim$-ball. It then follows that every C-swiss cheese is a $\precsim$-swiss cheese.

Conversely, assume that X is a $\precsim$-ball. Assume that X is given by the formula $x - a \precsim b$. If $0 \lnsim b$, then one can check that $x - a \precsim b$ is equivalent to $\neg C(x, b + a, a)$, which is a thick cone. Assume that $b \precsim 0$. If G^o is trivial, then $b = 0$ and $X = \{a\}$, so assume that G^o is not trivial. Then there is $g \in G^o$ with $-b \lnsim g$. We then have $0 \lnsim b + g$. It also follows from (Q_2) that for all x, $x - a \precsim b \Leftrightarrow x - a + g \precsim b + g$. Since $0 \lnsim b + g$, this brings us back to the case $0 \lnsim b$. This proves that every closed $\precsim$-ball is a thick cone. Finally, assume that X is defined by $x - a \lnsim b$. By the same argument as before, we can assume that $0 \lnsim b$. One can check that X is then defined by $C(b + a, x, a)$, so X is a cone. This shows that every $\precsim$-ball is a cone, and it immediately follows that every $\precsim$-swiss cheese is also a C-swiss cheese.

□

As a consequence, C-minimality is equivalent to quasi-order-minimality:

Proposition 3.4.4
Let $\mathcal{L}$ be a language containing $\{0, +, -, \precsim\}$ and let $\mathcal{L}'$ be the language obtained when we replace $\precsim$ by C in $\mathcal{L}$. Assume $(G, \precsim, \dots)$ is an $\mathcal{L}$-structure so that $(G, \precsim)$ is a compatible q.o.a.g. and let $(G, C, \dots)$ be the corresponding $\mathcal{L}'$-structure where C is interpreted

as the C-relation induced by $\lesssim$. Then $(G, \lesssim, \dots)$ is quasi-order-minimal if and only if $(G, C, \dots)$ is C-minimal.

Proof. Assume that $(G, \lesssim, \dots)$ is quasi-order-minimal and let $(H, C, \dots)$ be an $\mathcal{L}'$-structure such that $(G, C, \dots) \equiv (H, C, \dots)$. Because C is definable in $\mathcal{L}$ without parameters and $\lesssim$ is definable in $\mathcal{L}'$ without parameters (see Proposition 3.4.1), it follows that $(G, \lesssim, \dots) \equiv (H, \lesssim, \dots)$. By assumption, it follows that every $\mathcal{L}$-definable subset of H is a finite disjoint union of $\lesssim$-swiss cheeses. By Proposition 3.4.3, it follows that every $\mathcal{L}'$-definable subset of H is a finite disjoint union of C-swiss cheeses. This proves that $(G, C, \dots)$ is C-minimal. The proof of the other direction is similar. $\square$

As a concluding remark for this chapter, we would like to detail the connection between compatible q.o.a.g.'s and C-groups. Proposition 3.4.1 basically says that any compatible q.o.a.g. is a C-group. This allows us to give examples of C-groups which are neither order-type nor valuational: Indeed, any example from Example 3.1.2 can be seen as a C-group. This naturally raises the following question: Is every compatible C-relation on an abelian group induced by a compatible q.o.? In the next Chapter, we will see that the answer to this question is negative. However, it should be emphasized that my work on compatible q.o.a.g.'s played an essential role in the discovery of the main result of Chapter 4 (Theorem 4.3.33), which gives the structure of an arbitrary C-group. Indeed, seeing compatible q.o.a.g.'s as an example of C-groups enabled me to gain the right intuition on the structure of arbitrary C-groups. Theorem 4.3.33, as well as the methods used to prove it, were inspired by my work on compatible q.o.a.g.'s (Note that Theorem 4.3.33 bears some similarities with Theorem 3.1.26). It is interesting to note that our structure Theorem 3.1.26 shows that compatible q.o.a.g.'s form a particularly simple class of C-groups since the set of o-type elements is an initial segment (one can compare this setting to Proposition 4.3.38, which states that a C-group can *a priori* contain any arbitrary alternation of o-type and v-type parts). In a sense, compatible q.o.a.g.'s are the simplest examples of C-groups whose C-relation is neither order-type nor valuational. This makes compatible q.o.a.g.'s potentially useful for the study of C-groups.

Chapter 4

Quasi-orders and C-groups

Introduction

The goal of this chapter is to describe C-groups and to characterize C-minimal groups amongst them. We saw in Section 3.4 that the class of compatible q.o.a.g.'s is a subclass of the class of abelian C-groups. However, it is a strict subclass: we can give examples of abelian C-groups whose C-relation does not come from a compatible q.o. (see Examples 4.3.1). This shows that the class of compatible quasi-orders is not really adapted to the study of the whole class of C-groups. In this chapter, we introduce a new kind of quasi-orders, which we call C-quasi-orders (see Definition 4.1.2), which in some sense generalize compatible q.o.'s. C-quasi-orders are in a one-to-one correspondence with compatible C-relations on groups. The study of C-groups is made easier by the use of C-quasi-orders, which is why we will not be working directly with C-relations but with C-quasi-orders.

We start by introducing and axiomatizing C-quasi-orders in Section 4.1.1. We then explain how C-quasi-orders relate to compatible q.o.'s in Section 4.1.2. Section 4.1.3 gives a few results on the behavior of the group operation with respect to the C-q.o. These results will be particularly useful later when we describe the structure of a C-q.o. group. In Section 4.1.4, we describe the structure of fundamental C-q.o.'s, i.e C-q.o.'s induced by fundamental C-relations. As one might expect, a valuational C-relation induces a valuational C-q.o. The case of order-type C-q.o.'s is more complicated. In particular, it is important to note that order-type C-q.o.'s are not orders. However, order-type C-q.o.'s are all obtained from orders in the same way (see Proposition 4.1.17). If we start with a group order on G, then the corresponding C-q.o. is obtained by merging all the negative elements of G into one single equivalence class (which then becomes the class of o^--type elements), and then moving this class between 1 and the set of positive elements (which is the class of o^+-type elements). The order remains unchanged on the positive cone of G, which allows us to recover the original order from its corresponding C-q.o.

In Section 4.2, we are interested in finding C-q.o. group analogs of Theorems 2.7.9 and 2.3.1. In the process, we show that, as happens for compatible q.o.'s, a C-q.o. induces a C-q.o. on the quotient G/H if H is convex. This notion of induced q.o. on a quotient plays an important role later when we decompose C-q.o. groups into "valued" and "ordered" parts. We obtain an analog of Theorem 2.7.9 in Theorem 4.2.4. We then show that the class of C-q.o.'s is stable under lifting (Proposition 4.2.7). We then establish a "Baer-Krull theorem" for C-q.o. group (Theorem 4.2.11). We deduce two corollaries from it: a "Baer-Krull theorem" for ordered groups (Theorem 4.2.13) and a "Baer-Krull theorem" for valued groups (Theorem 4.2.12). We then recover the classical Baer-Krull theorem from our Baer-Krull theorem for ordered groups in Section 4.2.3.

Section 4.3 describes the structure of C-groups. Roughly speaking, a C-group is a generalization of a compatible q.o.a.g., in the sense that it is a mix of ordered and valued groups. More precisely, we can decompose the group into a family of strictly convex subsets, on each of which the C-q.o. is "fundamental-like" (see Remark 4.1.23 for this terminology). The difference between a compatible q.o.a.g. and an arbitrary C-group is that, whereas the ordered part of a compatible q.o.a.g. is always an initial segment (Proposition 3.1.14), a C-group can in general alternate between "order-type-like" parts and "valuational-like" parts in a more arbitrary way (see Proposition 4.3.38). The main result of this chapter is Theorem 4.3.33, which is a decomposition theorem for C-q.o. groups. Theorem 4.3.33 basically states that valued and ordered groups are the "building blocks" of C-groups. In analogy to what we did in Chapter 3, the main idea to obtain this result is to use the distinction between v-type and o-type elements (see Definition 2.7.10). To each $g \in G$, we associate a subset T_g of G called the type-component of G. This set T_g is characterized by two properties : T_g is strictly convex, and if g is v-type (respectively o-type), then the C-quasi-order $\precsim$ is valuational-like (respectively, order-type-like) on T_g. Moreover, T_g is maximal with these properties. We can then show that the family of all type-components form a partition of G.

We want to draw attention to a counter-intuitive phenomenon, which we call welding, which occurs in certain C-quasi-ordered groups (see Definition 4.3.2). Welding happens when the group contains an o-type element which is equivalent to a v-type element. This is counter-intuitive, since one would expect the quasi-order to separate elements of different types. If there is no welding in the group, then the T_g's are actually convex. However, if there is welding at a point g, then the maximum of T_g is equivalent to the minimum of a T_h, which means that the type-components are only strictly convex. This also means that a C-q.o. cannot in general be obtained by lifting fundamental C-q.o.'s. However, Theorem 4.3.33 states that any C-q.o. can be obtained by first lifting fundamental C-q.o.'s and then "welding" if necessary, i.e coarsening the quasi-order in a certain way (see Proposition 4.3.32).

We finish this chapter by a study of C-minimal groups via the decomposition given by Theorem 4.3.33. We start by reinterpreting Macpherson's and Steinhorn's results on C-minimal groups (see [MS96]) in terms of C-q.o.'s in Section 4.4.1. In Section 4.4.2, we show a "Feferman-Vaught" theorem for C-q.o. groups, i.e. we show that finite valuational products preserve elementary equivalence (see Theorem 4.4.13). We then show that any welding-free abelian C-group is a finite valuational product of fundamental C-minimal groups (see Theorem 4.4.37). Finally, we show that the valuational product of a C-minimal order-type C-q.o. group by a finite valued group is always C-minimal

(Proposition 4.4.41). This allows us to give an example of a C-minimal group which is neither order-type nor valuational (Example 4.4.43).

The groups considered in this chapter are generally non-abelian, which is why we adopt the multiplicative notation. We recall that, whenever a q.o. group $(G, \precsim)$ is fixed, $\mathcal{V}$ denotes the set of v-type elements, $\mathcal{O}$ denotes the set of o-type elements, $\mathcal{O}^-$ the set of o^--type elements and $\mathcal{O}^+$ the set of o^+-type elements.

4.1 C-quasi-orders

4.1.1 Definition and axiomatization

As mentioned in the introduction, we want to associate a quasi-order to every compatible C-relation. This idea originates from the following general fact:

Lemma 4.1.1
Let A be a set (not necessarily a group), C a C-relation on A and take $z \in A$. Then z induces a quasi-order on A by $a \precsim b \Leftrightarrow \neg C(a, b, z)$.

Proof. Note that $\neg C(z, z, z)$ follows from (C_2), so we have $z \precsim z$. Let $a \in A$ with $a \neq z$. By (C_4), we have $C(z, a, a)$. By (C_2), this implies $\neg C(a, z, a)$, which by (C_1) implies $\neg C(a, a, z)$. This proves that $\precsim$ is reflexive. Transitivity is the contra-position of axiom (C_3). Totality is given by axiom (C_2).

□

In the context of groups, the natural candidate for the parameter z is $z = 1$, hence the following definition:

Definition 4.1.2
Let G be a group. For any compatible C-relation C on G, we define the **q.o. induced by** C as the q.o. given by the formula $x \precsim y \Leftrightarrow \neg C(x, y, 1)$. A **C-quasi-order** (C-q.o.) on G is the q.o. induced by a compatible C-relation on G. A **C-quasi-ordered group** (C-q.o.g.) is a pair $(G, \precsim)$ consisting of a group G endowed with a C-q.o. $\precsim$.

Remark 4.1.3: If $\precsim$ is the q.o. induced by C, then we have $C(x, y, 1) \Leftrightarrow y \precnsim x$.

If $\precsim$ is a C-q.o. induced by the C-relation C, then we say that $\precsim$ is **order-type** (respectively **valuational/ fundamental**) if C is order-type (respectively valuational/ fundamental). These definitions make sense thanks to the following proposition:

Proposition 4.1.4
Let $\precsim$ be a C-q.o. Then there is only one compatible C-relation inducing it, namely the one given by the formula $C(x, y, z) \Leftrightarrow yz^{-1} \precnsim xz^{-1}$.

Proof. Let C be a compatible C-relation inducing $\precsim$. C is compatible, so we have $C(x, y, z) \Leftrightarrow C(xz^{-1}, yz^{-1}, 1) \Leftrightarrow yz^{-1} \precnsim xz^{-1}$. □

Remark 4.1.5: Note that the definition of valuational C-q.o.'s which we just gave coincides with the definition of valuational q.o.'s given in Section 2.7. Indeed, if v is a valuation on G, C the C-relation induced by v and $\precsim$ the C-q.o. corresponding to C, then it easily follows from the definition of C and $\precsim$ that $x \precsim y \Leftrightarrow v(x) \geq v(y)$ for all $x, y \in G$. In other words, a C-q.o. is valuational if and only if it is the q.o. induced by a valuation.

We now want to axiomatize the class of C-q.o.'s. Proposition 4.1.4 states that $\precsim$ is a C-q.o. if and only if the formula $yz^{-1} \precnsim xz^{-1}$ defines a compatible C-relation. We thus want to answer the question: When does this formula define a compatible C-relation?

Lemma 4.1.6
Let $\precsim$ be a quasi-order on a group G and define a ternary relation $C(x, y, z)$ by the formula $yz^{-1} \precnsim xz^{-1}$. Then the relation C satisfies (C_2) and (C_3).

Proof. C clearly satisfies (C_2). Assume $C(x, y, z)$ and $\neg C(w, y, z)$ hold. This means $yz^{-1} \precnsim xz^{-1}$ and $\neg(yz^{-1} \precnsim wz^{-1})$. Since $\precsim$ is total, this implies $wz^{-1} \precsim yz^{-1} \precnsim xz^{-1}$, hence $wz^{-1} \precnsim xz^{-1}$ i.e $C(x, w, z)$. This proves (C_3). □

This gives us an axiomatization of C-q.o.'s:

Proposition 4.1.7 (Axiomatization of C-q.o's)
Let G be a group and $\precsim$ a q.o. on G. Then $\precsim$ is a C-q.o. if and only if the following three axioms are satisfied:

(CQ_1) $\forall x \in G \backslash \{1\}$, $1 \precnsim x$.

(CQ_2) $\forall x, y (x \precsim y \Leftrightarrow xy^{-1} \precsim y^{-1})$.

(CQ_3) $\forall x, y, z \in G$, $x \precsim y \Leftrightarrow x^z \precsim y^z$.

Note that "$\Leftrightarrow$" can be replaced by "$\Rightarrow$" in (CQ_2) and (CQ_3) since $(xy^{-1})(y^{-1})^{-1} = x$, $(y^{-1})^{-1} = y$ and $(x^z)^{z^{-1}} = x$.

Proof. Define $C(x, y, z) := yz^{-1} \precnsim xz^{-1}$. By Proposition 4.1.4, $\precsim$ is a C-q.o. if and only if C is a compatible C-relation. Assume C is a compatible C-relation. By (C_4), we have $C(x, 1, 1)$ for any $x \neq 1$, which means $1 \precnsim x$. Take $x, y, z \in G$ with $x \precsim y$, which means $\neg C(x, y, 1)$. By (C_1), we then have $\neg C(x, 1, y)$. By compatibility, this implies $\neg C(xy^{-1}, y^{-1}, 1)$ i.e. $xy^{-1} \precsim y^{-1}$, hence (CQ_2). By compatibility we also have $\neg C(x^z, y^z, 1)$, hence (CQ_3). Conversely, assume $(CQ_1), (CQ_2), (CQ_3)$ hold. By Lemma 4.1.6, we already know that C satisfies (C_2) and (C_3). We first prove that C is compatible. Take $x, y, z, u, v \in G$ with $C(x, y, z)$. We thus have $yz^{-1} \precnsim xz^{-1}$. By (CQ_3), this implies $uyz^{-1}u^{-1} \precnsim uxz^{-1}u^{-1}$ i.e. $(uyv)(uzv)^{-1} \precnsim (uxv)(uzv)^{-1}$, so $C(uxv, uyv, uzv)$. This proves compatibility. Let $x \neq y$ in G. (CQ_1) implies $1 \precnsim xy^{-1}$ which means $C(x, y, y)$, so C satisfies (C_4). Now assume $\neg C(x, y, z)$, i.e. $xz^{-1} \precsim yz^{-1}$. By applying (CQ_2) to this inequality, we get $xy^{-1} \precsim zy^{-1}$, hence $\neg C(x, z, y)$, which proves that C satisfies (C_1). □

Remark 4.1.8: By combining (CQ_3) and (CQ_2) we obtain an improved version of (CQ_2): $x \precsim y \Rightarrow xy^{-1} \precsim y^{-1} \wedge y^{-1}x \precsim y^{-1}$. We will also often use the contra-position of (CQ_2):

(CQ_2') $\quad y \precnsim x \Rightarrow y^{-1} \precnsim xy^{-1}$.

4.1.2 Connection with compatible q.o's

We now want to establish the connection between the notion of C-q.o. and the notion of compatible q.o. developed in chapter 3. Proposition 3.4.1 shows that we can associate a compatible C-relation to any compatible quasi-order defined on an abelian group. However, this does not mean that compatible q.o.'s are C-q.o.'s. In fact, we have the following:

Proposition 4.1.9
Let $(G, \precsim)$ be a compatible quasi-ordered abelian group. Then $\precsim$ is a C-q.o. if and only if $\precsim$ is valuational.

Proof. We know that valuational q.o.'s are C-q.o.'s. Now assume that $\precsim$ is a compatible q.o. and a C-q.o. By Proposition 3.1.14, $(\mathcal{O}, \precsim)$ is an ordered abelian group. If $\{1\} \subsetneq \mathcal{O}$ were true, then there would be $g \in G$ with $g \precnsim 1$. This would contradicts axiom (CQ_1). Therefore, we have $\mathcal{O} = \{1\}$, hence $\mathcal{V} = G$. By Proposition 3.1.17, it follows that $\precsim$ is valuational. □

Now let $(G, \precsim)$ be a compatible quasi-ordered abelian group. Proposition 4.1.9 states that, if the subgroup $\mathcal{O}$ of o-type elements is non-trivial, then $\precsim$ is not a C-q.o. However, we can transform $\precsim$ into a corresponding C-q.o $\precsim^*$. We know that $\precsim$ coincides with an order $\leq$ on $\mathcal{O}$ and behaves like a valuation on $\mathcal{V}^*$. Now define $\precsim^*$ as follows: on $\mathcal{O}$, $\precsim^*$ is the order-type C-q.o corresponding to $\leq$. On $\mathcal{V}^*$, $\precsim^*$ coincides with $\precsim$. Finally, declare $\mathcal{O} \precnsim^* \mathcal{V}^*$. Then $\precsim^*$ is a C-q.o. Now denote by C^* the C-relation corresponding to the C-q.o $\precsim^*$ and denote by C the C-relation induced by the compatible q.o $\precsim$ as defined in Proposition 3.4.1. By distinguishing the cases $xz^{-1} \in \mathcal{V}^*$ and $xz^{-1} \notin \mathcal{V}^*$, one can show that $C(x, y, z)$ holds if and only if $yz^{-1} \precnsim^* xz^{-1}$. It then follows that $C = C^*$.

4.1.3 Some relations between the q.o. and the group operation

Here we investigate the relation between multiplication and $\precsim$. More precisely, we want to understand how the equivalence class of the product of two elements relates to the equivalence class of each factor. These results will play a fundamental role in the proofs of Section 4.3.3. We fix a C-q.o.g. $(G, \precsim)$. We first note that in many cases the order of the factors will not matter:

Lemma 4.1.10
For any $g, h \in G$, $hg \sim g \Leftrightarrow gh \sim g$.

Proof. It is a direct consequence of (CQ_3): take the inequalities $hg \precsim g \precsim hg$ and conjugate by g. □

Lemma 4.1.11
Let $g, h \in G$. The following holds:

(i) If $h \precsim g^{-1}$, then $g \sim hg \sim gh$.

(ii) Assume that $h \precnsim \{g^{-1}, g\}$. Then $h^{-1} \precnsim \{g, g^{-1}\}$ and we have $gh \sim g \sim gh^{-1}$ and $g^{-1} \sim hg^{-1} \sim h^{-1}g^{-1}$.

(iii) If $\{h, h^{-1}\} \precsim g^{-1} \precsim g$, then $g \sim gh \sim gh^{-1} \sim hg \sim h^{-1}g$ and
$g^{-1} \sim g^{-1}h^{-1} \sim g^{-1}h \sim h^{-1}g^{-1} \sim hg^{-1}$.

Proof. (i) By (CQ_2), $h \precsim g^{-1} \Rightarrow hg \precsim g$. By (CQ_2'), $h \precnsim g^{-1} \Rightarrow h^{-1} \precnsim g^{-1}h^{-1}$. By (CQ_2), $h^{-1} \precsim g^{-1}h^{-1} \Rightarrow g \precsim hg$, hence $g \sim hg$.

(ii) By (i), $g \sim gh$ and $g^{-1} \sim g^{-1}h$. By (CQ_2'), $h \precnsim gh \Rightarrow h^{-1} \precnsim g$ and $h \precnsim g^{-1}h \Rightarrow h^{-1} \precnsim g^{-1}$. In particular, h^{-1} satisfies $h^{-1} \precnsim \{g, g^{-1}\}$, so we get $g \sim gh^{-1}$ and $g^{-1} \sim g^{-1}h^{-1}$, hence the claim.

(iii) By (i), $\{h, h^{-1}\} \precnsim g$ implies $g^{-1} \sim g^{-1}h^{-1} \sim g^{-1}h$. By (CQ_2), $h \precsim g^{-1} \Rightarrow hg \precsim g$ and $h^{-1} \precsim g^{-1}h^{-1} \Rightarrow g \precsim hg$, hence $g \sim hg$. Analogously, $g \sim gh^{-1}$. The rest follows from Lemma 4.1.10.

□

We can summarize these results in the following proposition:

Proposition 4.1.12
Assume g is v-type. If $h \precnsim g$, then $h^{-1} \precnsim g$ and we have
$hg \sim h^{-1}g \sim gh^{-1} \sim gh \sim g \sim g^{-1} \sim g^{-1}h \sim g^{-1}h^{-1} \sim h^{-1}g^{-1} \sim hg^{-1}$.
Assume g is o^+-type. If $\{h, h^{-1}\} \precsim g^{-1}$, then we have
$hg^{-1} \sim h^{-1}g^{-1} \sim g^{-1}h^{-1} \sim g^{-1}h \sim g^{-1} \precnsim g \sim gh \sim gh^{-1} \sim hg \sim h^{-1}g$.

We now want to find an analog of axiom (Q_2) of compatible q.o's (see Definition 3.1.1).

Lemma 4.1.13
If $f \precsim g$ and $g^{-1} \precsim h^{-1}g^{-1}$, then $fh \precsim gh$ and $hf \precsim hg$.

Proof. By (CQ_2), $f \precsim g$ implies $fg^{-1} \precsim g^{-1}$. By assumption, this implies $fg^{-1} \precsim h^{-1}g^{-1}$. By (CQ_2) again, this implies $fh \precsim gh$. (CQ_3) then implies $hf \precsim hg$. □

Proposition 4.1.14
Let $f, g \in G$ such that $f \precsim g$ and assume that either $g \not\sim h^{-1}$ or $\{h, h^{-1}\} \precsim g \precnsim g^{-1}$ holds. Then we have $fh \precsim gh$ and $hf \precsim hg$.

Proof. If $h^{-1} \precnsim g$, then by 4.1.11 we have $g^{-1} \sim h^{-1}g^{-1}$. If $g \precnsim h^{-1}$, then (CQ_2') implies $g^{-1} \precnsim h^{-1}g^{-1}$. In both cases, we have $g^{-1} \precsim h^{-1}g^{-1}$, so we can apply the previous lemma. For the second claim, we use 4.1.12 to get $g^{-1} \sim h^{-1}g^{-1}$. □

Remark 4.1.15: We just showed that C-q.o.g.'s satisfy the formula: $\forall g, h, f \in G$, $f \precsim g \nsim h^{-1} \Rightarrow fh \precsim gh$. This formula is very similar to axiom (Q_2) of compatible q.o.'s and seems to be more practical to deal with than axiom (CQ_2) of C-q.o.'s. However, we don't know if we can actually replace (CQ_2) by this formula in our axiomatization of C-q.o.'s.

4.1.4 Fundamental C-q.o.'s

Before investigating the structure of an arbitrary C-q.o.g., we want to understand the structure of fundamental C-q.o.'s. We know that valuational C-q.o.'s are the q.o.'s induced by a valuation. Moreover, we can easily characterize valuational C-q.o.'s by looking at the type of elements:

Proposition 4.1.16
Let $(G, \precsim)$ be a C-q.o. group. Then $\precsim$ is valuational if and only if $\mathcal{V} = G$.

Proof. If $\precsim$ is valuational, then every element must obviously be v-type. Conversely, assume that every element is v-type. We use Remark 2.7.3. (i) and (iv) of Remark 2.7.3 follow from (CQ_1) and (CQ_3), and (iii) follows from the fact that every element is v-type, so we just have to show (ii). Now let $g, h \in G$. If $g \precsim h$, then $g \precsim h^{-1}$ (because $h \sim h^{-1}$). Then (CQ_2) implies $gh \precsim h$. ◻

The order-type case is a bit more complicated. Note first that if we start with an ordered group $(G, \leq)$, if C is the C-relation induced by $\leq$ and if $\precsim$ is the corresponding C-q.o., then there is no reason for $\leq$ and $\precsim$ to be the same. In fact, an order-type C-q.o. can never be an order. Let us have a closer look at order-type C-q.o.'s. Let $(G, \leq)$ be an ordered group with positive cone P, C the C-relation induced by $\leq$ and $\precsim$ the corresponding C-q.o. By looking at the definition of C in Example 2.6.1 and at Definition 4.1.2, we get the following formulas:

$$x \precsim y \Leftrightarrow (y \neq 1 \vee x = y) \wedge (yx^{-1} \in P \vee x^{-1} \in P) \tag{O_1}$$

$$y \precnsim x \Leftrightarrow (y = 1 \wedge x \neq y) \vee (yx^{-1} \notin P \wedge x^{-1} \notin P) \tag{O_2}$$

This allows us to describe order-type C-q.o's:

Proposition 4.1.17
Let $(G, \leq)$ be an ordered group with positive cone P and $\precsim$ the corresponding C-q.o. The following holds:

(1) $\mathcal{O} = G$, $\mathcal{O}^+ = P \backslash \{1\}$ and $\mathcal{O}^- = P^{-1} \backslash \{1\}$.

(2) $1 \precnsim \mathcal{O}^- \precnsim \mathcal{O}^+$.

(3) $\precsim$ is trivial on $\mathcal{O}^-$.

(4) $\precsim$ coincides with $\leq$ on $\mathcal{O}^+$. In particular, $\precsim$ is an order on $\mathcal{O}^+$.

Proof. Formula (O_2) immediately implies $P\backslash\{1\} = \mathcal{O}^+$. Since $\mathcal{O}^- = (\mathcal{O}^+)^{-1}$, it follows that $\mathcal{O}^- = P^{-1}\backslash\{1\}$. Finally, $P \cup P^{-1} = G$ then implies $\mathcal{O} = G$. This proves (1). Now let us prove (2). $1 \precnsim \mathcal{O}^-$ follows directly from (CQ_1). Take $g \in \mathcal{O}^-$ and $h \in \mathcal{O}^+$. By (1), we then have $h, g^{-1} \in P$, hence $hg^{-1} \in P \wedge h \in P$, hence $gh^{-1} \notin P \wedge h^{-1} \notin P$. By formula (O_2), this implies $g \precnsim h$. This proves (2). Now let us prove (3). Take g, h both in $\mathcal{O}^-$. By (1), we have $g^{-1} \in P$. It then follows immediately from formula (O_1) that $g \precsim h$. Similarly, we have $h \precsim g$, hence $g \sim h$. Now let us show (4). Take $g, h \in \mathcal{O}^+$. Since $g^{-1} \notin P$, it follows immediately from formula (O_1) that $g \precsim h$ holds if and only if $hg^{-1} \in P$. This in turn is equivalent to $g \leq h$. □

In example 4.3.1(c), we have a C-q.o group $(G, \precsim)$ satisfying conditions (1) of Proposition 4.1.17 but where conditions (2),(3) and (4) fail. Therefore, the condition $\mathcal{O} = G$ alone is not sufficient for $\precsim$ to be order-type. However, if $\mathcal{O} = G$, then we can associate an order to $\precsim$. For any C-q.o. $\precsim$ on G, let us denote by $P(\precsim)$ the set $\{1\} \cup \mathcal{O}^+$. It turns out that, if $G = \mathcal{O}$, then $P(\precsim)$ is a positive cone of G. To show this, we first need to show the following lemma, which will also be useful later in Section 4.3.3:

Lemma 4.1.18
Let $(G, \precsim)$ be a C-q.o. group and $g \in \mathcal{O}^+$. Let $h \in G$ with $g^{-1} \precnsim h \precsim g$. Then $h^{-1} \sim g^{-1}$, and in particular $h \in \mathcal{O}^+$.

Proof. By (CQ_2), $h \precsim g$ implies $hg^{-1} \precsim g^{-1}$, hence $hg^{-1} \precnsim h$. By (CQ_2) and (CQ_3), this implies $g^{-1} \precsim h^{-1}$. Now assume that $g^{-1} \precnsim h^{-1}$ holds. By Lemma 4.1.11, we then have $h \sim hg^{-1} \precsim g^{-1}$, which is a contradiction. Therefore, $g^{-1} \sim h^{-1}$. By assumption on h, it follows that $h^{-1} \precnsim h$, hence $h \in \mathcal{O}^+$. □

We then have the following:

Proposition 4.1.19
Let $(G, \precsim)$ be a C-q.o. group. The following holds:

(1) The set $P(\precsim)$ is stable under "$\sim$".

(2) If $G = \mathcal{O}$, then $P(\precsim)$ is a positive cone on G.

Proof. Set $P := P(\precsim)$. Lemma 4.1.18 implies that $\mathcal{O}^+$ is stable under "$\sim$". It then follows from (CQ_1) that P is also stable under "$\sim$". Assume $\mathcal{O} = G$ and let us prove that P is a positive cone on G. We clearly have $P \cap P^{-1} = \{1\}$ and $P \cup P^{-1} = G$. The fact that $zPz^{-1} = P$ for every $z \in G$ is a direct consequence of (CQ_3). We just have to show $P.P \subseteq P$. We actually show that P^{-1} is closed under the group operation. Let $g, h \in P^{-1} = \mathcal{O}^- \cup \{1\}$. Without loss of generality, we may assume $g \precsim h$. Since $h \in \mathcal{O}^-$, this implies $g \precnsim h^{-1}$. This implies $gh \sim h$ by Lemma 4.1.11, and since P is stable under "$\sim$" it follows that $gh \in P^{-1}$. □

Remark 4.1.20: Note that the only place in the proof where we used the assumption $\mathcal{O} = G$ is to prove $G = P \cup P^{-1}$. In particular, if G contains some non-trivial v-type elements, then $P(\precsim)$ is the positive cone of a partial order on G.

Proposition 4.1.19 gives us a map $\mathcal{P} : \{\text{C-q.o.'s with } \mathcal{O} = G\} \to \{\text{group orders on } G\}$ defined by $\precsim \mapsto P(\precsim)$. This map is surjective: Indeed, if P is any positive cone on G, take C to be the C-relation induced by P and $\precsim$ the corresponding C-q.o. Then it follows from Proposition 4.1.17 that $P(\precsim) = P$. Note however that $\mathcal{P}$ is not injective, as illustrated by example 4.3.1(c). Indeed, let $(G, \precsim)$ be as in Example 4.3.1(c). Then we have $\mathcal{O} = G$, but $\precsim$ is not order-type because it does not satisfy condition (2) of Proposition 4.1.17. Let $P := \mathcal{P}(\precsim)$, take C the C-relation induced by P and $\precsim^*$ the C-q.o. induced by C. Then $\mathcal{P}(\precsim^*) = \mathcal{P}(\precsim)$, but $\precsim^*$ and $\precsim$ don't coincide because $\precsim^*$ is order-type.

We can now use $\mathcal{P}$ to give a full characterization of order-type C-q.o.'s:

Proposition 4.1.21
Let $(G, \precsim)$ be a C-q.o. group. Then $\precsim$ is order-type if and only if the two following conditions hold:

(1) $G = \mathcal{O}$.

(2) $\mathcal{O}^- \precnsim \mathcal{O}^+$.

Moreover, if $\precsim$ is order-type, then $\precsim$ is trivial on $\mathcal{O}^-$ and $\precsim$ is an order on $\mathcal{O}^+$.

Proof. One direction is given by Proposition 4.1.17. Let us prove the converse. Assume that (1) and (2) are satisfied, set $P := \mathcal{P}(\precsim)$. We know that P is a positive cone on G thanks to Proposition 4.1.19. We will show that $\precsim$ is the C-q.o. induced by P. We just have to show formula (O_1) is satisfied. Assume that $x \precsim y$ holds. By (CQ_1), we must have $x = y \vee y \neq 1$. Assume that $x^{-1} \notin P$. We then have $x^{-1} \in \mathcal{O}^-$, so $x \in \mathcal{O}^+$. By assumption (2) and by $x \precsim y$, we then have $y \in \mathcal{O}^+$, so $y^{-1} \in \mathcal{O}^-$. By (CQ_2), $x \precsim y$ implies $xy^{-1} \precsim y^{-1}$. It then follows from (2) that $xy^{-1} \notin \mathcal{O}^+$, hence $xy^{-1} \notin P \backslash \{1\}$. But since P is a positive cone, this implies $yx^{-1} \in P$. This proves that $(x = y \vee y \neq 1) \wedge (yx^{-1} \in P \vee x^{-1} \in P)$ holds. Now assume that $\neg(x \precsim y)$ holds, i.e $y \precnsim x$. (CQ_1) immediately implies $x \neq 1$. Assume that $x = y \vee y \neq 1$ holds. Obviously, we cannot have $x = y$, so we must have $y \neq 1$. If x were in $\mathcal{O}^-$, then it would follows from (2) and from $y \precnsim x$ that $y = 1$, which we excluded. Therefore, $x \notin \mathcal{O}^-$. It follows that $x \in P$, which implies $x^{-1} \notin P$ because P is a positive cone. By (CQ_2'), $y \precnsim x$ implies $y^{-1} \precnsim xy^{-1}$. If xy^{-1} were in $\mathcal{O}^-$, then by (2) we would have $y^{-1} = 1$, which is impossible. Therefore, we must have $xy^{-1} \notin \mathcal{O}^-$. By (1), this implies $yx^{-1} \in \mathcal{O}^-$, hence $yx^{-1} \notin P$. This shows that $\neg((x = y \vee y \neq 1) \wedge (yx^{-1} \in P \vee x^{-1} \in P))$ holds. This proves that formula (O_1) is satisfied, so $\precsim$ is the C-q.o. induced by P. It then follows from Proposition 4.1.17 that $\precsim$ is trivial on $\mathcal{O}^-$ and that $\precsim$ is an order on $\mathcal{O}^+$. $\square$

All of this shows us how to construct $\precsim$ from $\leq$ and vice-versa. More precisely, we see that $\leq$ and $\precsim$ define the same sets:

Proposition 4.1.22
Let $(G, \leq)$ be an ordered group and $\precsim$ the corresponding C-q.o. The relation $\precsim$ is quantifier-free definable in the language $\{1, ., ^{-1}, \leq\}$ and $\leq$ is quantifier-free definable in $\{1, ., ^{-1}, \precsim\}$.

Proof. Formula (O_1) gives a definition of $\precsim$ using $\leq$. Conversely, it follows from Proposition 4.1.17 that $P = \{1\} \cup \mathcal{O}^+$, and we know that $\mathcal{O}^+$ is definable with $\precsim$, so P is definable in $\{1, ., ^{-1}, \precsim\}$. □

Remark 4.1.23: We just saw what fundamental C-q.o. groups look like. In Section 4.3, our work will consist in showing that any C-q.o. group is in some sense a "mix" of the fundamental ones. This means that we will identify parts of the group where the q.o. is "order-type-like" and parts where it is "valuational-like". Intuitively, we want to say that a q.o is "like" a fundamental C-q.o. on a subset T of G if it shares the important properties of this fundamental C-q.o. We will say that the q.o. $\precsim$ is **valuational-like on** T if $gh \precsim \max(\{g, h\})$ for any $g, h \in T$. We will say that $\precsim$ is **order-type-like on** T if T can be partitioned into two subsets, T^- and T^+, such that the following holds: $T^- = \{g^{-1} \mid g \in T^+\}$, $T^- \precnsim T^+$ and $\precsim$ is trivial on T^- (i.e $g \sim h$ for all $g, h \in T^-$). This definition is motivated by Proposition 4.1.17. We say that $\precsim$ is **fundamental-like on** T if it is either valuational-like or order-type-like on T.

4.2 A Baer-Krull theorem for C-q.o. groups

We want to give an analog of theorems 2.7.9 and 2.3.1 for quasi-ordered groups. We saw that compatible q.o.'s are not suited for a Baer-Krull theorem, but we will see that C-q.o.'s are.

4.2.1 characterization of compatibility

We first want to characterize compatibility between valuations and C-q.o.'s in analog of Theorem 2.7.9. This relates to the notion of induced q.o. on a quotient and to the notion of convexity of subgroups. Note first that axiom (CQ_1) immediately implies the following:

Proposition 4.2.1
Let $(G, \precsim)$ be a C-q.o.g. and H a subgroup of G. Then H is convex in $(G, \precsim)$ if and only if it is an initial segment of $(G, \precsim)$.

As happens with compatible q.o.'s, we have the following:

Proposition 4.2.2
Let $(G, \precsim)$ be a C-q.o.g. and H a normal subgroup of G. Then $\precsim$ induces a q.o. on G/H if and only if H is convex in G. If H is convex in G, then the q.o. induced by $\precsim$ on G/H is a C-q.o. and it is given by the formula:

$$gH \precsim hH \Leftrightarrow (g \in H) \vee (h \notin H \wedge g \precsim h). \tag{†}$$

Moreover, for any $g \in G\backslash H$, then g is v-type (respectively, o$^+$-type/o$^-$-type) if and only if gH is v-type (respectively, o$^+$-type/o$^-$-type).

Proof. By (CQ_1) and by Lemma 4.1.14, $\precsim$ satisfies condition $(*)$ of Lemma 2.7.20. If $\precsim$ induces a C-q.o. on G/H, then Lemma 2.7.20 immediately implies that H is convex in G. Conversely, assume that H is convex in G. Then Lemma 2.7.20 implies that $\precsim$ induces a q.o. on G/H with $\mathrm{cl}(1) = \{1\}$. Proposition 4.2.1 implies that H is an initial segment of G, which immediately implies that $(G/H, \precsim)$ satisfies (CQ_1). We will now show that the q.o. $\precsim$ on G/H is given by the formula $gH \precsim fH \Leftrightarrow (g \in H) \vee (f \notin H \wedge g \precsim f)$, and we will then show that $\precsim$ on G/H satisfies (CQ_2) and (CQ_3). Assume $gH \precsim fH$ and $g \notin H$. There is $h_1, h_2 \in H$ with $gh_1 \precsim fh_2$. Since $g \notin H$, we have $gh_1 \notin H$. Since H is an initial segment of G, it follows that $fh_2 \notin H$, hence also $f \notin H$. By convexity of H, we have $h_1 \precnsim g^{-1}$ and $h_2 \precnsim f^{-1}$. It follows from Lemma 4.1.11 that $gh_1 \sim g$ and $fh_2 \sim f$. Since $gh_1 \precsim fh_2$, it follows that $g \precsim f$. This shows that $gH \precsim fH \Rightarrow (g \in H) \vee (f \notin H \wedge g \precsim f)$ holds. Conversely, assume that $(g \in H) \vee (f \notin H \wedge g \precsim f)$ holds. We know that $(G/H, \precsim)$ satisfies (CQ_1), so $g \in H$ implies $gH \precsim hH$. Assume that $(f \notin H \wedge g \precsim f)$ holds. Then $g \precsim f$ immediately implies $gH \precsim fH$. This proves that $gH \precsim fH \Leftrightarrow (g \in H) \vee (f \notin H \wedge g \precsim f)$. Now let us prove $(CQ_2) \wedge (CQ_3)$. Assume $gH \precsim fH$, so $(g \in H) \vee (f \notin H \wedge g \precsim f)$. If $g \in H$, then $gf^{-1}H = f^{-1}H$, hence $gf^{-1}H \precsim f^{-1}H$. Moreover, since H is normal, $g^z \in H$, and by (CQ_1) on G/H this implies $g^zH \precsim f^zH$. Assume $f \notin H \wedge g \precsim f$. Then $f^{-1}, f^z \notin H$, and since (CQ_2) and (CQ_3) are satisfied on G we also have $gf^{-1} \precsim f^{-1}$ and $g^z \precsim f^z$. It follows that $gf^{-1}H \precsim f^{-1}H$ and $g^zH \precsim f^zH$. □

Remark 4.2.3: Assume that H is convex. Then it follows from formula (†) in Proposition 4.2.2 that, if $g \notin H$ or $h \notin H$, then $gH \precsim hH \Leftrightarrow g \precsim h$.

Proof. Assume $g \notin H$ or $h \notin H$. We know that $g \precsim h \Rightarrow gH \precsim hH$. Now assume that $gH \precsim hH$. If $h \in H$, then by (CQ_1) on G/H we also have $g \in H$, which contradicts the assumption. Therefore, we have $h \notin H$. If $g \in H$ then by convexity of H we must have $g \precsim h$. If $g \notin H$, then by formula (†), we must have $g \precsim h$. □

In Section 4.1.4, we saw that an order is not a C-q.o., but that there is a natural connection between orders and o-type C-q.o.'s, given by the map $\mathcal{P}$. The fact that $\mathcal{P}$ is surjective allows us to see orders as special case of C-q.o.'s. Through $\mathcal{P}$ we will be able to transform certain statements concerning C-q.o.'s into statements about orders. In other words, we can use C-q.o.'s as a uniform approach to ordered and valued groups.

Theorem 4.2.4
Let $\precsim$ be a C-q.o. on G and $v : G \to \Gamma \cup \{\infty\}$ a valuation. The following statements are equivalent:

(1) $\forall \gamma \in \Gamma, G^\gamma$ is convex in $(G, \precsim)$.

(2) $\forall \gamma \in \Gamma, G_\gamma$ is convex in $(G, \precsim)$.

(3) $\forall \gamma \in \Gamma$, $\precsim$ induces a C-q.o. $\precsim_\gamma$ on $B_\gamma := G^\gamma/G_\gamma$, which is given by the formula $gG_\gamma \precsim hG_\gamma \Leftrightarrow (g \in G_\gamma) \vee (h \notin G_\gamma \wedge g \precsim h)$.

(4) v is compatible with $\precsim$.

Moreover, $\mathcal{V} = G$ (respectively $\mathcal{O} = G$) if and only if for all $\gamma \in \Gamma$, every element of B_γ is v-type (respectively o-type).

Proof. By Lemma 2.7.18, we know that (1)⇔(2). By Lemma 2.7.19, (1)⇔(4). By Lemma 2.7.18, (2) holds if and only if G_γ is convex in G^γ. By Proposition 4.2.2, this holds if and only if (3) holds. □

4.2.2 Baer-Krull theorems for groups

We saw that an obstacle for the existence of a Baer-Krull theorem for compatible q.o.'s is the fact that compatible q.o.'s are not stable under lifting (see Section 3.2). The class C-q.o.'s does not suffer from the same problem.

Definition 4.2.5
Let (G, v) be a valued group with value chain Γ. Assume that for each $\gamma \in \Gamma$, $B_\gamma := G^\gamma/G_\gamma$ is endowed with a C-q.o. $\precsim_\gamma$. We say that the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ has the **q.o-conjugation property** if for each $\gamma \in \Gamma$ and each $z \in G$, the canonical homomorphism $G^\gamma/G_\gamma \to G^{\gamma^z}/G_{\gamma^z}$ induced by conjugation by z (see Remark 2.2.5(d)) is quasi-order-preserving.

Note that in the abelian setting, the q.o.-conjugation property is always satisfied. However, we can easily give a non-abelian example where it fails:

Example 4.2.6
Let $\precsim_o$ be the C-q.o. induced by the usual order of the additive group $\mathbb{Q}$ and let $\precsim_v$ be any valuational q.o. on the multiplicative group $\mathbb{Q}^*$. Define $\alpha : \mathbb{Q}^* \to \mathrm{Aut}((\mathbb{Q}, +))$ by $\alpha(a)(b) = ab$. Set $G := \mathbb{Q}^* \ltimes_\alpha \mathbb{Q}$. Define $v : G \to \{1, 2, \infty\}$ as follows: $v(a, b) = \begin{cases} 1 \text{ if } a \neq 1. \\ 2 \text{ if } a = 1 \wedge b \neq 0. \\ \infty \text{ if } (a, b) = (1, 0). \end{cases}$

We have $G^2/G_2 \cong \mathbb{Q}$ and $G^1/G_1 \cong \mathbb{Q}^*$, but the family $((\mathbb{Q}, \precsim_o), (\mathbb{Q}^*, \precsim_v))$ does not have the q.o.-conjugation property. To see this, take $z := (-1, 0)$. By conjugation, z induces the automorphism $a \mapsto -a$ on $\mathbb{Q}$, which does not preserve $\precsim_o$.

The q.o.-conjugation property ensures the existence of a lifting (see Definition 2.7.12):

Proposition 4.2.7
Let (G, v) be a valued group with value chain Γ. Assume that for each $\gamma \in \Gamma$, $B_\gamma := G^\gamma/G_\gamma$ is endowed with a C-q.o. $\precsim_\gamma$. If the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ has the q.o.-conjugation property, then the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ admits a unique C-q.o. lifting to G, which is the C-q.o. $\precsim$ defined by the formula

$g \precsim h \Leftrightarrow (gG_\gamma \precsim_\gamma hG_\gamma,\ \text{where } \gamma = \min(v(g), v(h))).$

Proof. Let $\precsim$ denote the binary relation on G defined by the formula $g \precsim h \Leftrightarrow (gG_\gamma \precsim_\gamma hG_\gamma,$ where $\gamma = \min(v(g), v(h)))$. Note that this formula makes sense because, if $\gamma = \min(v(g), v(h))$, then $g, h \in G^\gamma$, so $gG_\gamma, hG_\gamma \in B_\gamma$.

Claim 1: For any $g, h \in G$, we have $g \precsim h \Rightarrow v(h) \leq v(g)$.

Proof. Set $\gamma := \min(v(g), v(h))$ and assume $g \precsim h$. Assume for a contradiction that $v(h) > \gamma$. Then we have $v(g) = \gamma$ and $hG_\gamma = 1$. But since $g \precsim h$, the definition of $\precsim$ implies $gG_\gamma \precsim_\gamma hG_\gamma = 1$. Since $\precsim_\gamma$ satisfies (CQ_1), this implies $gG_\gamma = 1$ i.e $g \in G_\gamma$, hence $\gamma < v(g)$, which is a contradiction. Therefore, $v(h) = \gamma$, hence $v(h) \leq v(g)$. □

We now show that $\precsim$ is a q.o. $\precsim$ is reflexive and total because $\precsim_\gamma$ is reflexive and total for all $\gamma \in \Gamma$. Let us show transitivity. Assume that $f \precsim g$ and $g \precsim h$ hold. Set $\gamma := \min(v(f), v(g))$ and $\delta := \min(v(g), v(h))$. By Claim 1, we have $\delta = v(h) \leq v(g) = \gamma \leq v(f)$. In particular, we have $\delta = \min(v(f), v(h))$. Assume first that $\delta < v(f)$. We then have $fG_\delta = 1$, which by (CQ_1) implies $fG_\delta \precsim_\delta hG_\delta$. Assume now that $v(f) = \delta$. It then follows that $\delta = \gamma$. By definition of $\precsim$, the relations $f \precsim g$ and $g \precsim h$ imply $fG_\delta \precsim_\delta gG_\delta$ and $gG_\delta \precsim_\delta hG_\delta$. By transitivity of $\precsim_\delta$, this implies $fG_\delta \precsim_\delta hG_\delta$. In any case, we have $fG_\delta \precsim_\delta hG_\delta$, which proves $f \precsim h$. This proves that $\precsim$ is a q.o. Let us now prove that it is a C-q.o. Let $g \neq 1$. We then have $\gamma := v(g) < v(1)$. Moreover, since $\precsim_\gamma$ satisfies (CQ_1), we have $1G_\gamma \precnsim gG_\gamma$ (because $g \notin G_\gamma$). By definition of $\precsim$, this means $1 \precnsim g$, which proves that $\precsim$ satisfies (CQ_1). Now take $g, h, z \in G$ with $g \precsim h$. By Claim 1, we have $v(h) \leq v(g)$. Set $\gamma := v(h)$. We have $gG_\gamma \precsim_\gamma hG_\gamma$. Since $\precsim_\gamma$ satisfies (CQ_2), this implies $gh^{-1}G_\gamma \precsim_\gamma h^{-1}G_\gamma$. By definition of a valuation, we have $v(gh^{-1}) \geq \min(v(g), v(h))$ and $v(h) = v(h^{-1})$, hence $\gamma = \min(v(h^{-1}), v(gh^{-1}))$. It then follows from the definition of $\precsim$ that $gh^{-1} \precsim h^{-1}$. This proves that $\precsim$ satisfies (CQ_2). By definition of a valuation, $v(g) \geq v(h)$ implies $v(g^z) \geq v(h^z) = \gamma^z$, so $\gamma^z = \min(v(g^z), v(h^z))$. Since the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ satisfies the q.o-conjugation property, $gG_\gamma \precsim_\gamma hG_\gamma$ implies $g^zG_{\gamma^z} \precsim_{\gamma^z} h^zG_{\gamma^z}$. This implies $g^z \precsim h^z$, which proves that $\precsim$ satisfies (CQ_3).

Now let us show that $\precsim$ is indeed a lifting of $(\precsim_\gamma)_{\gamma\in\Gamma}$. By Claim 1, v is compatible with $\precsim$. Since $\precsim$ is a C-q.o., it then follows from Theorem 4.2.4 that $\precsim$ induces a C-q.o. on each quotient $B_\gamma := G^\gamma/G_\gamma$, and the induced C-q.o. is given by the formula $gG_\gamma \precsim hG_\gamma \Leftrightarrow (g \in G_\gamma) \vee (h \notin G_\gamma \wedge g \precsim h)$. We just have to show that this q.o. coincides with $\precsim_\gamma$. Let $g, h \in G^\gamma$. Assume that $gG_\gamma \precsim_\gamma hG_\gamma \wedge g \notin G_\gamma$ holds. Since $\precsim_\gamma$ satisfies (CQ_1), this implies $h \notin G_\gamma$. It follows that $v(g) = v(h) = \gamma$, and then by definition of $\precsim$ on G it follows that $g \precsim h$. This proves $gG_\gamma \precsim_\gamma hG_\gamma \Rightarrow (g \in G_\gamma) \vee (h \notin G_\gamma \wedge g \precsim h)$. Conversely, assume that $(g \in G_\gamma) \vee (h \notin G_\gamma \wedge g \precsim h)$ holds. If $g \in G_\gamma$, then since $\precsim_\gamma$ satisfies (CQ_1) we have $gG_\gamma \precsim_\gamma hG_\gamma$. Assume now that $g \notin G_\gamma$. Then we have $(h \notin G_\gamma \wedge g \precsim h)$. $h \notin G_\gamma$ implies $\gamma = v(h)$, and by assumption we have $\gamma \leq v(g)$, so $\gamma = \min(v(g), v(h))$. By definition of $\precsim$, the relation $g \precsim h$ then implies that $gG_\gamma \precsim_\gamma hG_\gamma$. This proves that $gG_\gamma \precsim_\gamma hG_\gamma \Leftrightarrow (g \in G_\gamma) \vee (h \notin G_\gamma \wedge g \precsim h)$, which proves that $\precsim$ is a lifting of $(\precsim_\gamma)_{\gamma\in\Gamma}$.

Now let us show uniqueness of the lifting. Let $\precsim^*$ be a C-q.o. lifting of $(\precsim_\gamma)_{\gamma\in\Gamma}$ and let us show that $\precsim^*$ and $\precsim$ coincide. Let $g, h \in G$ and set $\gamma := \min(v(g), v(h))$. Because $\precsim$ and $\precsim^*$ are liftings of $(\precsim_\delta)_{\delta\in\Gamma}$, they both induce the q.o. $\precsim_\gamma$ on B_γ. Because $\precsim$ and $\precsim^*$ are C-q.o.'s, it then follows from Proposition 4.2.2 that $gG_\gamma \precsim_\gamma hG_\gamma \Leftrightarrow g \in G_\gamma \vee (h \notin G_\gamma \wedge g \precsim h)$ and $gG_\gamma \precsim_\gamma hG_\gamma \Leftrightarrow g \in G_\gamma \vee (h \notin G_\gamma \wedge g \precsim^* h)$. It follows that $g \in G_\gamma \vee (h \notin G_\gamma \wedge g \precsim h) \Leftrightarrow g \in G_\gamma \vee (h \notin G_\gamma \wedge g \precsim^* h)$. If $g, h \notin G_\gamma$, it immediately follows that $g \precsim h \Leftrightarrow g \precsim^* h$. Now note that, by Theorem 4.2.4, G_γ is convex in $(G, \precsim)$ and in $(G, \precsim^*)$, so it is an initial segment in both C-q.o.g.'s. Since $\gamma = \min(v(g), v(h))$, we either have $g \notin G_\gamma$ or $h \notin G_\gamma$. If $h \in G_\gamma$, then $g \precsim h$ would imply $g \in G_\gamma$ by convexity, which is a contradiction. Similarly, $g \precsim^* h$ cannot hold. Now assume Assume $g \in G_\gamma$. Then $h \notin G_\gamma$, which by convexity implies $g \precsim h$ and $g \precsim^* h$. This shows that we always have $g \precsim h \Leftrightarrow g \precsim^* h$. □

Remark 4.2.8: Because of (CQ_3), the q.o.-conjugation property is necessary to ensure that the lifting is a C-q.o.

As a special case of lifting we can define a C-q.o. on semi-direct products:

Proposition 4.2.9
Let $(G, \precsim_G), (H, \precsim_H)$ be two C-q.o.g.'s and let $\alpha : G \to Aut(H)$ such that for any $g \in G$, $\alpha(g)$ preserves $\precsim_H$. Define a q.o. $\precsim$ on $G \ltimes_\alpha H$ by
$(g_1, h_1) \precsim (g_2, h_2) \Leftrightarrow (g_1 \precsim_G g_2) \wedge (g_2 \neq 1 \vee (g_2 = 1 \wedge h_1 \precsim_H h_2))$. Then $\precsim$ is a C-q.o.

Proof. Set $F := G \ltimes_\alpha H$, $\Gamma := \{1, 2\}$ and define $v : F \to \Gamma \cup \{\infty\}$ as follows:

$$v(g,h) := \begin{cases} 1 \text{ if } g \neq 1. \\ 2 \text{ if } g = 1 \neq h. \\ \infty \text{ if } g = h = 1. \end{cases}$$

This defines a valuation on F. We have $F_2 \cong \{1\}$, $F^2 = F_1 = \{1\} \times H$ and $F^1 = F$. Now take $h_1, h_2 \in H \cong F^2/F_2$ with $h_1 \precsim_H h_2$ and $z = (g, h) \in F$. Since H is normal in F, we have $F^2/F_2 = F^{2^z}/F_{2^z}$. We have $h_i^z = (\alpha(g)(h_i))^h$ for $i = 1, 2$. By assumption, $\alpha(g)$ preserves $\precsim_H$, hence $\alpha(g)(h_1) \precsim_H \alpha(g)(h_2)$. By (CQ_3), it then follows that $h_1^z \precsim_H h_2^z$. This proves that the isomorphism $F^2/F_2 \to F^{2^z}/F_{2^z}$ induced by z preserves $\precsim_H$. Now note that $G \cong F^1/F_1$, so F^1/F_1 is endowed with a C-q.o. $\precsim_G$ defined by $(g_1, 1).F_1 \precsim_G (g_2, 1).F_1 \Leftrightarrow g_1 \precsim_G g_2$. Take $(g_1, 1).F_1, (g_2, 1).F_1 \in F^1/F_1$ with $(g_1, 1).F_1 \precsim_G (g_2, 1).F_1$. By definition, we have $((g_i, 1).F_1)^z = (g_i, 1)^z.F_1 = (g_i^g, h\alpha(g_i^g)(h^{-1})).F_1 = (g_i^g, 1).F_1$. Because $(g_1, 1).F_1 \precsim_G (g_2, 1).F_1$, it follows from (CQ_3) on G that $(g_1^g, 1) \precsim_G (g_2^g, 1)$, hence $((g_1, 1).F_1)^z \precsim_G ((g_2, 1).F_1)^z$. This proves that the isomorphism $F^1/F_1 \to F^{1^z}/F_{1^z}$ induced by z preserves $\precsim_G$. Thus, the hypothesis of Proposition 4.2.7 are satisfied. Now note that $f_1 \precsim f_2$ is equivalent to $(\min(v(f_1), v(f_2)) = 1 \wedge f_1F_1 \precsim_G f_2F_1) \vee (\min(v(f_1), v(f_2)) = 2 \wedge f_1F_2 \precsim_H f_2F_2)$ and apply Proposition 4.2.7.

□

Another special case of lifting is the valuational product that we defined in Definition 2.7.14:

Proposition 4.2.10
Let $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$ be a family of C-q.o. groups index by a totally ordered set Γ. Then the valuational product of the family $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$ is a C-q.o. group.

Proof. Set $H := \underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma$, and let v be the usual valuation on H. Let us show that the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ has the q.o.-conjugation property. Take $z = \sum_{\gamma\in\Gamma} z_\gamma$ and $g = \sum_{\gamma\in\Gamma} g_\gamma$

both in H. Set $\delta := v(g)$. By definition of $\underset{\gamma\in\Gamma}{\mathrm{H}} B_\gamma$, we have $g^z = \sum_\gamma (g_\gamma)^{z_\gamma}$, hence $v(g^z) = \delta$. It follows that $H^{\delta^z}/H_{\delta^z} = H^\delta/H_\delta = B_\delta$. Now note that the automorphism induced by conjugation by z on B_δ is the map $h \mapsto h^{z_\delta}$. Since $\precsim_\delta$ satisfies (CQ_3), it immediately follows that the automorphism of B_δ induced by conjugation by z preserves the C-q.o. Therefore, by Proposition 4.2.7, $(\precsim_\gamma)_{\gamma\in\Gamma}$ admits a C-q.o. lifting to H which is given by the formula: $g \precsim h \Leftrightarrow (gG_\gamma \precsim_\gamma hG_\gamma,$ where $\gamma = \min(v(g), v(h)))$. This formula coincides with the definition of the valuational product. □

Unlike compatible q.o.'s, we proved in Proposition 4.2.7 that, modulo the q.o.-conjugation property, the class of C-q.o.'s is stable under lifting. This allows us to state a Baer-Krull theorem for C-q.o.'s. For any group G, $\mathcal{E}_C(G)$ denotes the set of all C-q.o.'s on G, and if v is a valuation, then $\mathcal{E}_C^v(G)$ denotes the set of all C-q.o.'s $\precsim$ of G such that v is compatible with $\precsim$:

Theorem 4.2.11 (Baer-Krull for C-q.o.'s)
Let (G, v) be a valued group. The map:

$$\begin{array}{ccc} \mathcal{E}_C^v(G) & \longleftrightarrow & \{\text{elements of } \prod_{\gamma\in\Gamma}\mathcal{E}_C(B_\gamma) \text{ with the q.o.conj. property}\} \\ \precsim & \longmapsto & (\precsim_\gamma)_{\gamma\in\Gamma}, \end{array}$$

where $\precsim_\gamma$ denotes the q.o. induced by $\precsim$ on the quotient $B_\gamma := G^\gamma/G_\gamma$, is a bijection. Moreover, for a fixed $\precsim$, we have $G = \mathcal{V}$ (respectively, $G = \mathcal{O}$) if and only if for all $\gamma \in \Gamma$, every element of $(B_\gamma, \precsim_\gamma)$ is v-type (respectively, o-type).

Proof. If $\precsim \in \mathcal{E}_C^v(G)$, then it follows from Theorem 4.2.4 that for all γ, $\precsim_\gamma$ induces a C-q.o. on G^γ/G_γ. Moreover, it follows from (CQ_3) that the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ has the q.o.-conjugation property. Therefore, the map given in the theorem is well-defined. Thanks to Proposition 4.2.7, we know that this map is bijective. The last statement follows directly from Proposition 4.2.2. □

Since valuational q.o.'s are in particular C-q.o.'s, we have as an immediate corollary:

Theorem 4.2.12
Let (G, v) be a valued group. The map:

$$\begin{array}{ccc} \{\text{refinement } w \text{ of } v\ \} & \longleftrightarrow & \{\text{ family } (w_\gamma)_{\gamma\in\Gamma} \text{ of valuations with the conj. property}\} \\ w & \longmapsto & (w_\gamma)_{\gamma\in\Gamma}, \end{array}$$

where $w_\gamma : B_\gamma \to w(G^\gamma \backslash G_\gamma) \cup \{\infty\}$ is defined by $w_\gamma(gG_\gamma) := w(g)$ for any $g \in G^\gamma \backslash G_\gamma$ and $w_\gamma(1) := \infty$, is a bijection.

Proof. Note that a valuational C-q.o. $\precsim$ is a refinement of v if and only if v is compatible with $\precsim$. Let ϕ denote the map from Theorem 4.2.11. It follows directly from the last statement of Theorem 4.2.11 and from Proposition 4.1.16 that a C-q.o. $\precsim$ is valuational if and only if the induced q.o.'s $\precsim_\gamma$ are all valuational. Therefore, if we restrict ϕ to the class of valuational C-q.o.'s, we obtain the bijection that we want.

□

For any group G, let $\mathcal{E}_o(G)$ denote the set of group orders on G, and $\mathcal{E}_o^v(G)$ the set of orders $\leq$ on G such that v is compatible with $\leq$. Using the map $\mathcal{P}$ from Section 4.1.4 and using Theorem 3.2.2, we obtain an analog for abelian ordered groups:

Theorem 4.2.13
Let (G, v) be an abelian valued group. The map:

$$\begin{array}{ccc} \mathcal{E}_o^v(G) & \longleftrightarrow & \prod_{\gamma\in\Gamma}\mathcal{E}_o(B_\gamma) \\ \leq & \longmapsto & (\leq_\gamma)_{\gamma\in\Gamma}, \end{array}$$

where each $\leq_\gamma$ is the order induced by $\leq$ on the quotient G^γ/G_γ, is a bijection.

Proof. Theorem 3.2.2 allows us to define the map:

$$\begin{array}{cccc} \phi: & \mathcal{E}_o^v(G) & \longleftrightarrow & \prod_{\gamma\in\Gamma}\mathcal{E}_o(B_\gamma) \\ & \leq & \longmapsto & (\leq_\gamma)_{\gamma\in\Gamma}, \end{array}$$

, where $\leq_\gamma$ denotes the order induced by $\leq$ on B_γ. The map ϕ is clearly injective, we just have to show that it is surjective. Let $(\leq_\gamma)_\gamma \in \prod_{\gamma\in\Gamma}\mathcal{E}_o(B_\gamma)$. For each γ, let $\precsim_\gamma \in \mathcal{P}^{-1}(\leq_\gamma)$. By Theorem 4.2.11, we can lift $(\precsim_\gamma)_\gamma$ to G and obtain a C-q.o. $\precsim$. Set $P := \mathcal{P}(\precsim)$. Since the lifting preserves the type of elements, we see that every element of $(G, \precsim)$ is o-type and that $(P \cap G^\gamma)/G_\gamma = P_\gamma$ for all $\gamma \in \Gamma$. It follows from Proposition 4.1.19 that P is a positive cone of G. Since $(P \cap G^\gamma)/G_\gamma = P_\gamma$ for all $\gamma \in \Gamma$, P is indeed a lifting of the family $(P_\gamma)_{\gamma\in\Gamma}$. □

4.2.3 q-sections and the classical Baer-Krull Theorem

We now want to show how one can recover the classical Baer-Krull theorem from Theorem 4.2.13. We fix a valued field (K, v) with value group $(G, \leq)$. Note that if $(\leq_g)_{g\in G}$ is a family of group orders on the quotients K^g/K_g, then Theorem 4.2.13 only tells us that this family lifts to a group order on $(K, +)$, but there is no reason to think that this lifting is a field order in general. In order to understand the connection between Theorem 4.2.13 and the classical Baer-Krull theorem, we need to characterize the families $(\leq_g)_{g\in G}$ whose lifting to K is a field order.

We can achieve this by using the the notion of q-section developed in [Pre84]. A q-section of the valued field (K, v) is a map $s : G \to K$ such that $s(0) = 1$, $v(s(g)) = g$ and $s(g + h) \equiv s(g)s(h) \mod K^2$. It was proved in [Pre84] that every valued field admits a q-section. We now fix a q-section s of (K, v). Then for any $g \in G$, the map $\phi_g : Kv \to K^g/K_g, a + K_0 \mapsto s(g)a + K_g$ defines an isomorphism from Kv to K^g/K_g. If we take a family $(\leq_g)_{g\in G}$ of orders on the quotients K^g/K_g then the behavior of the ϕ_g's with respect to $\leq_g$'s will tell us if the lifting of $(\leq_g)_g$ is a field order:

Proposition 4.2.14
Let $(\leq_g)_{g\in G}$ be a family of group orders on the quotients K^g/K_g. Then the lifting of $(\leq_g)_g$ to K is a field order if and only if the following conditions are satisfied:

(1) $\leq_0$ is a field order of Kv.

(2) there exists a group homomorphism $\epsilon : G \to \{-1, 1\}$ such that for any $g \in G$, ϕ_g is order-preserving when $\epsilon(g) = 1$ and ϕ_g is order-reversing when $\epsilon(g) = -1$.

Proof. Denote by $\leq$ the lifting. If $\leq$ is a field order, then $\leq_0$ must be a field order because it is the order induced by $\leq$ on Kv; moreover, we can define $\epsilon(g) = 1$ if $0 < s(g)$ and $\epsilon(g) = -1$ if $s(g) < 0$, and one easily sees that ϵ has the desired property.

Assume now that conditions (1) and (2) are satisfied. We already know from Theorem 4.2.13 that $\leq$ is a group order of $(K,+)$, so $\leq$ is a field order if and only if the set of positive elements of $(K,\leq)$ is stable under multiplication. By definition of $\leq$, this is equivalent to saying that for any $a,b\in K$ with $g=v(a)$ and $h=v(b)$, $0\leq_g a+K_g$ and $0\leq_h b+K_h$ imply $0\leq_{g+h} ab+K_{g+h}$. Note that since s is a q-section, we have $ab+K_{g+h}=d^2\phi_{g+h}(\phi_g^{-1}(a+K_g)\phi_h^{-1}(b+K_h))$ for some $d\in K$; in particular, $ab+K_{g+h}$ has the same sign as $\phi_{g+h}(\phi_g^{-1}(a+K_g)\phi_h^{-1}(b+K_h))$. Now assume for example that ϕ_g is order-preserving and ϕ_h order-reversing. If $a+K_g$ and $b+K_h$ are both positive,we then have $\phi_g^{-1}(a+K_g)\leq_0 0$ and $0\leq_0\phi_h^{-1}(b+K_h)$. Since $\leq_0$ is a field order, this implies that $\phi_g^{-1}(a+K_g)\phi_h^{-1}(b+K_h)\leq_0 0$. Since ϵ is a group homomorphism, then ϕ_{g+h} is order-reversing, hence $0\leq_{g+h}\phi_{g+h}(\phi_g^{-1}(a+K_g)\phi_h^{-1}(b+K_h))$. The other cases are treated similarly.

□

As a consequence of Proposition 4.2.14 we have the following variant of the Baer-Krull theorem:

Theorem 4.2.15 (Baer-Krull, variant)
Let $\mathcal{O}$ be the set of field orders of Kv and $\mathcal{E}$ the set of group homomorphisms from G to $\{-1,1\}$. Then $\mathcal{O}\times\mathcal{E}$ is in bijection with the set of families $(\leq_g)_{g\in G}$ whose lifting to K is a field order.

Proof. Assume that $\leq_0\in\mathcal{O}$ and $\epsilon\in\mathcal{E}$ are given. Then define $\leq_g$ on K^g/K_g as follows: If $\epsilon(g)=1$, define $a+K_g\leq_g b+K_g\Leftrightarrow\phi_g^{-1}(a+K_g)\leq_0\phi_g^{-1}(b+K_g)$; if $\epsilon(g)=-1$, define $a+K_g\leq_g b+K_g\Leftrightarrow\phi_g^{-1}(a+K_g)\geq_0\phi_g^{-1}(b+K_g)$. By Proposition 4.2.14, this family of orders lifts to a field order on K. Conversely, assume $(\leq_g)_{g\in G}$ is given and denote by $\leq$ the lifting of $(\leq_g)_{g\in G}$ to K. Then the existence of ϵ is given by Proposition 4.2.14. □

Now assume that $(\pi_i)_{i\in I}$ is a family of elements of K such that $(v(\pi_i)+2G)_{i\in I}$ is an $\mathbb{F}_2$-Basis of $G/2G$. In order to recover Theorem 2.3.1 from Theorem 4.2.15, we need to show that the set of homomorphisms from G to $\{-1,1\}$ is in bijection with the set of maps from I to $\{-1,1\}$. First note that we can see I as a subset of G if we identify $i\in I$ with $v(\pi_i)$. Thus, any homomorphism $\epsilon:G\to\{-1,1\}$ canonically induces a map $I\to\{-1,1\}$ (just take $\epsilon_{|I}$). For the converse, note that every $g\in G$ has a decomposition $g=\sum_{i\in I}n_iv(\pi_i)+2h$, where $h\in G$, $n_i\in\{0,1\}$ and $n_i=1$ only for finitely many i. If $\epsilon:I\to\{-1,1\}$ is given, we can extend ϵ to a homomorphism $G\to\{-1,1\}$ as follows. For $g=\sum_{i\in I}n_iv(\pi_i)+2h$, let l_g be the number of $i\in I$ such that $n_i=1$ and $\epsilon(i)=-1$. Set $\epsilon(g):=-1$ if l_g is odd and $\epsilon(g):=1$ if l_g is even.

4.3 Structure of C-q.o.g.'s

4.3.1 Examples of C-q.o.'s

In this section, we describe the structure of an arbitrary C-q.o.g. $(G,\precsim)$. We start by giving five different examples of C-q.o.'s. Examples (a),(c) and (d) are obtained by direct application of 4.2.10, and example (e) is proved from example (d) with Proposition 4.2.9. To prove example (b), one needs to use Proposition 4.3.32 on the C-q.o. group $(G, \precsim)$ from example (a) with $g := (-1, 0)$.

Examples 4.3.1
Set $G := \mathbb{Z}^2$. We let $\precsim_o$ denote the C-q.o. induced by the usual order of $\mathbb{Z}$ (which is characterized in Proposition 4.1.17) and $\precsim_v$ the C-q.o. induced by the trivial valuation on $\mathbb{Z}$.

(a) Choose $\precsim_1 := \precsim_o$ and $\precsim_2 := \precsim_v$. The valuational product $(G, \precsim)$ of the family $((\mathbb{Z}, \precsim_1), (\mathbb{Z}, \precsim_2))$ is given by :

$(0,0) \precnsim (\{0\} \times (\mathbb{Z}\backslash\{0\}), \precsim_t) \precnsim (-\mathbb{N} \times \mathbb{Z}, \precsim_t) \precnsim (\mathbb{N} \times \mathbb{Z}, \precsim)$,

where $\precsim_t$ denotes the trivial q.o. and $\precsim$ is defined on $\mathbb{N} \times \mathbb{Z}$ as follows: $(a, b) \precsim (c, d) \Leftrightarrow a \leq c$. In this example, $\precsim$ is valuational on $\{0\} \times \mathbb{Z}$ and order-type-like on $(\mathbb{Z}\backslash\{0\} \times \mathbb{Z})$. The set of v-type elements is $\{0\} \times \mathbb{Z}$, the set of o^--type elements is $-\mathbb{N} \times \mathbb{Z}$ and the set of o^+-type elements is $\mathbb{N} \times \mathbb{Z}$.

(b) Coarsen the C-q.o. of the previous example by declaring that $(\{0\} \times (\mathbb{Z}\backslash\{0\}), \precsim_t) \sim (-\mathbb{N} \times \mathbb{Z}, \precsim_t)$. This new C-q.o. is now given by:

$(0,0) \precnsim ((-\mathbb{N}_0 \times \mathbb{Z})\backslash\{(0,0)\}, \precsim_t) \precnsim (\mathbb{N} \times \mathbb{Z}, \precsim)$.

All elements of G in this example have the same type as in (a).

(c) Define $\precsim_1 = \precsim_2 = \precsim_o$. The valuational product of the family $((\mathbb{Z}, \precsim_1), (\mathbb{Z}, \precsim_2))$ is now given by :

$(0,0) \precnsim (\{0\} \times -\mathbb{N}, \precsim_t) \precnsim (\{0\} \times \mathbb{N}, \leq) \precnsim (-\mathbb{N} \times \mathbb{Z}, \precsim_t) \precnsim (\mathbb{N} \times \mathbb{Z}, \precsim)$,

where $\leq$ is the natural order of $\mathbb{Z}$ and $\precsim$ is defined on $\mathbb{N} \times \mathbb{Z}$ as follows: $(a, b) \precsim (c, d) \Leftrightarrow a \leq c$. Here $\precsim$ is order-type-like on $\{0\} \times \mathbb{Z}$ and on $(\mathbb{Z}\backslash\{0\} \times \mathbb{Z})$. The set of o^--type elements is $\{0\} \times -\mathbb{N} \cup -\mathbb{N} \times \mathbb{Z}$, the set of o^+-type elements is $\{0\} \times \mathbb{N} \cup \mathbb{N} \times \mathbb{Z}$, and $(0,0)$ is the only v-type element.

(d) Let $\precsim$ be the C-q.o. of example (a) on G. Let $(H, \precsim_H)$ be the valuational product of the family $((G, \precsim)_{n \in \mathbb{Z}})$ ($\mathbb{Z}$-many copies of $(G, \precsim)$). Here the C-q.o alternates infinitely many times between order-type-like parts and valuational-like parts. More precisely: Let w_H denote the valuation $w_H : H \to \mathbb{Z} \cup \{\infty\}, h \mapsto \min \operatorname{supp}(h)$. For any $h = \sum_{n \in \mathbb{Z}} g_n \in H$ ($g_n \in G$) with $m := w_H(h)$, then h is v-type if and only if $g_m \in \{0\} \times \mathbb{Z}$, h is o^--type if and only if $g_m \in -\mathbb{N} \times \mathbb{Z}$ and h is o^+-type if and only if $g_m \in \mathbb{N} \times \mathbb{Z}$. For any $m \in \mathbb{Z}$, $\precsim_H$ is valuational-like on $\{h = \sum_{n \in \mathbb{Z}} g_n \tau_n \in H \mid w_H(h) = m, g_m \in \{0\} \times \mathbb{Z}\}$ and is order-type-like on $\{h = \sum_{n \in \mathbb{Z}} g_n \tau_n \in H \mid w_H(h) = m, g_m \in (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z}\}$.

We can also give a non-abelian example:

(e) Let $(H, \precsim_H)$ be as in the previous example. For any $k \in \mathbb{Z}$, let α_k be the k-th shift on H (i.e $\alpha_k(\sum_{n\in\mathbb{Z}} g_n) = \sum_{n\in\mathbb{Z}} g_{n-k}$). This is a group automorphism of H. Set $F := \mathbb{Z} \ltimes_\alpha H$ and define $\precsim_F$ by:
$(k, h_1) \precsim_F (l, h_2) \Leftrightarrow (k \precsim_v l) \wedge (l \neq 0 \vee (l = 0 \wedge h_1 \precsim_H h_2))$. Here the elements of H have the same type as in (d). Elements of the form (l, h) with $l \neq 0$ are v-type.

We see on each of these examples that G can be partitioned into strictly convex subsets on each of which $\precsim$ is fundamental-like. We will show that this is true for any arbitrary C-q.o.g. As the terminology and Examples 4.3.1 suggest, and similarly to what happened with compatible q.o.'s in chapter 3, it will turn out that $\precsim$ is valuational-like on the set of v-type elements and order-type-like around o-type elements.

Example (b) deserves a special attention. The C-q.o. in example (b) seems counter-intuitive. Indeed, we would expect the C-q.o. to separate o-type elements from v-type elements, but we see that $(0, 1) \sim (-1, 1)$. This means that the C-q.o. does not distinguish between the v-type element $(0, 1)$ and the o-type element $(-1, 1)$. This phenomenon is what we call "welding".

Definition 4.3.2
We say that $(G, \precsim)$ is **welded** at h, or that h is a **welding point** of $(G, \precsim)$, if there exists an element g such that g and h are of different type and $g \sim h$. We then say that g and h are **welded**. We say that $(G, \precsim)$ **has welding** if G has a welding point, and we say that $(G, \precsim)$ is **welding-free** if it has no welding. Finally, a **welding class** of $(G, \precsim)$ is a $\sim$-class containing a welding point.

We will see that the existence of welding in certain groups makes things technically slightly more difficult but does not fundamentally change the structure of a C-q.o.g. It is important to note that welding only concerns v-type and o$^-$-type elements (but not o$^+$-type elements):

Proposition 4.3.3
Let $(G, \precsim)$ be a C-q.o. group. Then the set $\mathcal{O}^+$ is stable under "$\sim$".

Proof. This follows directly from Lemma 4.1.18. □

Proposition 4.3.4
Let $(G, \precsim)$ be a C-q.o. group and take $g, h \in G$. Assume that g and h are welded. Then one of them is v-type and the other one is o$^+$-type.

Proof. This follows directly from Proposition 4.3.3. □

Notation
If $g \in \mathcal{O}^-$, we set $\mathcal{W}_g := \{h \in \mathcal{V} \mid h \sim g\}$, and if $g \in \mathcal{V}$ then we set
$\mathcal{W}_g := \{h \in \mathcal{O}^- \mid h \sim g\}$.

The following Proposition shows that example (b) cannot have been obtained directly by lifting fundamental C-q.o.'s, as opposed to the other four examples:

Proposition 4.3.5
Let $(G, \precsim)$ be a C-q.o. group. Assume that there is a valuation $v : G \to \Gamma \cup \{\infty\}$ and a family $(\precsim_\gamma)_{\gamma\in\Gamma}$ of fundamental C-q.o.'s on the quotients $B_\gamma := G^\gamma/G_\gamma$ such that $\precsim$ is the C-q.o. lifting of the family $(\precsim_\gamma)_{\gamma\in\Gamma}$ (see Proposition 4.2.7). Then $(G, \precsim)$ is welding-free.

Proof. Take $g \in \mathcal{O}^-$, and let $h \in G$ with $h \sim g$. Looking at the formula for $\precsim$ given in Proposition 4.2.7, we see that $g \sim h$ implies $v(g) = v(h)$. Set $\gamma := v(g)$. It follows from Proposition 4.2.2 that gG_γ has the same type as g, so it is o-type. Since $\precsim_\gamma$ is fundamental, it follows that every element of B_γ is also o-type. It follows from Proposition 4.2.2 that hG_γ has the same type as h, so h is o-type. This shows that there is no welding in G. □

It follows from Proposition 4.3.5 that lifting is not sufficient to construct all C-q.o's from the fundamental ones. However, we will show that lifting and welding are sufficient to construct any C-q.o. from the fundamental ones, as suggested by Examples 4.3.1.

4.3.2 Quotient by strictly convex subgroups

In order to decompose a C-q.o. groups into fundamental ones, we will consider quotients on which $\precsim$ induces a fundamental C-q.o. However, because of the existence of welding, it can happen that the subgroup which we need to consider is only strictly convex, and not convex. This seems problematic at first. Indeed, Proposition 4.2.2 states that, if H is not convex, then $\precsim$ does not induce a C-q.o. on G/H in the sense given in Definition 2.7.11. We solve this problem by showing that the formula given in Proposition 4.2.2 still makes sense when H is only strictly convex. The idea goes as follows: if $\precsim$ is a C-q.o. such that H is strictly convex in $(G, \precsim)$, then we can refine $\precsim$ into a C-q.o $\precsim^*$ in a way that makes H convex in $(G, \precsim^*)$. We then know from Proposition 4.2.2 that $\precsim^*$ induces a C-q.o. on G/H. By abuse of terminology, we will then call this induced q.o. the C-q.o. induced by $\precsim$ on G/H. Note however that, strictly speaking, this induced C-q.o. is not the induced q.o. as defined in Definition 2.7.11.

Note that, because every convex subgroup is an initial segment, any non-convex strictly convex subgroup of G is in case (iii) of Lemma 2.7.5. We need the following Lemma:

Lemma 4.3.6
Let $(G, \precsim_1)$ be a C-q.o.g. and let H be a strictly convex normal subgroup of $(G, \precsim_1)$ with convexity complement $F \neq \emptyset$. We are then in case (iii) of Lemma 2.7.5, so we have $H \precsim_1 F$. Let $\precsim_2$ be the refinement of $\precsim_1$ defined by declaring that $H \precnsim_2 F$. Then $\precsim_2$ is a C-q.o. and H is $\precsim_2$-convex.

Proof. The fact that H is $\precsim_2$-convex is clear, as is the fact that $1 \precnsim_2 x$ for every $x \in G$. Since $F \neq \emptyset$, it follows from Lemma 2.7.5 that $\max(H), \min(F)$ are non-empty and $\min(F) \sim_1 \max(H)$. Note that the notation $\max(H)$ is unambiguous, since the max of H in $(G, \precsim_1)$ is the same as in $(G, \precsim_2)$; similarly for $\min(F)$. Now assume $x \precnsim_2 y$ and take $z \in G$. We want to show $xy^{-1} \precsim_2 y^{-1}$ and $x^z \precsim_2 y^z$. Since $\precsim_1$ is a coarsening of $\precsim_2$, we have $x \precsim_1 y$. Because $\precsim_1$ is a C-q.o., this implies $xy^{-1} \precsim_1 y^{-1}$ and $x^z \precsim_1 y^z$. Towards a contradiction, assume $y^{-1} \precnsim_2 xy^{-1}$. Because $\precsim_1$ and $\precsim_2$ coincide outside of $\max(H)$ and $\min(F)$, this implies $y^{-1} \in \max(H)$ and $xy^{-1} \in \min(F)$. Since $x \precsim_2 y \in H$, the $\precsim_2$-convexity of H then implies $x \in H$. This implies $xy^{-1} \in H$, which contradicts $xy^{-1} \in F$. Therefore, we must have $xy^{-1} \precsim_2 y^{-1}$. By the same reasoning (using the fact that H is normal), we get $x^z \precsim_2 y^z$.

□

Lemma 4.3.7
Take the same notations as in Lemma 4.3.6. Both $\precsim_1$ and $\precsim_2$ induce a q.o. on G/H, and for every $g, h \in G\backslash H$ we have $gH \precsim_1 hH \Leftrightarrow gH \precsim_2 hH$. In particular, the refinement of $\precsim_1$ on G/H obtained by declaring $0 \precnsim \pi_H(F)$ is the C-q.o. $\precsim_2$.

Proof. Since H is convex in $(G, \precsim_2)$, we already know from Proposition 4.2.2 that $\precsim_2$ induces a C-q.o. on G/H. Define $\precsim_1$ on G/H as in Definition 2.7.11, i.e. $g_1H \precsim_1 g_2H \Leftrightarrow (\exists h_1, h_2 \in H, g_1h_1 \precsim_1 g_2h_2)$. Since H is in case (iii) of Lemma 2.7.5, we have $1.H \precsim_1 gH$ for every $g \in G$. The fact that $gH \precsim_1 hH \Leftrightarrow gH \precsim_2 hH$ holds for every $g, h \notin H$ follows from the fact that $\precsim_1$ and $\precsim_2$ coincide on $G\backslash H$. It follows immediately that $\precsim_1$ is transitive on $G/H\backslash\{1.H\}$. Because $H \precsim_1 (G\backslash H)$, $1.H$ is minimal in $(G/H, \precsim_1)$. We then immediately have that $\precsim_1$ is transitive on G/H. This shows that $\precsim_1$ induces a q.o. on G/H. Note however that $1.H \sim_1 f.H$ for every $f \in F$, so (CQ_1) is not satisfied. However, because $\precsim_1$ and $\precsim_2$ coincide on $G/H\backslash\{1.H\}$, then $\precsim_2$ is the refinement of $\precsim_1$ obtained by declaring that $1 \precnsim \pi_H(F)$. □

We can now show the following:

Proposition 4.3.8
Let H be a strictly convex normal subgroup of G. Then the following formula defines a C-q.o. on G/H: $gH \precsim hH \Leftrightarrow (g \in H) \vee (h \notin H \wedge g \precsim h)$. Moreover, π_H preserves the type of elements in $G\backslash H$.

Proof. Set $\precsim_1 := \precsim$ and consider the q.o. $\precsim_2$ as in Lemma 4.3.6. Since H is $\precsim_2$-convex, we know from Proposition 4.2.2 that $\precsim_2$ induces a C-q.o. on G/H given by the formula: $gH \precsim hH \Leftrightarrow (g \in H) \vee (h \notin H \wedge g \precsim_2 h)$. We also know from Proposition 4.2.2 that π_H preserves the type of elements. It is easy to see that $(g \in H) \vee (h \notin H \wedge g \precsim_2 h)$ is equivalent to $(g \in H) \vee (h \notin H \wedge g \precsim h)$, since for any $h \notin H$ and any $g \in G$, $g \precsim h \Leftrightarrow g \precsim_2 h$. □

Remark 4.3.9: (i) By abuse of terminology, we call the C-q.o. $\precsim$ defined on G/H as in Proposition 4.3.8 the **C-q.o. induced by $\precsim$ on G/H**. Note that $\precsim$ is different from the q.o. induced by $\precsim$ on G/H as defined in Definition 2.7.11. If $\precsim^*$ denote the q.o. induced by $\precsim$ on G/H as defined in Definition 2.7.11, then $\precsim$ is the refinement of $\precsim^*$ obtained by declaring $1.H \precnsim g.H$ for every $g \in G\backslash H$ (this follows from Lemma 4.3.7).

(ii) If H is convex, then the induced C-q.o. on G/H really is the q.o. induced on G/H as defined in Definition 2.7.11.

(iii) Note that, for any $h \notin H$ and any $g \in G$, we have $g \precsim h \Leftrightarrow gH \precsim hH$.

4.3.3 Type-components

In this section, we introduce the "type-components" T_g mentioned in the introduction of this chapter. For $g \neq 1$, we want to find a set T_g which is the biggest strictly convex subset of G containing g on which $\precsim$ is fundamental-like. If $g \in \mathcal{O}^+$, one can see that the set of $h \in \mathcal{O}^+$ such that every element strictly between g and h are also o$^+$-type is the greatest strictly convex subset of o$^+$-type elements containing g. We can define in the same way such a "strictly convex closure" for an o$^-$-type element or a v-type element. Now, since by definition T_g contains g and g^{-1}, in the o-type cases T_g cannot be this closure. We will show that T_g is the union of the strictly convex closures of g and g^{-1}. In the v-type case the strictly convex closures of g and g^{-1} are equal. We also introduce the set G_g which should be thought of as the set of elements of G which are "below" T_g . We then introduce the set G^g which should be thought of as the set of elements which are not bigger than T_g. We will show that G_g and G^g are subgroups. For proving the properties of T_g, G^g, G_g, and the welding properties, it is more convenient to define T_g by means of formulas with inequalities instead of strict inequalities. This motivates the following definitions. For an element $1 \neq g \in G$, we define the type-component T_g of g as follows:

- If $g \in \mathcal{V}$, then $T_g := \{1 \neq h \in \mathcal{V} \mid \text{there is no } o^+\text{-type element between } h \text{ and } g\}$.
- If $g \in \mathcal{O}^+$, then $T_g^+ := \{h \in \mathcal{O}^+ \mid \text{every element between } g \text{ and } h \text{ is } o^+\text{-type}\}$. We then set $T_g^- := (T_g^+)^{-1}$ and $T_g := T_g^+ \bigcup T_g^-$.
- If $g \in \mathcal{O}^-$, then $T_g := T_{g^{-1}}$.

We also define two sets G^g and G_g as follows:

- If $g \in \mathcal{V}$, then define $G_g := \{h \mid h \precnsim T_g\}$.
- If $g \in \mathcal{O}^+$, then define $G_g := \{h \mid \{h, h^{-1}\} \precnsim g^{-1}\}$.
- If $g \in \mathcal{O}^-$, then define $G_g := G_{g^{-1}}$.

In all cases we set $G^g := G_g \bigcup T_g$. For $g = 1$, we set $T_g = G^g = G_g = \{1\}$. We will show later that G^g and G_g are actually subgroups of G (see Propositions 4.3.13 and 4.3.24). Note that for any $g \in G$, $1 \in G_g$, so G_g and G^g are non-empty.

Example 4.3.10
Let us have a look again at the groups given in Examples 4.3.1. Set $g = (0, 1)$ and $h = (1, 0)$. In examples (a), (b) and (c) we have $T_g = (\{0\} \times \mathbb{Z})\backslash\{(0, 0)\}$, $T_h^+ = \mathbb{N} \times \mathbb{Z}$ and $T_h = (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z}$. We also have $G_g = \{0\}, G^g = G_h = \{0\} \times \mathbb{Z}, G^h = G$. It is easy

to see that the q.o. induced on the quotients G^g/G_g and G^h/G_h are exactly the q.o's $\precsim_1$ and $\precsim_2$ which we lifted to construct the q.o. on G. Note that the only difference between cases (a) and (b) is that T_g, T_h, G_h, G^g are convex in case (a) but are only strictly convex in case (b) due to welding. Note also that in example (b), each element of the form (x, y) with $x < 0$ is an o$^-$-type welding point with $(x, y) \sim (0, z)$ for every $z \neq 0$. In particular, there is an o$^-$-type element (for example $(-1, 0)$) which is contained between g and g^{-1}, even though $g^{-1} \in T_g$. This explains why we restrict to o$^+$-type elements in the definition of T_g when g is v-type.

In the next two sections, we describe some properties of the sets T_g, G^g and G_g for $g \neq 1$. As announced in the introduction, we are going to show that T_g is a maximal subset of G with the properties that T_g is strictly convex and that $\precsim$ is fundamental-like (of the same type as g) on T_g (see Propositions 4.3.11 and 4.3.23). We will also show that G_g and G^g are subgroups of G and that G_g is normal in G^g. We first show these properties for the case where g is o-type and then do the same for the case where g is v-type.

T_g in the o-type case

We now want to describe T_g, G_g, G^g in the case where $g \neq 1$ is o-type. By definition of T_g, we can assume without loss of generality that $g \in \mathcal{O}^+$. We recall that $\mathcal{W}_{g^{-1}}$ denotes the set $\{h \in \mathcal{V} \mid h \sim g^{-1}\}$, and that g^{-1} is a welding point if and only if $\mathcal{W}_{g^{-1}} \neq \emptyset$. The following proposition states the main properties of T_g:

Proposition 4.3.11 (Characteristics of T_g)
The set T_g has the following properties:

(1) T_g is right-convex in G with convexity complement $\mathrm{cl}(g^{-1}) \backslash T_g = \mathcal{W}_{g^{-1}}$.

(2) T_g is convex if and only if g^{-1} is not a welding point of G.

(3) T_g is the biggest strictly convex subset of G containing g with the following properties:

  (i) Every element of T_g is o-type.

  (ii) T_g contains exactly one class of o^--type elements, and this class is smaller than every o^+-type element.

(4) for any $f_1, f_2, h \in T_g^+$, we have $f_1 \precsim f_2 \Rightarrow f_1 h \precsim f_2 h \wedge h f_1 \precsim h f_2$.

Remark 4.3.12: (i) Proposition 4.3.11(3) basically says that T_g is the biggest strictly convex subset of G containing g on which $\precsim$ is order-type-like.

(ii) It follows from Proposition 4.3.11(1) and from Lemma 2.7.5(ii) that $\min(T_g) = \mathrm{cl}(g^{-1}) \cap T_g$.

(iii) If g^{-1} is not a welding point, then we can replace “strictly convex” by “convex” in Proposition 4.3.11(3).

(iv) Example 4.3.1(b) shows that T_g is not always convex.

(v) It is interesting to note that property (4) in 4.3.11 is the property satisfied by ordered groups (see axiom (OG) in Section 2.2).

We now state the main properties of G_g and G^g:

Proposition 4.3.13 (Quotient for o-type elements)
Both G^g and G_g are subgroups of G. Moreover, G^g is convex in G and G_g is the smallest normal strictly convex subgroup of G^g such that the C-q.o. induced by $\precsim$ on G^g/G_g is order-type.

Remark 4.3.14: If g^{-1} is not a welding point, then G_g is actually convex. However, Example 4.3.1(b) shows that G_g is not convex in general. We see that the existence of welding makes the structure of G less smooth, since it prevents the type-components from being convex.

Our goal is now to prove Propositions 4.3.11 and 4.3.13. We start by characterizing the elements of T_g^+ in the next lemma:

Lemma 4.3.15
For any $h \in G$, $h \in T_g^+$ if and only if $h \in \mathcal{O}^+$ and $h^{-1} \sim g^{-1}$. In particular, $g \in T_g^+$ and $T_g^- \subseteq \mathrm{cl}(g^{-1})$.

Proof. Assume $h \in T_g^+$. $h \in \mathcal{O}^+$ by definition of T_g. If $h^{-1} \precnsim g^{-1}$, then $h^{-1} \precnsim g^{-1} \precnsim g$. By Lemma 4.1.11(ii), this implies $h \precnsim g^{-1} \precnsim g$, so there is an o^--type element between h and g, which contradicts $h \in T_g$. If $g^{-1} \precnsim h^{-1}$, then by the same reasoning we get $g \precnsim h^{-1} \precnsim h$, which also contradicts $h \in T_g$. This proves that $h^{-1} \sim g^{-1}$. Conversely, assume that h is o^+-type and $h^{-1} \sim g^{-1}$. We want to show that every f between h and g is o^+-type. Since f is between h and g and since $h^{-1} \sim g^{-1}$, we either have $h^{-1} \precnsim f \precsim h$ or $g^{-1} \precnsim f \precsim g$. By Lemma 4.1.18, this implies that f is o^+-type. $\square$

As a direct consequence of these two lemmas, we have that the q.o is order-type-like on T_g:

Proposition 4.3.16
We have $T_g \subseteq \mathcal{O}$. Moreover, T_g contains exactly one class of o^--type elements, which is T_g^-. Moreover, $T_g^- \precnsim T_g^+$ and there is no h such that $T_g^- \precnsim h \precnsim T_g^+$

Proof. The fact that $T_g \subseteq \mathcal{O}$ is a direct consequence of the definition of T_g. The fact that there is exactly one class of o^-type elements is a consequence of Lemma 4.3.15. If h satisfies $T_g^- \precnsim h \precsim T_g^+$, then by Lemma 4.1.18 $h \in T_g^+$, so we don't have $h \precnsim T_g^+$. $\square$

We can now show Proposition 4.3.11:

proof of 4.3.11. We first prove (1). It is clear from the definition of T_g^+ and from Proposition 4.3.3 that T_g^+ is convex. We also know from Proposition 4.3.16 that $\min(T_g) = T_g^- \subseteq \mathrm{cl}(g^{-1})$ and that there is no element strictly between T_g^- and T_g^+. It follows that

$T_g \cup \mathrm{cl}(g^{-1})$ is convex. By Lemma 2.7.5(ii), it follows that T_g is right-convex and that $F_g := \mathrm{cl}(g^{-1})\backslash T_g$ is the convexity complement of T_g. By definition of $\mathcal{W}_{g^{-1}}$, we have $\mathcal{W}_{g^{-1}} \subseteq F_g$. To show equality, we just need to show that every element of F_g is v-type. Let $h \in F_g$. Then $h \sim g^{-1}$. By Proposition 4.3.3, $h \notin \mathcal{O}^+$. If h were o^--type, then by Lemma 4.3.15 we would have $h \in T_g^-$, which is excluded, so $h \notin \mathcal{O}^-$. Thus, $h \in \mathcal{V}$. This shows that $\mathcal{W}_{g^{-1}} = \mathrm{cl}(g^{-1})\backslash T_g$ is the convexity complement of T_g. It follows that T_g is convex if and only if $\mathcal{W}_{g^{-1}} = \varnothing$. This in turn holds if and only if g^{-1} is not a welding point. This proves (2). Now let us prove (3). We know by Proposition 4.3.16 that T_g satisfies (i) and (ii). It only remains to prove that there is no strictly convex set bigger than T_g satisfying (i) and (ii). Towards a contradiction, let $S \supsetneq T_g$ be such a set and take $h \in S\backslash T_g$. Because T_g is right-convex and because $T_g^- \precnsim T_g^+$, we either have $T_g^+ \precnsim h$ or $h \precsim T_g^-$. Assume first that $T_g^+ \precnsim h$. By condition (ii), we must then have $h \in \mathcal{O}^+$ and $h^{-1} \precsim T_g^-$. Let $g \precsim f \precsim h$. We have $h^{-1} \precnsim f \precsim h$. By Lemma 4.1.18, this implies $f \in \mathcal{O}^+$. Thus, every element between g and h is o^+-type, so $h \in T_g^+$, which is a contradiction. Assume that $h \precnsim T_g^-$. By condition (ii), we must have $h \in \mathcal{O}^-$ and $T_g^- \precnsim h^{-1}$. Because T_g is right-convex, it follows that $T_g^+ \precnsim h^{-1}$. We then have $h \precnsim g \precsim h^{-1}$. By 4.1.18, this implies $g^{-1} \sim h$. By Lemma 4.3.15, this implies $h^{-1} \in T_g^+$: contradiction. (4) is a direct consequence of Proposition 4.1.14, since $h^{-1} \nsim f_2$. $\square$

We mentioned in Remark 4.3.12 that the q.o. $\precsim$ on T_g is order-type-like. In fact, the only difference between the structure of T_g and the group in Proposition 4.1.17 is that $\precsim$ is not an order on T_g^+ (see for example T_h^+ in Example 4.3.10). However, we have the following:

Lemma 4.3.17
Let $f, h \in T_g^+$ and $f \sim h$. Then $fh^{-1} \in G_g$.

Proof. By applying (CQ_2) to the inequalities $f \precsim h \precsim f$ we obtain $fh^{-1} \precsim h^{-1}$ and $hf^{-1} \precsim f^{-1}$. By Lemma 4.3.15, $g^{-1} \sim h^{-1} \sim f^{-1}$, hence $\{fh^{-1}, hf^{-1}\} \precsim g^{-1}$, hence $fh^{-1} \in G_g$. $\square$

Intuitively, we see from Lemma 4.3.17 that the C-q.o. induced by $\precsim$ on the quotient G^g/G_g will look exactly like the C-q.o. of Proposition 4.1.17. It still remains to show that G^g and G_g have the desired properties:

Proposition 4.3.18
We have $G_g = \{h \precnsim g^{-1}\} \cup \mathcal{W}_{g^{-1}}$. In particular, G_g is left-convex in G, and it is convex if and only if g^{-1} is not a welding point. This in turn holds if and only if T_g is convex in G. Moreover, if G_g is not convex, then its convexity complement is T_g^- and we have $\mathcal{W}_{g^{-1}} = \max(G_g)$.

Proof. Let $h \in G_g$. Then in particular $\{h, h^{-1}\} \precsim g^{-1}$. If $h \sim g^{-1}$, then we have $h \in \mathrm{cl}(g^{-1}) = T_g^- \cup \mathcal{W}_{g^{-1}}$. Since $h^{-1} \precsim g^{-1}$, we cannot have $h \in T_g^-$ (otherwise we would have $h^{-1} \in T_g^+$), so $h \in \mathcal{W}_{g^{-1}}$. Conversely, assume $h \precnsim g^{-1}$. By Lemma 4.1.11(ii), this implies $h^{-1} \precnsim g^{-1}$, hence $h \in G_g$. Assume $h \in \mathcal{W}_{g^{-1}}$. Then $h \sim g^{-1}$ and h is v-type, so $h^{-1} \sim h \precsim g^{-1}$, hence $h \in G_g$. If $\mathcal{W}_{g^{-1}} = \varnothing$, then $G_g = \{h \precnsim g^{-1}\}$ is clearly convex. Now assume that $\mathcal{W}_{g^{-1}} \neq \emptyset$. By definition of $\mathcal{W}_{g^{-1}}$, we have $f \sim g^{-1}$ for every $f \in \mathcal{W}_{g^{-1}}$, hence $h \precsim f$ for every $f \in \mathcal{W}_{g^{-1}}$ and $h \in G_g$. Since $\mathcal{W}_{g^{-1}} \subseteq G_g$, it follows that $\mathcal{W}_{g^{-1}} = \max(G_g)$. Now take any $f \in \mathcal{W}_{g^{-1}}$. We then have $f \in G_g$, $f \sim g^{-1}$, but $g^{-1} \notin G_g$, so G_g is not convex. Moreover, we have $T_g^- = \mathrm{cl}(f) \backslash G_g$. By Lemma 2.7.5(iii), this implies that T_g^- is the convexity complement of G_g. □

Proposition 4.3.19
G^g is an initial segment of G.

Proof. It follows from Proposition 4.3.18 that $G_g \cup T_g^-$ is an initial segment of G. Since T_g^+ is convex and since there is no element strictly contained between T_g^- and T_g^+, it follows that $G^g = G_g \cup T_g^- \cup T_g^+$ is an initial segment.

□

We can now show Proposition 4.3.13:

proof of 4.3.13. Let $h_1, h_2 \in G_g$. We have $h_1 \precsim g^{-1}$ and $\{h_2, h_2^{-1}\} \precsim g^{-1} \precnsim g$, so we can apply Propositions 4.1.14 and 4.1.12 and get $h_1 h_2^{-1} \precsim g^{-1} h_2^{-1} \sim g^{-1}$. By a similar argument, we also have $h_2 h_1^{-1} \precsim g^{-1}$, hence $h_1 h_2^{-1} \in G_g$. This proves that G_g is a subgroup of G. Now let us show that G^g is a subgroup of G. Note that by Propositions 4.3.16, 4.3.18 and 4.3.19 we have $G_g \precsim T_g^- \precnsim T_g^+$. Since G^g is moreover an initial segment of G^g, it follows that an element $h \in G$ is in G^g if and only if there exists $f \in T_g^+$ with $h \precsim f$. Let $h_1, h_2 \in G^g$. There exists $f \in T_g^+$ with $\{h_1, h_2\} \precsim f$. Assume $h_2 \sim f$. By convexity of T_g^+, this implies that $h_2 \in T_g^+$. We then have $h_1 \precsim h_2$. By (CQ_2), this implies $h_1 h_2^{-1} \precsim h_2^{-1} \precnsim h_2 \in T_g^+$, hence $h_1 h_2^{-1} \in G^g$. Assume $h_2 \precnsim f$. By Proposition 4.1.14, this implies $h_1 h_2^{-1} \precsim f h_2^{-1}$. If $f h_2^{-1} \precsim f^{-1}$, then $h_1 h_2^{-1} \precsim f \in T_g^+$, hence $h_1 h_2^{-1} \in G^g$. Assume then that $f^{-1} \precnsim f h_2^{-1}$. By Lemma 4.1.11 (i), $h_2 \precnsim f$ implies $f^{-1} \sim h_2 f^{-1}$, hence $h_2 f^{-1} \precnsim f h_2^{-1}$ which means that $f h_2^{-1}$ is o^+-type. Since $f \in T_g^+$, we have $f^{-1} \sim g^{-1}$ by Lemma 4.3.15. We thus have $g^{-1} \sim h_2 f^{-1}$ and $f h_2^{-1}$ is o^+-type. By Lemma 4.3.15, we then have $f h_2^{-1} \in T_g^+$. Since $h_1 h_2^{-1} \precsim f h_2^{-1}$, it follows that $h_1 h_2^{-1} \in G^g$. This proves that G^g is a subgroup of G. Now let us show that G_g is normal in G^g. Let $h \in G_g$ and $z \in G^g$. By (CQ_3), we have $\{(h^{-1})^z, h^z\} \precsim (g^{-1})^z$. It is enough to show that $(g^{-1})^z \precsim g^{-1}$. Note that by (CQ_3), conjugation preserves types, so $(g^{-1})^z$ is o^--type. Since G^g is a group, we have $(g^{-1})^z \in G^g$, and since there is no o^--type element above g^{-1} in G^g we must have $(g^{-1})^z \precsim g^{-1}$.

Now let us prove that the C-q.o. induced on G^g/G_g is order-type. By Proposition 4.3.8, the canonical projection π_{G_g} from G^g to G^g/G_g preserves the type of elements. Moreover, by definition of G^g, we have $\pi_{G^g}(T_g) = G^g/G_g \backslash \{1.G_g\}$. It then follows from Proposition 4.3.16 that $(G^g/G_g, \precsim)$ satisfies the conditions of Proposition 4.1.21, so $\precsim$ is order-type on G^g/G_g. Now assume that $H \subsetneq G_g$ is another strictly convex normal subgroup of G^g. Take $h \in G_g \backslash H$. By Proposition 4.3.18, we either have $h \in \mathcal{W}_{g^{-1}}$ or $h \precnsim g^{-1}$. If $h \in \mathcal{W}_{g^{-1}}$, then $h \in \mathcal{V}$. Since π_H preserves the types of elements, $\pi_H(h)$ is also v-type. But then condition (1) of Proposition 4.1.21 fails for $(G^g/H, \precsim)$. If $h \precnsim g^{-1}$, then condition (2) of Proposition 4.1.21 fails for $(G^g/H, \precsim)$. In any case, this shows that $\precsim$ is not order-type on G^g/H. □

T_g in the v-type case

Assume now that $g \neq 1$ is v-type.

Lemma 4.3.20
We have $g \in T_g$.

Proof. It follows from Proposition 4.3.3. □

Lemma 4.3.21
Let $h \in \mathcal{O}$. Then either $T_h \precnsim T_g$ or $T_g \precnsim T_h$ holds.

Proof. By Proposition 4.3.11, T_h is right-convex and contains an o$^+$-type element f. Assume first that $g \precsim h$ and let $g' \in T_g, h' \in T_h$. Assume for a contradiction that $h' \precsim g'$. Since $T_g \subseteq \mathcal{V}$ and $T_h \subseteq \mathcal{O}$, we have $g' \notin T_h$. By right-convexity of T_h, $g \precsim h$ implies $g \precsim T_h$ and $h' \precsim g'$ implies $T_h \precnsim g'$. Therefore, we have $g \precsim f \precsim g'$. But since $f \in \mathcal{O}^+$, this contradicts the fact that $g' \in T_g$. Therefore, we must have $g' \precsim h'$. Since this holds for arbitrary $g' \in T_g$ and $h' \in T_h$, this means $T_g \precsim T_h$.

Now assume that $h \precsim g$ and let $g' \in T_g, h' \in T_h$. Assume for a contradiction that $g' \precsim h'$. By right-convexity of T_h, we have $g' \precsim T_h \precnsim g$, hence $g' \precsim f \precnsim g$, which contradicts $g' \in T_g$. This proves that $h' \precnsim g'$. This proves $T_h \precnsim T_g$.

□

Proposition 4.3.22
Define $F_g := \{h \in \mathcal{O}^- \mid h \sim \max(T_g)\}$ if $\max T_g \neq \varnothing$ and $F_g := \varnothing$ otherwise. Then T_g is left-convex with convexity complement F_g. In particular, T_g is convex if it has no maximum.

Proof. Assume T_g is not convex. Then there exists $h_1, h_2 \in T_g$ and $f \notin T_g$ such that $h_1 \precsim f \precsim h_2$. If f were v-type, then since $f \notin T_g$ there would an o^+-type element between g and f. This would imply that there is an o^+-type element either between g and h_1 or between g and h_2, which is a contradiction. For the same reason f cannot be o^+-type. Thus, $f \in \mathcal{O}^-$. It follows Lemma 4.3.21 that $T_g \precsim f$, so $h_2 \precsim f$, hence $h_2 \sim f$. It follows that $h_2 \in \max(T_g)$. Now let us show that $T_g \cup \mathrm{cl}(h_2)$ is convex. Let $f_1, f_2 \in T_g \cup \mathrm{cl}(h_2)$ and $f_1 \precsim f \precsim f_2$. With the same reasoning as above, f cannot be o^+-type so it must either be v-type or o^--type. If it is v-type, then $f \in T_g$. If it is o^--type, then $f \sim h_2$. □

We can now state a v-type analogue of Proposition 4.3.11:

Proposition 4.3.23
The set T_g is the biggest strictly convex subset of $G \backslash \{1\}$ containing g such that every element of T_g is v-type. If G is welding-free, then T_g is even convex.

Proof. Let $S \supsetneq T_g$ be strictly convex and let $h \in S \backslash T_g$ with $h \neq 1$ be v-type. Since $h \notin T_g$, then by definition of T_g there must be an o^+-type element f between g and h. By Proposition 4.3.3, we have $f \nsim h$ and $f \nsim g$, so f is strictly between g and h. Since S is strictly convex, it follows that $f \in S$. Thus, S must contain o-type elements. □

We now want to establish the v-type analogue of Proposition 4.3.13.

Proposition 4.3.24
Both G^g and G_g are subgroups of G, G^g is strictly convex in G with convexity complement F_g and G_g is convex in G. Moreover, G_g is normal in G^g.

Proof. G_g is clearly an initial segment by definition, so it is convex. Moreover, we know that T_g is left-convex (Proposition 4.3.22) and that there is no element strictly contained between G_g and T_g (by definition of G_g), so it follows immediately that $G^g = G_g \cup T_g$ is left-convex. We also know that F_g is the convexity complement of T_g (Proposition 4.3.22) so it is also the convexity complement of G^g.

Let us show that G_g is a group. Let $f_1, f_2 \in G_g$ and $h \in T_g$, so $\{f_1, f_2\} \precnsim h$. Assume $h \precsim f_1 f_2^{-1}$. We then have $f_1 \nsim f_1 f_2^{-1}$. Applying Proposition 4.1.14, we get $f_1^{-1} h \precsim f_2^{-1}$. However, by Proposition 4.1.12, we have $f_2^{-1} \precnsim h$ and $f_1^{-1} h \sim h$, so this is a contradiction. Thus, we must have $f_1 f_2^{-1} \precnsim h$. Since h is arbitrary in T_g, this means $f_1 f_2^{-1} \in G_g$. Now let us show that G^g is a group. Let $f_1, f_2 \in G^g$. This implies that there is $h \in T_g$ with $\{f_1, f_2\} \precsim h$. If $h \sim f_2$, then it follows from the left-convexity of T_g that $f_2 \in T_g$, so $f_2 \in \mathcal{V}$ and we have $f_1 \precsim f_2$. By (CQ_2), it follows that $f_1 f_2^{-1} \precsim f_2^{-1} \sim h \in T_g$. If $h \nsim f_2$, then $f_1 \precsim h$ implies $f_1 f_2^{-1} \precsim h f_2^{-1}$ by Proposition 4.1.14. Since $h \in \mathcal{V}$, $f_2 \precnsim h$ implies $h \sim h f_2^{-1}$ by Proposition 4.1.12, hence $f_1 f_2^{-1} \precsim h$. In any case we have $f_1 f_2^{-1} \precsim h$, which means $f_1 f_2^{-1} \in G^g \cup F_g$. We can show with the same reasoning that $f_2 f_1^{-1} \precsim h$. This implies that $f_1 f_2^{-1} \notin F_g$. Indeed, if $f_1 f_2^{-1}$ were in F_g, then it would be o^--type, so we would have $h \sim f_1 f_2^{-1} \precnsim f_2 f_1^{-1}$.

Take $1 \neq h \in G_g$ and $z \in T_g$. Since $h \notin T_g$, there exists an o^+-type element f between h and g. We then have $h^z \precnsim f^z \precnsim g^z \in G^g$, so there is an o^+-type element between h^z and g^z, hence $h^z \in G_g$. □

Proposition 4.3.25
The group G_g is the smallest normal convex subgroup of G^g such that the C-q.o. induced by $\precsim$ on G^g/G_g is valuational.

Proof. By Proposition 4.3.8, π_{G_g} preserves the type of elements. It follows that every element of G^g/G_g is v-type. By Proposition 4.1.16, this implies that $\precsim$ is valuational on G^g/G_g. Now let H be a normal convex subgroup of G^g strictly contained in G_g. Take $h \in G_g \backslash H$. If $h \in \mathcal{V}$, then since $h \notin T_g$, there must be an o$^+$-type element f between h and g. By convexity of H, we have $f \notin H$. This shows that $G_g \backslash H$ must contain an o-type element. It then follows that G^g/H contains an o-type element which is not 1. By Proposition 4.1.16, this shows that $\precsim$ is not valuational on G^g/H. □

Remark 4.3.26: (i) As happens in the o-type case, welding is the only thing preventing T_g and G^g from being convex. If G has no welding point, then we can replace "strictly convex" by "convex" in Proposition 4.3.24.

(ii) In the o-type case as well as in the v-type case, it can happen that G_g and G^g are not normal in G (see Example 4.3.36 below).

Type-valuation

We can now show that the T_g's form a partition of G:

Proposition 4.3.27
The following holds for any $g, h \in G$:
$g \in T_h \Leftrightarrow h \in T_g \Leftrightarrow T_g = T_h \Leftrightarrow T_g \cap T_h \neq \emptyset \Leftrightarrow G_g = G_h \Leftrightarrow G^g = G^h$.

Proof. Assume $g \in T_h$. If h, g are v-type, then we use Proposition 4.3.23. We know that T_g is the biggest strictly convex subset of G containing g whose every element is v-type. Since T_h is strictly convex and only contains v-type elements and $g \in T_h$, it follows that $T_h \subseteq T_g$. This implies $h \in T_g$. By a similar argument, it also follows that $T_g \subseteq T_h$, hence $T_g = T_h$. The case where they are o-type is similar by using Proposition 4.3.11. This proves the first two equivalences. The third one follows immediately: if $T_g \cap T_h \neq \emptyset$, then there is $f \in G$ with $f \in T_g \cap T_h$, which implies $T_g = T_f = T_h$. Assume $T_g = T_h$. In the v-type case we obviously have $G_h = G_g$ by definition of G_g. If they are o^+-type, then $g^{-1} \sim h^{-1}$. But then, for any $f \in G$, $\{f, f^{-1}\} \precsim g^{-1}$ is equivalent to $\{f, f^{-1}\} \precsim h^{-1}$, hence $G_g = G_h$. Assume $G_g = G_h$. Without loss of generality $g \precsim h$. Since $G_g = G_h$, we have $g \notin G_h$. Note that there is no element strictly contained between G_h and T_h (otherwise, there would be an element f with $f \notin G^h = G_h \cup T_h$ and $1 \precnsim f \precnsim h$. This would contradict the fact that G^h is strictly convex). Thus, we have $g \in T_h$, hence $T_h = T_g$, which also implies $G^g = G^h$. Finally, assume $G^g = G^h$, i.e $T_g \cup G_g = T_h \cup G_h$, and let us show $g \in T_h$. Towards a contradiction, assume $g \notin T_h$. We already proved that this implies $h \notin T_g$. Since $T_g \cup G_g = T_h \cup G_h$, we have $g \in G_h$ and $h \in G_g$. It follows that $g \precsim h$ and $h \precsim g$, i.e $h \sim g$. If $h \in \mathcal{V}$ were true, then by definition of G_h in the v-type case we would have $g \precnsim h$, which is a contradiction. For the same reason, we cannot have $g \in \mathcal{V}$. Therefore, we have $g, h \in \mathcal{O}$. By definition of G_g in the o-type case, and since $h \in G_g$, it follows that $\{h, h^{-1}\} \precnsim \{g, g^{-1}\}$. Similarly, since $g \in G_h$, we have $\{g, g^{-1}\} \precnsim \{h, h^{-1}\}$. But these inequalities imply $h \sim h^{-1}$, which contradicts $h \in \mathcal{O}$. □

We have thus reached the goal we announced in the introduction: we showed that G is partitioned into a family of sets on each of which the C-q.o is fundamental-like. Our next objective is to reformulate this statement by showing that $\precsim$ can be obtained by lifting fundamental C-q.o's. To do this we need to define a valuation on G whose fibers are the type-components. We first notice that $\precsim$ naturally induces an order on the set of type-components:

Proposition 4.3.28
Define $\leq^*$ on the set of all type-components by $T_g \leq^* T_h \Leftrightarrow T_g = T_h \vee T_g \precsim T_h$. This is a total order on the set of all type-components of G.

Proof. The fact that $\leq^*$ is total follows from the fact that the type-components are strictly convex and pairwise disjoint. The relation $\leq^*$ is clearly reflexive and transitive, let us prove that is is antisymmetric. If $T_g \precsim T_h \precsim T_g$, then all elements of $T_g \cup T_h$ are equivalent to one another. It follows that h, g must both be v-type. Since $g \sim h$, this implies $T_g = T_h$. □

Remark 4.3.29: If S is a subset of G which contains elements $s, t \in S$ such that $s \lnsim t$, then $S \lnsim S$ does not hold (remember that $S \lnsim T$ means that $s \lnsim t$ for *any* pair $(s,t) \in S \times T$). Hence the condition $T_g = T_h$ does not imply $T_g \lnsim T_h$. Therefore, the condition "$T_g = T_h$" in the definition of $\leq^*$ is essential for reflexivity.

Proposition 4.3.30
Set $\Gamma := \{T_g \mid g \in G\}$ and let $\leq$ be the reverse order of the order $\leq^*$ given in Proposition 4.3.28. We define a valuation on G called the **type-valuation associated to** $\precsim$ by

$$\begin{aligned} v: \quad & G \to (\Gamma, \leq) \\ & g \mapsto T_g. \end{aligned}$$

Proof. Clearly, T_1 is a maximum of $(\Gamma, \leq)$ and $v(g) = v(g^{-1})$ for any $g \in G$. Let $g, h \in G$ with $v(g) \leq v(h)$. By definition of $\leq$, it follows that $h \in G^g$. Since G^g is a group, we then have $gh \in G^g$. This implies $T_{gh} = T_g$ or $T_{gh} \lnsim T_g$, which means $v(g) \leq v(gh)$, hence $\min(v(g), v(h)) \leq v(gh)$. Now let $z \in G$. If $T_h \lnsim T_g$, then in particular $h \lnsim g$, so $h^z \lnsim g^z$. This implies $v(g^z) \leq v(h^z)$. Now assume $T_g = T_h$. If g, h are both v-type, then so are g^z and h^z (this follows from (CQ_3)). Since $h \in T_g$, there is no o^+-type element between g and h. Therefore, by (CQ_3), there cannot be an o^+-type elements between g^z and h^z. This proves $T_{g^z} = T_{h^z}$. The same kind of argument shows $T_g^+ = T_h^+$ in the case where g, h are both o^+-type. If one of them is o^--type, then take their inverse and we are back to the o^+-type case. ◻

Remark 4.3.31: For any $g \in G$, we have $G^g = G^{v(g)}$ and $G_g = G_{v(g)}$, i.e. $G^g = \{h \in G \mid v(h) \geq v(g)\}$ and $G_g = \{h \in G \mid v(h) > v(g)\}$.

4.3.4 Structure theorems

We now want to summarize the results of Section 4.3.3 into a structure theorem of C-q.o.g.'s. We saw that Example 4.3.1(b) is not obtained by lifting fundamental C-q.o.'s. This forces us to introduce another way of constructing C-q.o.'s, which we call welding. We will then show that every C-q.o. is obtained from fundamental C-q.o.'s by lifting and welding.

We now introduce the welding construction. Let $g \in \mathcal{O}^-$, and assume that the maximum M_g of G_g is non-empty. We noted in Proposition 4.3.18 that, if $\mathcal{W}_g \neq \varnothing$, then $M_g = \mathcal{W}_g$, and so, by Proposition 4.3.11(1), we have $M_g \subseteq \mathrm{cl}(g)$. If $\mathcal{W}_g = \varnothing$, then by Proposition 4.3.18 we have $G_g = \{h \in G \mid h \lnsim g\}$. In any case, there is no element strictly between M_g and $\mathrm{cl}(g)$. This means that we can coarsen $\precsim$ by joining the sets $\mathrm{cl}(g)$ and M_g. In other words, we define a coarsening $\precsim_2$ of $\precsim$ by declaring that $h \sim_2 f$ for any $f, h \in M_g \cup \mathrm{cl}(g)$ and $h \precsim_2 f \Leftrightarrow h \precsim f$ whenever $h \notin M_g \cup \mathrm{cl}(g)$ or $f \notin M_g \cup cl(g)$. Note that, in example 4.3.1(b), if we set $g := (-1, 0)$, then we have $G_g = \{0\} \times \mathbb{Z}$ and $M_g = \{0\} \times (\mathbb{Z} \backslash \{0\}) \subseteq \mathrm{cl}(g)$. Therefore, it can happen that $M_g \subseteq \mathrm{cl}(g)$, in which case nothing changes. But if T_g is convex, then by 4.3.18 we have $M_g \cap \mathrm{cl}(g) = \varnothing$, and then $\precsim_2$ is different from $\precsim$. If we apply this coarsening operation simultaneously at each g^z for $z \in G$, then we will obtain a new C-q.o., as the next proposition shows:

Proposition 4.3.32 (Construction by welding)
Let $(G, \precsim)$ be a C-q.o.g. and $g \in \mathcal{O}^-$ such that $M_g := \max(G_g)$ is non-empty. Then for any $z \in G$, $M_{g^z} := \max(G_{g^z})$ is also non-empty. We can then define a coarsening $\precsim_2$ of $\precsim$ by declaring $M_{g^z} \sim_2 g^z$ for every $z \in G$. Moreover, this coarsening is a C-q.o.

Proof. Note that by (CQ_3), we have $g \precnsim g^{-1} \Rightarrow g^z \precnsim (g^{-1})^z = (g^z)^{-1}$, hence $g^z \in \mathcal{O}^-$. The fact that M_{g^z} is non-empty is also a direct consequence of (CQ_3). It also follows from (CQ_3) that $\mathcal{W}_g \neq \emptyset \Leftrightarrow \mathcal{W}_{g^z} \neq \emptyset$. Note also that if $\mathcal{W}_g \neq \emptyset$, then by Proposition 4.3.18 we have $M_g = \mathcal{W}_g$, so we already have $M_g \sim g$. By (CQ_3), this implies $M_{g^z} \sim g^z$ for all $z \in G$. It then follows that $\precsim = \precsim_2$, so there is nothing to prove. Therefore, we can assume without loss of generality that $\mathcal{W}_{g^z} = \emptyset$ for all $z \in G$.

Set $\precsim_1 := \precsim$. We want to show that $\precsim_2$ is a C-q.o. Let $x, y, z \in G$ with $x \precsim_2 y$. If $x \precsim_1 y$, then we have $xy^{-1} \precsim_1 y^{-1}$ and $x^z \precsim_1 y^z$. Since $\precsim_2$ is a coarsening of $\precsim_1$, this implies $xy^{-1} \precsim_2 y^{-1}$ and $x^z \precsim_2 y^z$. Now assume $y \precnsim_1 x$. This can only happen if there is $w \in G$ with $y \in M_{g^w}$ and $x \sim_1 g^w$. Since we assumed that $\mathcal{W}_{g^w} = \emptyset$, it follows that $x \in \mathcal{O}^-$. By maximality of y, we have $y^{-1} \precsim_1 y$. We thus have $\{y, y^{-1}\} \precnsim_1 x \precnsim_1 x^{-1}$. By Lemma 4.1.11(iii), this implies $xy^{-1} \sim_1 x$. By (CQ_2'), $y^{-1} \precnsim_1 y$ would imply $y \precnsim_1 y^2$, which would contradict the maximality of y. It follows that $y \in \mathcal{V}$. We thus have $xy^{-1} \sim_1 g^w$ and $y^{-1} \in M_{g^w}$. By definition of $\precsim_2$, this implies $xy^{-1} \sim_2 y^{-1}$. Moreover, we have $y^z \in M_{g^{wz}}$ and $x^z \sim_1 g^{wz}$, which also implies $x^z \sim_2 y^z$. □

We see that, if we lift a family of fundamental C-q.o.'s as in Proposition 4.2.7 and then apply welding, then the q.o. which we obtain is again a C-q.o. The next theorem states that any C-q.o. is obtained through this process:

Theorem 4.3.33 (Structure theorem of a C-q.o. group)
Let $(G, \precsim)$ be a C-q.o. group. There exists a valuation v on G with value set $\Gamma \cup \{\infty\}$, called the type-valuation associated to $\precsim$, such that the following holds:

(1) For any $\gamma \in \Gamma$, G^γ and G_γ are strictly convex subgroups of $(G, \precsim)$.

(2) The C-q.o. $\precsim_\gamma$ induced by $\precsim$ on $B_\gamma := G^\gamma / G_\gamma$ is a fundamental C-q.o.

(3) If $\gamma \leq \delta$, if $\precsim_\gamma, \precsim_\delta$ are both valuational, then there exists α between γ and δ such that $\precsim_\alpha$ is order-type.

Moreover, the q.o. $\precsim$ can be obtained by lifting the family $(\precsim_\gamma)_{\gamma \in \Gamma}$ to G and then applying (possibly several) weldings if necessary.

Proof. We already defined the type-valuation v in Proposition 4.3.30. Following Remark 4.3.31, we will write G^g and G_g instead of $G^{v(g)}$ and $G_{v(g)}$. We will also write $\precsim_g$ instead of $\precsim_{v(g)}$. (1) and (2) follow from Propositions 4.3.13, 4.3.24 and 4.3.25, (3) follows from 4.3.23. Denote by $\precsim^*$ the lifting of $(\precsim_\gamma)_{\gamma \in \Gamma}$ to G. Note that an element $g \in G$ is v-type (respectively, o$^-$-type) with respect to $\precsim$ if and only if it is v-type (respectively, o$^-$-type) with respect to $\precsim^*$ (this follows easily from Propositions 4.3.13 and 4.3.25 and from the definition of the the lifting). Therefore, we can use the notations $\mathcal{V}$ and $\mathcal{O}$ without ambiguity. We first show that $\precsim$ is a coarsening of $\precsim^*$. Let $g, h \in G$ with $g \precsim^* h$. By definition of $\precsim^*$, we either have $v(h) > v(g)$ or $v(g) = v(h) \wedge gG_g \precsim_g hG_g$. In the first case we have by definition of v: $g \precsim h$. In the second case, since $G_h = G_g$, we have $h \notin G_g$. It then follows from Remark 4.3.9(iii) that $g \precsim h$. This proves that $\precsim$ is a coarsening of $\precsim^*$. Now let $g, h \in G$ be such that $g \precsim h$ but $h \precnsim^* g$. We will show that $h \in \mathcal{V}$, $g \in \mathcal{O}$ and $h \in \max(G_g, \precsim^*)$. It will then follow that $\precsim$ is obtained from $\precsim^*$ by welding g and $\max(G_g, \precsim^*)$. By definition of $\precsim^*$, $h \precnsim^* g$ means either $v(h) > v(g)$ or $v(g) = v(h)$ and $hG_g \precnsim_g gG_g$. By Remark 4.3.9(iii), the latter case would imply $h \precnsim g$, so we must have $v(h) > v(g)$ i.e $h \in G_g$. This implies $h \precsim g$, so $g \sim h$. If h were o-type, then by Proposition 4.3.13 G^h would be convex with respect to $\precsim$. The inequality $g \precsim h$ would then imply $g \in G^h$, which contradicts $v(g) < v(h)$. Therefore, $h \in \mathcal{V}$. Assume for a contradiction that $g \in \mathcal{V}$. Since $v(h) > v(g)$, we have $g \notin T_h$. By definition of T_h, it follows that there is $f \in \mathcal{O}^+$ between g and h. But since $g \sim h$, it follows that $f \sim h$. This contradicts Lemma 4.1.18. Therefore, $g \in \mathcal{O}$. Since $h \sim g$, h is in the convexity complement of T_g. By Proposition 4.3.18, we thus have $h \in \max(G_g, \precsim)$. Now let $f \in G_g$ with $h \precsim^* f$. Since $\precsim$ is a coarsening of $\precsim^*$, we then have $h \precsim f$, hence $h \sim f$ by maximality of h. Now $h \precsim^* f$ implies $v(f) \leq v(h)$ and $f \in G_g$ implies $v(f) > v(g)$. Since $h \in \max(G_g, \precsim)$, there is no element strictly contained between T_h and T_g, so we must have $v(h) = v(f)$. By definition of $\precsim_h$ (see Proposition 4.3.8), it then follows from $h \sim f$ that $hG_h \sim_h fG_h$, hence $h \sim^* f$ by definition of $\precsim^*$. This shows that h is maximal in $(G_g, \precsim^*)$. Thus, the only point on which $\precsim$ and $\precsim^*$ disagree are welding points, so $\precsim$ is obtained from $\precsim^*$ by welding. ◻

Corollary 4.3.34
Let G be a group and $\precsim$ a q.o. on G. Then $\precsim$ is a C-q.o. if and only if it is obtained by lifting a family of fundamental C-q.o.'s with the q.o.-conjugation property and then possibly welding.

Proof. If $\precsim$ is a C-q.o., it follows from Theorem 4.3.33 that $\precsim$ is obtained by lifting a family of fundamental C-q.o.'s and then possibly welding. It then follows from (CQ_3) that this family has the q.o.-conjugation property. The converse follows from Propositions 4.2.7 and 4.3.32. ◻

Remark 4.3.35: (i) The C-q.o.-groups $(B_\gamma, \precsim_\gamma)$ in Theorem 4.3.33 are called the **fundamental components** of $(G, \precsim)$.

(ii) Let $\precsim^*$ denote the lifting of $(\precsim_\gamma)_{\gamma\in\Gamma}$. Then $(G, \precsim^*)$ is welding-free. More precisely, $\precsim^*$ is the unique welding-free C-q.o. which has the same type-valuation as $(G, \precsim)$. $\precsim^*$ is called the **unwelding** of $\precsim$.

(iii) If $(G, \precsim)$ is welding-free, then $\precsim$ and $\precsim^*$ coincide.

(iv) In Theorem 4.3.33, Γ becomes a colored chain with two colors $\mathcal{O}$ and $\mathcal{V}$: say $\gamma \in \mathcal{V}$ (respectively, $\gamma \in \mathcal{O}$) if $g \in \mathcal{V}$ (respectively, $g \in \mathcal{O}$) for all $g \in G$ with $v(g) = \gamma$. By (3), this colored chain satisfies the condition:

$$(\gamma \in \mathcal{V} \wedge \delta \in \mathcal{V} \wedge \gamma < \delta) \Rightarrow \exists \alpha, (\gamma < \alpha < \delta \wedge \alpha \in \mathcal{O}). \tag{CC}$$

The two-colored chain $(\Gamma, \leq, \mathcal{O}, \mathcal{V})$ is called the **type-chain of the C-q.o. group** $(G, \precsim)$.

(v) condition (CC) says that Γ cannot have two consecutive v-type elements. However, example 4.3.1(b) shows that Γ can have two consecutive o-type elements.

Example 4.3.36
We take notations from Examples 4.3.1. We are going to give an explicit definition to the type-valuation associated to the C-q.o's $\precsim_H$ and $\precsim_F$ of examples (d) and (e). We already defined a valuation $w_H : H \to \mathbb{Z}$ on H. Define the valuation v_G on G by $v_G(a,b) = \begin{cases} 1 & \text{if } a \neq 0. \\ 2 & \text{if } a = 0 \neq b. \\ \infty & \text{if } a = b = 0. \end{cases}$ Now set $\Gamma := \mathbb{Z} \times \{1,2\}$ and order Γ lexicographically, i.e $(x,y) \leq (x',y') \Leftrightarrow (x < x' \vee (x = x' \wedge y \leq y'))$. Define $v_H : H \to \Gamma$ by $v(\sum_{n\in\mathbb{Z}} g_n) := (k, v_G(g_k))$, where $k = w_H(\sum_{n\in\mathbb{Z}} g_n)$. Then v_H is a valuation on H such that, for any $g \in G$, $T_g = \{h \in H \mid v_H(h) = v_H(g)\}$. If we assimilate an element γ of Γ with $v_H^{-1}(\{\gamma\})$, it follows that v_H is the type-valuation associated to $\precsim_H$. Now we extend v_H to a valuation $v_F : F \to \Gamma \cup \{a, \infty\}$, where a is a new element such that $a < \Gamma$, as follows: $v_F(k,h) = \begin{cases} a \text{ if } k \neq 0. \\ v_H(h) \text{ if } k = 0. \end{cases}$
If we assimilate elements of $\Gamma \cup \{a\}$ with their v_F-fiber, then v_F is the type-valuation associated to $\precsim_F$. Now take $z := (-1, \sum_{n\in\mathbb{Z}}(0,0)) \in F$ and $f := (0, \sum_{n\in\mathbb{Z}} g_n) \in F$, where $g_0 = (1,0)$ and $g_n = (0,0)$ for $n \neq 0$. We have $v_F(f) = (0,1)$ but $v_F(z+g-z) = (-1,1) < v_F(f)$. In particular, $F^{(0,1)} = F_{(-1,2)}$ is not normal in F. This shows that the groups G^γ and G_γ of theorem 4.3.33 are not always normal in G.

We can also reformulate Theorem 4.3.33 in terms of C-relations:

Theorem 4.3.37
Let (G, C) be a C-group. There exists a valuation $v : G \to \Gamma \cup \{\infty\}$ such that the following holds:

(1) For any $\gamma \in \Gamma$, C induces a C-relation C_γ on the quotient G^γ/G_γ defined by the formula $C_\gamma(fG_\gamma, gG_\gamma, hG_\gamma) \Leftrightarrow fh^{-1} \notin G_\gamma \wedge (gh^{-1} \in G_\gamma \vee C(f,g,h))$.

(2) For each $\gamma \in \Gamma$, C_γ is a fundamental C-relation.

(3) If $\gamma \leq \delta$, if C_γ, C_δ are both valuational, then there exists α between γ and δ such that C_α is order-type.

Finally, we show that any two-colored chain satisfying condition (CC) of Remark 4.3.35 can be realized as the type-chain of some C-q.o. group.

Proposition 4.3.38
Let $(\Gamma, \leq, \mathcal{O}, \mathcal{V})$ be a two-colored chain satisfying condition (CC). Then there exists a C-q.o. group $(G, \precsim)$ such that $(\Gamma, \leq, \mathcal{O}, \mathcal{V})$ is the type-chain of $(G, \precsim)$.

Proof. For every $\gamma \in \Gamma$, let $(B_\gamma, \precsim_\gamma)$ be :

1. A valuationally quasi-ordered group if $\gamma \in \mathcal{V}$.
2. A group endowed with an order-type C-q.o. if $\gamma \in \mathcal{O}$.

Now let $(G, \precsim)$ be the valuational product of the family $((B_\gamma, \precsim_\gamma))_{\gamma\in\Gamma}$. It follows from Proposition 4.2.10 that $(G, \precsim)$ is a C-q.o. group. By construction, the type-chain of $(G, \precsim)$ is $(\Gamma, \leq, \mathcal{O}, \mathcal{V})$. □

4.4 C-minimality

The object of this section is to describe C-minimal groups. More precisely, we want to characterize C-minimal groups in term of their type-valuations and of their fundamental components. Roughly speaking, we prove that, at least in the abelian case, welding-free C-minimal groups are obtained as a finite valuational product of C-minimal fundamental C-groups.

It will be convenient to work with a weaker notion of minimality:

Definition 4.4.1
Given a C-group (G, C), we say that (G, C) is **weakly C-minimal** if every subset of G definable with parameters in the language $\{1, ., ^{-1}, C\}$ is quantifier-free definable with parameters in the language $\{C\}$.

Therefore, (G, C) is C-minimal if and only if every C-group which is elementarily equivalent to (G, C) is weakly C-minimal. We will use the following characterization of C-minimality:

Proposition 4.4.2
Let (G, C) be a C-group. Then (G, C) is C-minimal if and only if there exists a weakly C-minimal ω-saturated C-group which is elementarily equivalent to (G, C).

Proof. If (G, C) is C-minimal, then any ω-saturated elementary extension of (G, C) is C-minimal. Conversely, assume that there exists a weakly C-minimal ω-saturated C-group (H, C) with $(G, C) \equiv (H, C)$. Let (F, C) be a C-group with $(F, C) \equiv (G, C)$ and let us show that (F, C) is weakly C-minimal. Let $\bar{a} \subseteq F$ and let $A \subseteq F$ be defined by the formula $\phi(x, \bar{a})$. By saturation, there exists a partial elementary embedding $\iota : \bar{a} \hookrightarrow H$. Because H is weakly C-minimal, there exists a quantifier-free formula $\psi(x, \bar{y})$ in the language $\{C\}$ such that $H \vDash \exists\bar{y}\forall x(\phi(x, \iota(\bar{a})) \Leftrightarrow \psi(x, \bar{y}))$. Since ι is a partial elementary embedding, we also have $F \vDash \exists\bar{y}\forall x(\phi(x, \bar{a}) \Leftrightarrow \psi(x, \bar{y}))$. This proves that A is quantifier-free definable in $\{C\}$. □

Given a C-group (G, C) and $\precsim$ the induced C-q.o., it clearly follows from Definition 4.1.2 and Proposition 4.1.4 that $\precsim$ is quantifier-free definable in $\{1, C\}$ and that C is quantifier-free definable in $\{., ^{-1}, \precsim\}$. This allows us to use C-q.o.'s to study model-theoretic properties of C-groups. Therefore, we will study C-groups in the language

$\mathcal{L} := \{1, ., ^{-1}, \precsim\}$, where $\precsim$ will be interpreted by a C-q.o. If $A \subseteq G$, then $\mathcal{L}_A$ denotes the language $\mathcal{L}$ to which we add parameters in A. For us, "definable" means "definable in $\mathcal{L}$ with parameters". If we want to say that a set is definable in $\mathcal{L}$ without parameters, then we say that it is $\emptyset$-definable. If $\phi(x, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{a} \subseteq G$, then we denote by $\phi(G, \bar{a})$ the set $\{g \in G \mid G \vDash \phi(g, \bar{a})\}$. All structures considered in this section are C-q.o. groups. When there is no ambiguity on the C-q.o., we write $G \equiv H$ to mean that G and H are elementarily equivalent as $\mathcal{L}$-structures. If F is a strictly convex normal subgroup of G, then the quotient G/F is always endowed with the C-q.o. induced by the C-q.o. of G (see Proposition 4.3.8). If $(G_1, \precsim_1), \ldots, (G_n, \precsim_n)$ is a family of C-q.o. groups, then the direct product $\prod_{i=1}^n G_i$ will always be endowed with the valuational product of the family $(\precsim_i)_{i\in\{1,\ldots,n\}}$ (see Definition 2.7.14). Note that, if $(F, \precsim_F), (H, \precsim_H)$ are C-q.o. groups and $G := H \times F$, then F is convex in $(G, \precsim)$, we have $H \cong G/F$, and the C-q.o. induced by $\precsim$ on G/F coincides with $\precsim_H$.

Note that every atomic formula in $\mathcal{L}$ is equivalent to a formula of the form $P(\bar{x}) \precsim Q(\bar{x})$, where $P(\bar{x}), Q(\bar{x})$ are terms of the language of groups. Indeed, by (CQ_1), the formula $P(\bar{x}) = 1$ is equivalent to $P(\bar{x}) \precsim 1$. Remember that equality between formulas is denoted by "$\equiv$". For $g \in G$, we keep the notations T_g, G_g, G^g defined in Section 4.3.3.

4.4.1 Macpherson and Steinhorn's results revisited

We first want to interpret the results on C-minimal groups given in [MS96] in view of our structure theorem 4.3.33. Note that the C-relations considered in [MS96] are dense, i.e they satisfy the extra axioms: $x \neq y \Rightarrow \exists z, (z \neq y \wedge C(x, y, z))$ and $\exists x \exists y, y \neq x$. The authors of [Del11] and [AN98] described how to obtain the canonical tree associated to a given C-structure (see Proposition 1.5 in [Del11] and Theorem 12.4 in [AN98]). If (M, C) is a C-structure, then we can define a partial quasi-order (i.e. a reflexive and transitive binary relation, not necessarily total) $\precsim$ on the set M^2 by $(x, y) \precsim (u, v) \Leftrightarrow \neg C(u, x, y) \wedge \neg C(v, x, y)$. We then define the canonical tree $(\mathfrak{T}, \leq)$ of (M, C) as the quotient $\mathfrak{T} := M^2/\sim$ endowed with the partial order $\leq$ induced by $\precsim$. To simplify notations, we will refer to elements of $\mathfrak{T}$ by one of their representatives in M^2. Note that $(x, y) = (y, x)$ for any x, y.

If (G, C) is a C-group with canonical tree $\mathfrak{T}$, then we see that G induces a right action on $\mathfrak{T}$ by $(x, y).g := (xg, yg)$. Note that the partial order on $\mathfrak{T}$ is compatible with this action in the sense that $(x, y) \leq (u, v) \Rightarrow (x, y).g \precsim (u, v).g$ (this follows directly from the fact that C is compatible). In [MS96], Macpherson and Steinhorn described dense C-minimal groups by looking at the orbits of this action. They distinguished three cases:

1. All orbits are antichains.
2. One orbit is a non-trivial chain.
3. No orbit is a non-trivial chain and there exists one non-trivial orbit which is not an antichain.

Now let $\precsim$ be the C-q.o. associated to C. We want to translate this trichotomy into the language of C-q.o. groups. More precisely, we want to see how the type of elements

x and y influences the orbit of (x, y). Note that the partial order $\leq$ of $\mathfrak{T}$ is given by $(x, y) \leq (u, v) \Leftrightarrow \{uy^{-1}, vy^{-1}\} \precsim xy^{-1}$. We first want to describe the structure of the tree $\mathfrak{T}$ in the order-type case:

Lemma 4.4.3
Assume $(G, \precsim)$ is an order-type C-q.o.g. and set $\mathcal{C} := \{(x, y) \in \mathfrak{T} \mid x \neq y\}$. Then $\mathcal{C}$ is a non-trivial chain and an orbit under the action of G.

Proof. Denote by $\leq$ the underlying order on G. Let $(x, y), (u, v) \in \mathcal{C}$. Note that since $(x, y) = (y, x)$, we can assume that $x < y$ and $u < v$. We have $(x, y) \leq (u, v) \Leftrightarrow \{uy^{-1}, vy^{-1}\} \precsim xy^{-1}$. Because $\mathcal{O}^+ \cup \{1\}$ is the positive cone of $(G, \leq)$ (see Proposition 4.1.19), $x < y$ is equivalent to $xy^{-1} \in \mathcal{O}^-$. Since $\precsim$ is trivial on $\mathcal{O}^-$ (see Proposition 4.1.21), $\{uy^{-1}, vy^{-1}\} \precsim xy^{-1}$ is equivalent to $uy^{-1}, vy^{-1} \in \mathcal{O}^- \cup \{1\}$. This in turn is equivalent to $u \leq y \wedge v \leq y$. Since $u < v$, this is equivalent to $v \leq y$. Thus, we have $(x, y) \leq (u, v) \Leftrightarrow v \leq y$ and it follows that $\mathcal{C}$ is a chain. Note that it also shows:
$(*)$ $(u < v \wedge x < y \wedge y = v) \Rightarrow (x, y) = (u, v)$.
Now we want to show that (x, y) and (u, v) are in the same orbit. Set $g := y^{-1}v$. Note that by definition of order-type C-relations in Example 2.6.1(a), $<$ is compatible with the group operation, so we have $xg < yg$. Moreover, we have $u < v$ and $yg = v$. By $(*)$, this implies that we have $(x, y).g = (u, v)$. $\square$

Lemma 4.4.4
Let $(G, \precsim)$ be a C-q.o.g. (not necessarily minimal) and $g \in G$. Let $(\mathfrak{T}, \leq)$ be the canonical tree associated to G^g and $(\mathfrak{T}', \leq')$ the canonical tree associated to G^g/G_g. If $x, y, u, v \in T_g$ are such that $xy^{-1}, uv^{-1} \in T_g$, then $(x, y) \leq (u, v)$ if and only if $(xG_g, yG_g) \leq' (uG_g, vG_g)$.

Proof. Since $xy^{-1} \notin G_g$, it follows from Remark 4.3.9(iii) that $\{uy^{-1}, vy^{-1}\} \precsim xy^{-1}$ if and only if $\{uy^{-1}G_g, vy^{-1}G_g\} \precsim xy^{-1}G_g$. $\square$

Lemma 4.4.5
Let $(G, \precsim)$ be a C-q.o.g. Let $x \in G$ and $y \in G^x$. The following holds:

(i) If $xy^{-1} \in \mathcal{V}$, then the orbit of (x, y) under the action of G is an antichain.

(ii) If $xy^{-1} \notin T_x$, then the orbit of (x, y) under the action of G^x is not a chain.

(iii) The orbit of (x, y) under the action of G^x is a non-trivial chain if and only if $x \in \mathcal{O}$ and $xy^{-1} \in T_x$.

Proof. (i) Assume that xy^{-1} is v-type and let $g \in G$. We want to show that (x, y) and (xg, yg) are either incomparable or equal. Assume $(xg, yg) \leq (x, y)$. This means $\{xg^{-1}y^{-1}, yg^{-1}y^{-1}\} \precsim xy^{-1}$. Since xy^{-1} is v-type, $yg^{-1}y^{-1} \precsim xy^{-1}$ implies $ygy^{-1} \precsim xy^{-1}$ (indeed, if $yg^{-1}y^{-1}$ is v-type, then $ygy^{-1} \sim yg^{-1}y^{-1}$. If $yg^{-1}y^{-1}$ is o-type, then we have $yg^{-1}y^{-1} \in G_{xy^{-1}}$. Since $G_{xy^{-1}}$ is a group, this implies $ygy^{-1} \in G_{xy^{-1}}$, hence $ygy^{-1} \precsim xy^{-1}$). Moreover, if we conjugate the inequality $yg^{-1}y^{-1} \precsim xy^{-1}$ by xy^{-1}, then we obtain $xgx^{-1} \precsim xy^{-1} \sim yx^{-1}$. By (CQ_2),

$xgx^{-1} \precsim yx^{-1}$ implies $xgy^{-1} \precsim xy^{-1}$. Thus, we have $\{xgy^{-1}, ygy^{-1}\} \precsim xy^{-1}$, which means $(x, y) \leq (xg, yg)$, so (x, y) and (xg, yg) are equal. Now if we assume that $(x, y) \leq (x, y).g$ instead of $(x, y).g \leq (x, y)$ at the beginning, then by compatibility of the action we have $(x, y).g^{-1} \leq (x, y)$, which brings us back to the previous case.

(ii) Assume $xy^{-1} \notin T_x$. Since $x \notin G_x$ and $x, y^{-1} \in G^x$, it follows that $y \notin G_x$, hence $y \in T_x$. We thus have $xy^{-1} \precnsim \{x, y^{-1}\}$. Taking $g := y^{-1}$, we cannot have $ygy^{-1} \precsim xy^{-1}$ and we also cannot have $xg^{-1}y^{-1} \precsim xy^{-1}$. Therefore, neither $(x, y) \leq (xg, yg)$ nor $(xg, yg) \leq (x, y)$ is true.

(iii) If the orbit of (x, y) under G^x is a chain, then by (ii) we must have $xy^{-1} \in T_x$. By (i), x cannot be v-type. Conversely, assume x is o-type with $xy^{-1} \in T_x$. Since G^x/G_x is order-type (Proposition 4.3.13), and since $xG_x \neq yG_x$, it follows from Lemma 4.4.3 that the orbit of (xG_x, yG_x) under the action of G^x/G_x is a non-trivial chain. It then follows from lemma 4.4.4 that the orbit of (x, y) under G^x is also a non-trivial chain.

□

Proposition 4.4.6
Let $(G, \precsim)$ be a C-q.o.g. with type-valuation $v : G \to (\Gamma \cup \{\infty\}, \leq)$ The following holds:

(1) All orbits of $\mathfrak{T}$ are antichains if and only if $G = \mathcal{V}$.

(2) There exists an orbit of $\mathfrak{T}$ which is a chain if and only if there exists $x \in \mathcal{O}$ such that $v(x) = \min(\Gamma)$.

Proof. If every orbit is an antichain, then by Lemma 4.4.5(iii) every element of G must be v-type (otherwise we can always choose $x, y \in \mathcal{O}$ with $xy^{-1} \in T_x$, for example choose any $x \in \mathcal{O}^+$ and $y := x^2$). The converse follows from 4.4.5(i). Now assume that $x \in \mathcal{O}$ and $v(x) = \min(\Gamma)$. Take $y \in T_x$ with $xy^{-1} \in T_x$. It follows from Lemma 4.4.5(iii) that the orbit of (x, y) under G is a chain. Conversely, assume there is an orbit of an element (x, y) which is a chain. Since $(x, y) = (y, x)$, we can assume without loss of generality that $y \precsim x$, hence $y \in G^x$. By Lemma 4.4.5(iii), this implies in particular that x is o-type with $xy^{-1} \in T_x$. Assume that there is some $g \notin G^x$. We then have $xgy^{-1}, ygy^{-1}, xg^{-1}y^{-1}, yg^{-1}y^{-1} \notin G^x$, so neither $(x, y) \leq (xg, yg)$ nor $(xg, yg) \leq (x, y)$ can be true. This contradicts the fact the the orbit of (x, y) is a chain. Therefore, $G^x = G$, which means $v(x) = \min(\Gamma)$. □

Proposition 4.4.6 allows us to translate Macpherson and Steinhorn's approach into the language of C-q.o. groups. The case where all orbits of $\mathfrak{T}$ are antichains corresponds to the case where $\precsim$ is valuational. The case where one orbit is a chain corresponds to a C-q.o. group having an order-type-like final segment, i.e. the case where Γ has a minimum γ such that $x \in \mathcal{O}$ for every $x \in G$ with $v(x) = \gamma$. Finally, the case where no orbit is a chain and some orbits are not antichains corresponds to the case where G has a valuational-like final segment and an order-type-like part beneath it. This last case means that Γ has a minimum γ such that $x \in \mathcal{V}$ for every $x \in G$ with $v(x) = \gamma$, but

G also contains some o-type elements. Our theory of C-q.o. groups then allows us to reinterpret the theorems of [MS96] on C-minimal groups:

Theorem 4.4.7 ([MS96, Theorems 4.4, 4.8 and 4.9])
Let $(G, \precsim)$ be a C-minimal C-q.o.g. and assume that C is a dense C-relation. Let $v : G \to (\Gamma \cup \{\infty\}, \leq)$ be the type-valuation of $(G, \precsim)$. Then exactly one of the following holds:

(1) $\precsim$ comes from a valuation $w : G \to \Psi \cup \{\infty\}$. In that case, we have the following:

   (i) For any $\lambda \in \Psi$, G_λ and G^λ are normal in G.

   (ii) The quotient G^λ / G_λ is abelian for all but finitely many $\lambda \in \Psi$.

   (iii) If G^λ / G_λ is infinite, then it is elementary abelian or divisible abelian. If it is divisible, then G^λ is also abelian.

   (iv) There is a definable abelian subgroup H of G such that G/H has finite exponent.

(2) There exists $g \in \mathcal{O}$ such that $v(g) = \min(\Gamma)$. In that case G is abelian and divisible, G_g is C-minimal and G^g / G_g is o-minimal.

(3) $G \neq \mathcal{V}$ and for all $g \in \mathcal{O}$, $v(g) \neq \min(\Gamma)$. In that case there exists $g \in \mathcal{V}$ with $v(g) = \min(\Gamma)$, and there is a definable subgroup H of G such that G/H has finite exponent.

Proof. Cases (1) and (2) are direct reformulations of [MS96, theorems 4.4 and 4.8] using our Proposition 4.4.6. For (3), we know from Proposition 4.4.6 and from [MS96, Theorem 4.9] that there exists $s := (g, 1) \in \mathfrak{T}$, $g \in G$, such that for any $t \leq s$, the orbit of t under G is an antichain. Since $g.1^{-1} \in T_g$, and since the orbit of s under G is an antichain, Lemma 4.4.5(iii) implies that g is v-type. Now let $u \in G$ with $g \precsim u$. We have $\{1, g\} \precsim u$. By definition of $\leq$, this implies $(u, 1) \leq (g, 1) = s$, so the orbit of $(u, 1)$ under G is an antichain. Since moreover $u \in T_u$, it follows from Lemma 4.4.5(iii) that u is v-type. □

Remark 4.4.8:

1. Theorem 4.4.7 shows in particular that, if G is C-minimal, then Γ has a minimum. Thus, the “ordered” parts cannot alternate indefinitely with the “valued” parts. Eventually, the group has to either stay valuational-like or stay order-type-like.

2. Theorem 4.4.7 leaves open the question of welding in the case of C-minimality. More precisely, it does not say if it is possible to have welding in case (ii).

The rest of this chapter is devoted to improving Theorem 4.4.7, i.e. giving a complete description of C-minimal groups in terms of their type-valuation and of their fundamental components. We will use a different approach than the one in [MS96]: instead of considering the canonical tree of a structure and the action induced by the group on the tree, we will use the theory of C-q.o.'s developed in this thesis, and in particular our theorem 4.3.33. Unlike the authors of [MS96], we will not restrict to dense C-relations.

4.4.2 A "Feferman-Vaught" theorem

This section is the C-q.o. analog of Section 3.3.3. More precisely, we want to see how elementary equivalence behaves with respect to quotients and valuational products. In the spirit of Theorem 3.3.13 for compatible q.o.a.g.'s, we prove a "Feferman-Vaught" theorem for C-q.o. groups, i.e. we prove that the valuational product of finitely many C-q.o. groups preserves elementary equivalence. Note that the proofs are similar to those of Section 3.3.3.

Lemma 4.4.9
Let $(G, \precsim)$ be a C-q.o. group, F a $\emptyset$-definable strictly convex normal subgroup of G and $H := G/F$. Let $\theta(x)$ be a formula defining F. Let $\phi(\bar{x})$ be a formula of $\mathcal{L}$. Then there exists two formulas $\phi^F(\bar{x}), \phi^H(\bar{x})$ in $\mathcal{L}$, each of the same arity as ϕ, such that, if $G' \equiv G$, if $F' = \theta(G')$ and $H' = G'/F'$, then we have:

(i) For any $\bar{f} \subseteq F'$, $F' \vDash \phi(\bar{f})$ if and only if $G' \vDash \phi^F(\bar{f})$.

(ii) For any $\bar{h} \subseteq H'$, $H' \vDash \phi(\bar{h})$ if and only if for all $\bar{g} \subseteq G'$, $\bar{g}F' = \bar{h} \Rightarrow G' \vDash \phi^H(\bar{g})$ if and only if there exists $\bar{g} \subseteq G'$ with $\bar{g}F' = \bar{h}$ and $G' \vDash \phi^H(\bar{g})$.

Proof. For (i): write ϕ in prenex form: $\phi(\bar{x}) \equiv Q_1 y_1 \dots Q_n y_n \psi(\bar{y}, \bar{x})$, where each Q_i is a quantifier and ψ is quantifier-free. Since F is $\emptyset$-definable in G, we can define the formula $\phi^F(\bar{x}) \equiv Q_1 y_1 \in F \dots Q_n y_n \in F \psi(\bar{y}, \bar{x})$, and it is then easy to see that ϕ^F has the desired property.

For (ii): We proceed by induction on ϕ. Assume first that ϕ is atomic: $\phi(\bar{x}) \equiv P(\bar{x}) \precsim Q(\bar{x})$. Define $\phi^H(\bar{x})$ as $\theta(P(\bar{x})) \vee (\neg\theta(Q(\bar{x})) \wedge \phi(\bar{x}))$. By definition of $\precsim$ on H' (see Proposition 4.3.8), this formula satisfies the desired condition. Assume now that $\phi \equiv \neg\psi$ and set $\phi^H :\equiv \neg\psi^H$. If $H' \vDash \phi(\bar{h})$, then $H' \nvDash \psi(\bar{h})$, so by induction hypothesis we have $G' \nvDash \psi^H(\bar{g})$ for all $\bar{g} \subseteq G'$ with $\bar{g}F' = \bar{h}$, hence $G' \vDash \phi^H(\bar{g})$. Conversely, if there is $\bar{g} \subseteq G'$ with $\bar{g}F' = \bar{h}$ and $G' \vDash \phi^H(\bar{g})$, then $G' \nvDash \psi^H(\bar{g})$ which by induction hypothesis means $H' \nvDash \psi(\bar{h})$ hence $H' \vDash \phi(\bar{h})$. If $\phi \equiv \phi_1 \wedge \phi_2$, one can easily show that $\phi^H :\equiv \phi_1^H \wedge \phi_2^H$ satisfies the desired property and if $\phi \equiv \exists y \psi(y, \bar{x})$, it is also easy to see that $\phi^H \equiv \exists y \psi^H(y, \bar{x})$ is suitable. □

Remark 4.4.10: If $A \subseteq G$, and if F is A-definable, then we can find $\phi^F(\bar{x}), \phi^H(\bar{x}) \in \mathcal{L}_A$ satisfying the conditions of Lemma 4.4.9 for $G' = G$ (the proof is the same).

Proposition 4.4.11
Let $(G_1, \precsim_1)$ and $(G_2, \precsim_2)$ be two C-q.o.g.'s such that $G_1 \equiv G_2$. Let F_1 be a $\emptyset$-definable strictly convex normal subgroup of G_1 and F_2 the strictly convex normal subgroup of G_2 defined by the same formula as F_1. Then we have $F_1 \equiv F_2$ and $G_1/F_1 \equiv G_2/F_2$.

Proof. Set $H_i := G_i/H_i$ and let ϕ be a sentence of $\mathcal{L}$. Take ϕ^F, ϕ^H as in Lemma 4.4.9. If $F_1 \vDash \phi$, then $G_1 \vDash \phi^F$, hence by assumption $G_2 \vDash \phi^F$, hence by choice of ϕ^F: $F_2 \vDash \phi$. We could show similarly that $H_1 \vDash \phi$ implies $H_2 \vDash \phi$, hence $F_1 \equiv F_2$ and $H_1 \equiv H_2$. □

Lemma 4.4.12
Let $\phi(\bar{x})$ be a formula of $\mathcal{L}$. Then there is $n \in \mathbb{N}$ such that there are $2n$ formulas $\phi_1^F(\bar{x}), \ldots, \phi_n^F(\bar{x}), \phi_1^H(\bar{x}), \ldots, \phi_n^H(\bar{x})$, each having the same arity as ϕ, such that the following holds:

For any C-q.o. groups $(H, \precsim)$ and $(F, \precsim)$, for any $\bar{g} = \bar{h}.\bar{f}$ in $G := H \times F$, we have: $G \vDash \phi(\bar{g})$ if and only if there exists $i \in \{1, \ldots, n\}$ such that $F \vDash \phi_i^F(\bar{f})$ and $H \vDash \phi_i^H(\bar{h})$.

Proof. We identify H with G/F. We proceed by induction on ϕ. We first assume that ϕ is an atomic formula: $P(\bar{x}) \precsim Q(\bar{x})$. Set $n = 2$ and define $\phi_1^F(\bar{x}) :\equiv (\bar{x} = \bar{x})$, $\phi_1^H(\bar{x}) :\equiv (Q(\bar{x}) \neq 1 \wedge \phi(\bar{x}))$, $\phi_2^F(\bar{x}) :\equiv \phi(\bar{x})$ and $\phi_2^H(\bar{x}) :\equiv (Q(\bar{x}) = P(\bar{x}) = 1)$. We must check that these formulas satisfy the desired condition. Note that for any $F, H, \bar{g}$ as above, we have $P(\bar{g}) = P(\bar{h})P(\bar{f})$ (because f and h commute) with $P(\bar{f}) \in F$ and $P(\bar{h}) \in H$, and in particular we have $P(\bar{g})F = P(\bar{h})$ and $P(\bar{g}) \in F$ if and only if $P(\bar{h}) = 1$. With this remark in mind, it follows directly from the definition of the valuational product that the formulas $\phi_1^F, \phi_2^F, \phi_1^H, \phi_2^H$ satisfy the condition we want. This settles the case where ϕ is atomic. If $\phi \equiv \psi \vee \chi$, and if $\psi_1^F, \ldots, \psi_k^F, \psi_1^H, \ldots, \psi_k^H, \chi_1^F, \ldots, \chi_l^F, \chi_1^H, \ldots, \chi_l^H$ are the desired formulas for ψ and χ, we simply set $n := k + l$, $\phi_i^F :\equiv \psi_i^F, \phi_i^H :\equiv \psi_i^H$ for $1 \leq i \leq k$ and $\phi_i^F :\equiv \chi_i^F, \phi_i^H :\equiv \chi_i^H$ for $k < i \leq n$. Now assume that $\phi \equiv \exists y \psi(y, \bar{x})$ and let $\psi_1^F, \ldots, \psi_k^F, \psi_1^H, \ldots, \psi_k^H$ be the desired formulas for ψ. Define $n := k$, $\phi_i^F :\equiv \exists y \psi_i^F(y, \bar{x})$ and $\phi_i^H :\equiv \exists y \psi_i^H(y, \bar{x})$ for every $i \in \{1, \ldots, n\}$. If $G \vDash \phi(\bar{g})$, then there is $a = a_H a_F \in G$ with $G \vDash \psi(a, \bar{g})$, which implies by induction hypothesis that there is i with $F \vDash \psi_i^F(a_F, \bar{f})$ and $H \vDash \psi_i^H(a_H, \bar{h})$, hence $F \vDash \phi_i^F(\bar{f})$ and $H \vDash \phi_i^H(\bar{f})$. Conversely, if we assume that $F \vDash \phi_i^F(\bar{f})$ and $H \vDash \phi_i^H(\bar{h})$, then there is some $a_F \in F$ and $a_H \in H$ with $F \vDash \psi_i^F(a_F, \bar{f})$ and $H \vDash \psi_i^H(a_H, \bar{h})$, and by induction hypothesis we then have $G \vDash \psi(a_H a_F, \bar{g})$ hence $G \vDash \phi(\bar{g})$. This shows that the formulas $\phi_1^F, \ldots, \phi_n^F, \phi_1^H, \ldots, \phi_n^H$ have the desired property.

Now we just have to consider the case $\phi \equiv \neg\psi$. Let $\psi_1^F, \ldots, \psi_k^F, \psi_1^H, \ldots, \psi_k^H$ be given. Let $P := \mathcal{P}(\{1, \ldots, k\})$ denote the power set of $\{1, \ldots, k\}$. For any $I \in P$, we define ϕ_I^F, ϕ_I^H as follows: $\phi_I^F \equiv \bigwedge_{i \in I} \neg\psi_i^F$ and $\phi_I^H \equiv \bigwedge_{i \notin I} \neg\psi_i^H$. Now let us check that the formulas $(\phi_I^F)_{I \in P}$ and $(\phi_I^H)_{I \in P}$ satisfy the desired property. Assume that $G \vDash \phi(\bar{g})$, so $G \nvDash \psi(\bar{g})$. By induction hypothesis, this means that for all $i \in \{1, \ldots, k\}$, either $F \nvDash \psi_i^F(\bar{f})$ or $H \nvDash \psi_i^H(\bar{h})$. Choose $I \in P$ as the set of all i with $F \nvDash \psi_i^F(\bar{f})$. Then $F \vDash \phi_I^F(\bar{f})$ and $H \vDash \phi_I^H(\bar{h})$. Conversely, assume there is $I \in P$ with $F \vDash \phi_I^F(\bar{f})$ and $H \vDash \phi_I^H(\bar{h})$. Then for any $i \in \{1, \ldots, k\}$, we either have $F \nvDash \psi_i^F(\bar{f})$ (when $i \in I$) or $H \nvDash \psi_i^H(\bar{h})$ (when $i \notin I$). By induction hypothesis, this means that $G \nvDash \psi(\bar{g})$.

□

Theorem 4.4.13
Let $(G_1, \precsim_1^G), \ldots, (G_n, \precsim_n^G)$ and $(H_1, \precsim_1^H), \ldots, (H_n, \precsim_n^H)$ be C-q.o. groups. Let $(G, \precsim^G)$ be the valuational product of the family $(G_i, \precsim_i^G)_{1 \leq i \leq n}$ and $(H, \precsim^H)$ the valuational product of the family $(H_i, \precsim_i^H)_{1 \leq i \leq n}$. Assume that for each i, we have $H_i \equiv G_i$. Then $H \equiv G$.

Proof. By induction, it is sufficient to show the case $n = 2$, which follows directly from Lemma 4.4.12. □

4.4.3 Basic formulas and definable sets of a C-minimal group

In all the rest of this section, $(G, \lesssim)$ is a C-q.o. group with type-valuation v, type-chain $(\Gamma, \leq, \mathcal{O}, \mathcal{V})$ and fundamental components $(B_\gamma, \lesssim_\gamma)_{\gamma\in\Gamma}$ (see Remark 4.3.35(i) and (iv)). We recall that, if $v(g) \neq v(h)$, then $v(g) < v(h)$ if and only if $h \lnsim g$. If $\gamma = v(g)$, then we will sometimes denote by G^γ, G_γ and T_γ the sets G^g, G_g and T_g (this notation does not generate confusion thanks to Remark 4.3.31).

We want to describe the definable sets of $(G, \lesssim)$ when the group is C-minimal. For any $a, b \in G$, we define:

(i) $[a, b] := \{g \in G \mid a \lesssim g \lesssim b\}$

(ii) $[a, b) := \{g \in G \mid a \lesssim g \lnsim b\}$

(iii) $(a, b] := \{g \in G \mid a \lnsim g \lesssim b\}$

(iv) $(a, b) := \{g \in G \mid a \lnsim g \lnsim b\}$

(v) $(a, \infty) := \{g \in G \mid a \lnsim g\}$

(vi) $[a, \infty) := \{g \in G \mid a \lesssim g\}$

A subset of G of this form is called an **interval** of $(G, \lesssim)$. Intervals of the form (iv) or (v) are called **open intervals**, whereas intervals of the form (i) or (vi) are called **closed intervals**. The elements a and b are called the **extremities** of the interval. Note that intervals are convex. It is easy to see that the intersection of an interval is an interval, for example $[a, b) \cap [c, d)$ is equal to either $[a, b), [a, d), [c, b)$ or $[c, d)$.

We define the **basic formulas** of $(G, \lesssim)$ as the formulas of the following type, where a, b are parameters in G:

Type 1: $xa^{-1} \lesssim b$.

Type 2: $xa^{-1} \lnsim b$.

Type 3: $b \lesssim xa^{-1}$.

Type 4: $b \lnsim xa^{-1}$.

Formulas of type 1 are called **closed balls** and formulas of type 2 are called **open balls**. Note that formulas of type 3 and 4 are negations of balls. It follows that an $\mathcal{L}$-formula is a boolean combination of balls if and only if it is a disjunction of conjunctions of basic formulas. Note also that a set with exactly one element is a ball: the set $\{a\}$ is defined by $xa^{-1} \lesssim 1$. The basic formulas are useful to characterize C-minimality:

Proposition 4.4.14
The C-group $(G, \lesssim)$ is weakly C-minimal if and only if every definable subset of G is a boolean combination of balls.

Proof. The atomic formulas of the language $\{C\}$ with parameters are formulas of the form $C(x,a,b)$, $C(a,x,b)$, $C(a,b,x)$ and $x=a$, where $a,b\in G$ are parameters. the formula $x=a$ is equivalent to $xa^{-1}\precsim 1$. By (C1), the formulas $C(a,x,b)$ and $C(a,b,x)$ are equivalent. By compatibility of C, $C(a,x,b)$ is equivalent to $C(ab^{-1},xb^{-1},1)$, which by definition of $\precsim$ is equivalent to $xb^{-1}\precnsim ab^{-1}$. Similarly, $C(x,a,b)$ is equivalent to $C(xb^{-1},ab^{-1},1)$, which is equivalent to $ab^{-1}\precnsim xb^{-1}$. Now each of the formulas $xa^{-1}\precsim 1$, $xb^{-1}\precnsim ab^{-1}$ and $ab^{-1}\precnsim xb^{-1}$ is a basic formula. It follows that any quantifier-free formula of the language $\{C\}$ is a disjunction of conjunctions of basic formulas. Conversely, the formula $xa^{-1}\precsim b$ is equivalent to $\neg C(xa^{-1},b,1)$ by definition of $\precsim$. By compatibility of C, this is equivalent to $\neg C(x,ba,a)$. Similarly, we have $xa^{-1}\precnsim b\Leftrightarrow C(ba,x,a)$. It follows that every boolean combination of balls is equivalent to a quantifier-free formula of $\{C\}$ with parameters. □

In order to understand the definable sets of G, it is therefore essential to understand what the balls in G look like. For this, we need the following lemmas:

Lemma 4.4.15
If $v(g)<v(h)$, then $gh\sim hg\sim g$. If $v(gh^{-1})>v(h)$, then $g\sim h$.

Proof. Assume $v(g)<v(h)$. We then have $\{h,h^{-1}\}\precnsim\{g^{-1},g\}$. If $g\in\mathcal{O}$, then it follows from Proposition 4.1.12 that $gh\sim g\sim hg$. If $g\in\mathcal{V}$, then T_g is left-convex (see Proposition 4.3.22), so $h\precnsim g$. It then follows from Proposition 4.1.12 that $gh\sim hg\sim g$. Now assume $v(gh^{-1})>v(h)$. Then by what we just proved, we have $gh^{-1}.h\sim h$, hence $g\sim h$. □

Lemma 4.4.16
If $a\in\mathcal{V}$, then for any $g\in G$, the following holds:

(i) $g\precnsim a\Rightarrow ag\sim ga\sim a$.

(ii) $g\sim a\Rightarrow\{ag,ga\}\precsim a$.

(iii) $a\precnsim g\Rightarrow ga\sim ag\sim g$.

Proof. The cases $a\precnsim g$ and $g\precnsim a$ directly follow from Proposition 4.1.12. Assume $a\sim g$. Because $a\in\mathcal{V}$, we have $g\precsim a^{-1}$, hence by (CQ_2): $ga\precsim a$ and $ag\precsim a$. □

Lemma 4.4.17
Let $g\in\mathcal{O}$ and take $h,f\in T_g^+$. The following holds:

(i) $\{h,f\}\precnsim\{hf,fh\}$.

(ii) $\{h^{-1}f,fh^{-1}\}\precsim f$.

(iii) For any $a\in G$, $a\sim f$ if and only if $fa^{-1}\in G_g$.

(iv) $fh^{-1}\in T_g^-\Leftrightarrow f\precnsim h$.

Proof. All these results follow directly from the fact that the C-q.o. on G^g/G_g is order-type (Proposition 4.3.13) with positive cone $\pi_{G_g}(T_g^+)$. □

These lemmas allow us to prove that any ball is either an initial segment or is contained in one $\sim$-class:

Proposition 4.4.18
Let $\phi(x) :\equiv xa^{-1} \precsim b$ and $\psi(x) :\equiv xa^{-1} \precnsim b$. Then $\phi(G)$ is either equal to $[1, a]$, $[1, b]$, $[1, ba]$ or to a subset of $\mathrm{cl}(a)$. Moreover, $\psi(G)$ is either equal to $[1, a)$, $[1, b)$, $[1, ba)$ or to a subset of $\mathrm{cl}(a)$.

Proof. Set $A := \phi(G)$ and $B := \psi(G)$. Note that $B \subseteq A$. We exhaustively study each case for the type of a and for the relative position of a and b.

(1) Case $a \in \mathcal{V}$: note that in this case, $\mathrm{cl}(a) = \mathrm{cl}(a^{-1})$ and $[1, a] = [1, a^{-1}]$.

- (i) Case $b \precnsim a$: If $g \precsim a^{-1}$, then Lemma 4.4.16(i) implies $ga^{-1} \sim a^{-1}$. If $a^{-1} \precnsim g$, then 4.4.16(iii) implies $ga^{-1} \sim g$. In both cases, we have $b \precnsim ga^{-1}$, hence $g \notin A$. This shows that $A \subseteq \mathrm{cl}(a^{-1}) = \mathrm{cl}(a)$, which also implies $B \subseteq \mathrm{cl}(a)$.
- (ii) Case $a \sim b$: If $a^{-1} \precnsim g$ then Lemma 4.4.16(iii) implies $b \precnsim ga^{-1}$, hence $g \notin A$. Conversely, if $g \precsim a^{-1}$, then Lemma 4.4.16(ii) implies $ga^{-1} \precsim b$, hence $g \in A$. This shows $A = [1, a^{-1}]$. If $g \precnsim a^{-1}$, then Lemma 4.4.16(i) implies $b \sim ga^{-1}$, hence $g \notin B$. This shows $B \subseteq \mathrm{cl}(a^{-1})$.
- (iii) Case $a \precnsim b$: If $g \precsim a$, then Lemma 4.4.16(ii) implies $ga^{-1} \precsim a^{-1}$, hence $ga^{-1} \precnsim b$, hence $g \in B$. This shows $[1, a] \subseteq B$. If $a \precnsim g$, then Lemma 4.4.16(iii) implies $ga^{-1} \sim g$, so $ga^{-1} \precsim b$ if and only if $g \precsim b$, and $ga^{-1} \precnsim b$ if and only if $g \precnsim b$. It follows that $A = [1, b]$ and $B = [1, b)$.

(2) Case $a \in \mathcal{O}^+$:

- (i) Case $v(b) > v(a)$: By Proposition 4.3.13, we know that G_a is strictly convex. Since $b \in G_a$, it follows that $ga^{-1} \precsim b \Rightarrow ga^{-1} \in G_a$, so $ga^{-1} \precnsim b \Rightarrow v(ga^{-1}) > v(a)$. By Lemma 4.4.15, it then follows that $B \subseteq \mathrm{cl}(a)$. If $b \nsim a^{-1}$, then the same argument shows that $A \subseteq \mathrm{cl}(a)$. Now assume that $b \sim a^{-1}$, so we have $g \in A \Leftrightarrow ga^{-1} \precsim a^{-1}$. It then follows from (CQ_2) that $g \in A \Leftrightarrow g \precsim a$, hence $A = [1, a]$.
- (ii) Case $v(b) < v(a)$: Because T_a is right-convex (Proposition 4.3.11), we have $G^a \precnsim b$. If $g \in G^a$, then $ga^{-1} \in G^a$, hence $ga^{-1} \precnsim b$. This shows $G^a \subseteq B$. If $g \notin G^a$, then $v(g) < v(a^{-1})$, and it then follows from Lemma 4.4.15 that $ga^{-1} \sim g$, so $ga^{-1} \precsim b$ if and only if $g \precsim b$, and $ga^{-1} \precnsim b$ if and only if $g \precnsim b$. This shows that $A = [1, b]$ and $B = [1, b)$.
- (iii) Case $b \in T_a^-$: By (CQ_2), $g \precsim a \Rightarrow ga^{-1} \precsim a^{-1}$. By Lemma 4.3.15, $a^{-1} \sim b$. This shows $[1, a] \subseteq A$. Moreover, by (CQ_2'), $a \precnsim g$ implies $a^{-1} \precnsim ga^{-1}$, which in turn implies $g \notin A$. This shows $A = [1, a]$. If $ga^{-1} \precnsim b$, then $v(ga^{-1}) > v(a)$, so Lemma 4.4.15 implies $g \in \mathrm{cl}(a)$. This shows $B \subseteq \mathrm{cl}(a)$.

(iv) Case $b \in T_a^+$: By (CQ_2), we have $ga^{-1} \precsim b \Leftrightarrow g(ba)^{-1} \precsim b^{-1}$. Since $ba \in T_a^+$, we have in particular $ba \in \mathcal{O}^+$ and $T_a^- = T_{ba}^-$, hence $b^{-1} \in T_{ba}^-$. Replacing a by ba and b by b^{-1}, this brings us back to Case (2)(iii), which implies $A = [1, ba]$. Now let $g \in A$. It follows from Lemma 4.4.17(iii) that $ga^{-1} \sim b \Leftrightarrow ga^{-1}b^{-1} \in G_a \Leftrightarrow g \sim ba$, hence $B = [1, ba)$.

(3) Case $a \in \mathcal{O}^-$:

(i) Case $v(b) > v(a)$: If $b \precsim a$, then $ga^{-1} \precsim b \Rightarrow v(ga^{-1}) > v(a)$ (this is because G_a is strictly convex). Lemma 4.4.15 then implies $A \subseteq \mathrm{cl}(a)$. Now assume $b \sim a$. This means $b \in \mathcal{W}_a$. If $v(g) > v(a)$, then by Lemma 4.4.15 we have $ga^{-1} \sim a^{-1}$, and since $a \in \mathcal{O}^-$ we have $a \precsim a^{-1}$, hence $b \precsim ga^{-1}$. It follows that $g \notin A$. If $v(g) < v(a)$, then by Lemma 4.4.15, we have $ga^{-1} \sim g$. Since G^a is convex, and since $b \in G^a$, it follows that $b \precsim ga^{-1}$, hence $g \notin A$. Therefore, $A \subseteq T_a$. If $g \in T_a^+$, then $ga^{-1} \in T_a^+$, hence $b \precsim ga^{-1}$, hence $g \notin A$. Therefore, $A \subseteq T_a^- \subseteq \mathrm{cl}(a)$. It then follows that $B \subseteq \mathrm{cl}(a)$.

(ii) Case $v(b) < v(a)$: Lemma 4.4.15 implies $A = [1, b]$ and $B = [1, b)$ (the proof is similar to case(2)(ii)).

(iii) Case $b \in T_a^-$: If $v(g) > v(a)$, then Lemma 4.4.15 implies $a^{-1} \sim ga^{-1}$. Since $a^{-1} \in T_a^+$, it follows that $b \precsim ga^{-1}$, so $g \notin A$. If $v(g) < v(a)$, then Lemma 4.4.15 implies $g \sim ga^{-1}$, which also implies $g \notin A$. Therefore, $A \subseteq T_a$. If $g \in T_a^+$, then $ga^{-1} \in T_a^+$, hence $b \precsim ga^{-1}$. Therefore, $A \subseteq T_a^- \subseteq \mathrm{cl}(a)$, hence also $B \subseteq \mathrm{cl}(a)$.

(iv) Case $b \in T_a^+$: By (CQ_2), we have $xa^{-1} \precsim b \Leftrightarrow x(ba)^{-1} \precsim b^{-1}$. Note that $b^{-1} \in T_a^-$. If $ba \in T_a^-$, then in particular we have $ba \in \mathcal{O}^-$ and $b^{-1} \in T_{ba}^-$. Case (3)(iii) then implies $A \subseteq \mathrm{cl}(ba) = \mathrm{cl}(a)$, which also implies $B \subseteq \mathrm{cl}(a)$. Assume $ba \in T_a^+$. Then we have $ba \in \mathcal{O}^+$ and $b^{-1} \in T_{ba}^-$. Case (2)(iii) then implies $A = [1, ba]$. It follows from Lemma 4.4.17 that, for any $g \in A$, $ga^{-1} \sim b \Leftrightarrow ga^{-1}b^{-1} \in G_a \Leftrightarrow g \sim ba$. It follows that $B = [1, ba)$. Finally, assume $ba \in G_a$. Then we have $v(b^{-1}) < v(ba)$. Cases (1)(ii),(2)(ii) and (3)(ii) then imply $A = [1, b^{-1}] = [1, a]$. Now take $g \in G_a$. Then by Lemma 4.4.15, we have $ga^{-1} \sim a^{-1}$. Moreover, since $ba \in G_a$, Lemma 4.4.17 implies $b \sim a^{-1}$. Therefore, $g \notin B$. Since $A = G_a \cup \mathrm{cl}(a)$, this shows $B \subseteq \mathrm{cl}(a)$.

□

Proposition 4.4.18 allows us to describe the definable subsets of a C-minimal group.

Proposition 4.4.19
Assume that $(G, \precsim)$ is weakly C-minimal and let A be a definable subset of G. There exists finitely many intervals $J_1, \ldots, J_n \subseteq G$ and a set $X \subseteq G$ such that $\mathrm{cl}(X)$ is finite and $A = J_1 \cup \cdots \cup J_n \cup X$.

Proof. Let E denote the class of all subsets of G of the form $J_1 \cup \cdots \cup J_n \cup X$ as above. By Proposition 4.4.14, we just have to show that E contains every boolean combination of balls. If A is a ball, then we know by Proposition 4.4.18 that either A is an interval or $A \subseteq \mathrm{cl}(a)$ for some $a \in G$, so $A \in E$. If A is an interval, then $G\backslash A$ is a union of two intervals (for example, $G\backslash[a,b] = (1,a) \cup (b,\infty)$). If $A \subseteq \mathrm{cl}(a)$, we have $G\backslash A = [1,a) \cup (a,\infty) \cup (\mathrm{cl}(a)\backslash A)$. This shows that the complement of a ball is always in E. Clearly, E is stable under finite union, so it only remains to show that E is stable under finite intersection. Let I and J be two finite unions of intervals and $X, Y \subseteq G$ such that $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$ are finite. We have $(I \cup X) \cap (J \cup Y) = (I \cap J) \cup (I \cap Y) \cup (J \cap X) \cup (X \cap Y)$. Clearly, $\mathrm{cl}((I \cap Y) \cup (J \cap X) \cup (X \cap Y))$ is finite, so we just have to show that $I \cap J$ is a finite union of intervals. Write $I = I_1 \cup \cdots \cup I_n$ and $J = J_1 \cup \cdots \cup J_k$, where I_i and J_i are intervals. We know that an intersection of intervals is an interval, so for any i, j, $I_i \cap J_j$ is an interval. It follows that $I \cap J = \bigcup_{i,j}(I_i \cap J_j)$ is a finite union of intervals. □

Remark 4.4.20: Note that, when we write a definable set A as $A = J_1 \cup \cdots \cup J_n \cup X$ as in Proposition 4.4.19, we can always assume the following:

(i) $J_1 \precsim J_2 \precsim \cdots \precsim J_n$.

(ii) For any $x \in X$, $\mathrm{cl}(x) \nsubseteq X$. Indeed, if $\mathrm{cl}(x) \subseteq X$, then $[x,x]$ is an interval contained in A, so set $J_{n+1} := [x,x]$ and $X' := X\backslash \mathrm{cl}(x)$ and we have $A = J_1 \cup \cdots \cup J_{n+1} \cup X'$.

Moreover, with these assumptions, A is convex if and only if $X = \varnothing$ if and only if A is a finite union of intervals.

With Proposition 4.4.19, we can describe the definable subgroups of a C-minimal group. For this, we will also need the following Lemma:

Lemma 4.4.21
Let H be a subgroup of G and $h \in G$. If $\mathrm{cl}(h^{-1}) \subseteq H$, then $[1,h) \subseteq H$.

Proof. Let $g \precsim h$. By Lemma 4.1.11, this implies $gh^{-1} \sim h^{-1}$, hence by assumption $gh^{-1} \in H$. Since $h \in H$, it follows that $g \in H$. □

Lemma 4.4.22
Let F be a definable subgroup of G and write $F = J_1 \cup \cdots \cup J_n \cup X$ as in Proposition 4.4.19 with the assumptions of Remark 4.4.20. Then $H := J_1 \cup \cdots \cup J_n$ is a normal subgroup of F and H is convex in G.

Proof. Let $g, h \in H$. Without loss of generality, $g \precsim h$. By (CQ_2), we have $gh^{-1} \precsim h^{-1}$. Because H is a union of intervals and because $h^{-1} \in H$, we have $\mathrm{cl}(h^{-1}) \subseteq H$. By Lemma 4.4.21, $[1,h) \subseteq H$. Since $h \in H$, we even have $[1,h] \subseteq H$, hence $gh^{-1} \in H$. This shows that H is a group. Lemma 4.4.21 shows that H is convex. Now let us show that H is normal in F. Let $g \in H$ and $z \in F$. By (CQ_3), we have $\mathrm{cl}(g^z) = \{h^z \mid h \sim g\}$. By assumption (ii) of Remark 4.4.20, there must be $h^z \in \mathrm{cl}(g^z)$ such that $h^z \notin X$, hence $h^z \in H$. Since H is convex and $g^z \sim h^z$, it follows that $g^z \in H$. □

The next result will be useful to describe C-minimal groups in Section 4.4.4:

Lemma 4.4.23
Assume $(G, \precsim)$ is weakly C-minimal, let F be a definable subgroup of G and $g \in F \cap \mathcal{O}$. Then $G^g \subseteq F$.

Proof. Without loss of generality, $g \in \mathcal{O}^+$. Write $F = J_1 \cup \cdots \cup J_n \cup X$ as in Lemma 4.4.22. Because $g \in \mathcal{O}^+$, the sequence $(g^l)_{l\in\mathbb{N}}$ is strictly increasing. Therefore, $\{g^l \mid l \in \mathbb{N}\}$ cannot be contained in X, so there is $l \in \mathbb{N}$ with $g^l \in H := J_1 \cup \cdots \cup J_n$. By Lemma 4.4.22, H is a convex subgroup of F. In particular, since H is convex and $T_g^- \precsim g^l$, we have $T_g^- \subseteq H$, hence $T_g^- \subseteq F$. This implies $T_g^+ \subseteq F$, hence $T_g \subseteq F$. Now take $f \in G_g$. By Lemma 4.4.15, we have $gf \in T_g$, hence $gf \in F$, hence $f \in F$. Therefore, $G^g = G_g \cup T_g \subseteq F$. □

4.4.4 Structure of a C-minimal group

We now assume that $(G, \precsim)$ is weakly C-minimal and we want to describe the structure of $(G, \precsim)$.

Proposition 4.4.24
Γ is finite.

Proof. Towards a contradiction, assume that Γ is infinite. Then without loss of generality, there exists an infinite sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{O}^+$ such that $v(g_n) > v(g_{n+1})$ for all $n \in \mathbb{N}$. The set $\mathcal{O}^+$ is definable, so we can write $\mathcal{O}^+ = J_1 \cup \cdots \cup J_n \cup X$ as in Proposition 4.4.19. Because $\mathrm{cl}(X)$ is finite, and because $(g_n)_n$ is strictly increasing, there must be $i, n, m \in \mathbb{N}$ with $n < m$ and $g_n, g_m \in J_i$. Since $v(g_m) < v(g_n)$, we have $g_n \precsim g_m^{-1} \precsim g_m$. Since J_i is an interval, it follows that $g_m^{-1} \in J_i \subseteq \mathcal{O}^+$: contradiction. □

Before describing the structure of a C-minimal group, we want to show that we can assume that the group is welding-free without losing generality. This will make later proofs easier. Note that, since Γ is finite, then in particular $(G, \precsim)$ has finitely many welding classes. Let $g_1, \ldots, g_n \in \mathcal{O}^-$ be a set of representatives of the welding classes and let $h_1, \ldots, h_n \in \mathcal{V}$ with $h_i \sim g_i$. Finally, let $\precsim^*$ be the unwelding of $\precsim$ (see Remark 4.3.35). Then $\precsim$ and $\precsim^*$ are inter-definable with the parameters $g_1, \ldots, g_n, h_1, \ldots, h_n$, via the following formulas (be careful that "cl" denotes the $\sim$-class and not the $\sim^*$-class):

$$g \precsim h \Leftrightarrow g \precsim^* h \vee (\bigvee_{i=1}^{n} (g \sim^* g_i \wedge h \sim^* h_i)) \tag{F1}$$

$$g \precsim^* h \Leftrightarrow g \precsim h \wedge \neg(\bigvee_{i=1}^{n} (g \in \mathrm{cl}(g_i) \cap \mathcal{O} \wedge h \in \mathrm{cl}(h_i) \cap \mathcal{V})) \tag{F2}$$

Lemma 4.4.25
Set $\mathcal{L}^* := \{1, ., ^{-1}, \precsim^*\}$. The following holds:

(i) For any $\phi(\bar{x}) \in \mathcal{L}^*$, there exists $\phi^*(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{L}$ such that for any $\bar{f} \subseteq G$, $(G, \precsim^*) \vDash \phi(\bar{f})$ if and only if $(G, \precsim) \vDash \phi^*(\bar{f}, g_1, \ldots, g_n, h_1, \ldots, h_n)$.

(ii) For any $\phi(\bar{x}) \in \mathcal{L}$, there exists $\phi^*(\bar{x}, \bar{y}, \bar{z}) \in \mathcal{L}^*$ such that for any $\bar{f} \subseteq G$, $(G, \precsim^*) \vDash \phi^*(\bar{f}, g_1, \ldots, g_n, h_1, \ldots, h_n)$ if and only if $(G, \precsim) \vDash \phi(\bar{f})$.

Proof. If ϕ is atomic, then this is given by formulas (F1) and (F2). Both claims then generalize to all formulas by induction. □

Lemma 4.4.26
If A is a boolean combination of balls in $(G, \precsim)$, then A is also a boolean combination of balls in $(G, \precsim^*)$.

Proof. It suffices to show the case where A is a ball in $(G, \precsim)$. Assume then that A is defined by the formula $xa^{-1} \precsim b$. By formula (F1), this is equivalent to $xa^{-1} \precsim^* b \vee (\bigvee_{i=1}^{n}(xa^{-1} \sim^* g_i \wedge b \sim^* h_i))$. This is indeed a boolean combination of balls in $(G, \precsim^*)$. Moreover, the formula $xa^{-1} \not\precsim b$ is equivalent to $\neg(b \precsim^* xa^{-1} \vee (\bigvee_{i=1}^{n}(b \sim^* g_i \wedge xa^{-1} \sim^* h_i))))$, which is also a boolean combination of balls in $(G, \precsim^*)$. □

Proposition 4.4.27
The C-q.o. group $(G, \precsim^*)$ is weakly C-minimal. If $(G, \precsim)$ is C-minimal, then $(G, \precsim^*)$ is also C-minimal.

Proof. Let $A \subseteq G$ be definable in $\mathcal{L}^*$. By Lemma 4.4.25, A is also definable in $\mathcal{L}$. Because $(G, \precsim)$ is weakly C-minimal, A is a boolean combination of balls in $(G, \precsim)$. It then follows from Lemma 4.4.26 that A is a boolean combination of balls in $(G, \precsim^*)$.

Now assume that $(G, \precsim)$ is C-minimal. Let $(H, \precsim)$ be an ω-saturated elementary extension of $(G, \precsim)$. Define $\precsim^*$ on H by formula (F2). Then $(H, \precsim^*)$ is an ω-saturated elementary extension of $(G, \precsim^*)$. Because $(G, \precsim)$ is C-minimal, $(H, \precsim)$ is also C-minimal. We just showed that this proves that $(H, \precsim^*)$ is weakly C-minimal. It then follows from Proposition 4.4.2 that $(G, \precsim^*)$ is C-minimal. □

Proposition 4.4.27 shows that, when studying the properties of C-minimal groups, we can always assume the group to be welding-free.

We now want to show that the fundamental components of $(G, \precsim)$ are weakly C-minimal.

Lemma 4.4.28
Let $(G, \precsim)$ be a C-q.o.g., $F \subseteq G$ a convex subgroup of G and $\psi(x, \bar{b})$ a boolean combination of balls with parameter $\bar{b} \subseteq G$. Then there is a boolean combination of balls $\chi(x, \bar{c})$ with parameter $\bar{c} \subseteq F$ such that for any $g \in F$, $G \vDash \psi(g, \bar{b})$ if and only if $G \vDash \chi(g, \bar{c})$.

Proof. We just have to show the case where $\psi(x, \bar{b})$ is a ball. Assume then that $\psi(x, a, b) \equiv xa^{-1} \precsim b$ with $a, b \in G$. Assume that $a \notin F$. Then by convexity of F, we have $g \precnsim a$ for every $g \in F$. It then follows from Lemma 4.1.11 that $ga^{-1} \sim a^{-1}$ for all $g \in F$. It follows that $\psi(F, a, b)$ is either empty or F, so in particular it is a ball of F. Assume $a \in F$. If

$b \in F$ then there is nothing to do. If $b \notin F$, then we have $\psi(F, a, b) = F$ by convexity of F. The case $\psi(x, a, b) \equiv xa^{-1} \precsim b$ is similar. □

Lemma 4.4.29
Let F be a convex subgroup of G and denote by π the canonical projection $\pi : G \to G/F$. Let $A \subseteq G$ be a boolean combination of balls. Then $\pi(A)$ is also a boolean combination of balls.

Proof. Because $\pi(X \cup Y) = \pi(X) \cup \pi(Y)$ for any $X, Y \subseteq G$, we only have to consider the case where A is a conjunction of basic formulas. Let $\phi(x, \bar{a}, \bar{b}) \equiv \bigwedge_{i \in I} \phi_i(x, a_i, b_i)$ be a formula defining A, where each $\phi_i(x, a_i, b_i)$ is a basic formula. We partition I in the following manner. Set $I_1 := \{i \in I \mid b_i \notin F\}$, $I_2 := \{i \in I \mid \phi_i(x, a_i, b_i)$ is of type 1 or 2 and $b_i \in F\}$ and $I_3 := \{i \in I \mid \phi_i(x, a_i, b_i)$ is of type 3 or 4 and $b_i \in F\}$. We thus have $I = I_1 \cup I_2 \cup I_3$. Assume first that $I_2 \neq \emptyset$. Let $g \in A$ and $i \in I_2$. Then we have $ga_i^{-1} \precsim b_i$. By definition of I_2, we have $b_i \in F$. By convexity of F, it follows that $ga_i^{-1} \in F$, hence $gF = a_iF$. This shows that $\pi(A) \subseteq \{a_iF\}$, so $\pi(A)$ is a ball. Therefore, we can assume that $I_2 = \emptyset$. Now set $J := \{i \in I_3 \mid a_iF \notin \pi(A)\}$. For any $i \in J$, set $\psi_i(x, a_iF) :\equiv x \neq a_iF$. Finally, set $\psi(x, \bar{a}F, \bar{b}F) :\equiv \bigwedge_{i \in I_1} \phi_i(x, a_iF, b_iF) \wedge \bigwedge_{i \in J} \psi_i(x, a_iF)$. Then $\psi(x, \bar{a}F, \bar{b}F)$ is a boolean combination of balls, and we will now show that it defines $\pi(A)$. Let $h \in \pi(A)$ and take $g \in A$ with $gF = h$. We thus have $G \vDash \phi(g, \bar{a}, \bar{b})$. Let $i \in I_1$. We have $G \vDash \phi_i(g, a_i, b_i)$. By definition of I_1, $b_i \notin F$. It then follows from Remark 4.2.3, that $H \vDash \phi_i(h, a_iF, b_iF)$. Now let $j \in J$. Since $h \in \pi(A)$, it follows from the definition of J that $h \neq a_jF$, hence $H \vDash \psi_j(h, a_jF)$. This shows that $H \vDash \psi(h, \bar{a}F, \bar{b}F)$. Conversely, assume that $H \vDash \psi(h, \bar{a}F, \bar{b}F)$, and let us prove that $h \in \pi(A)$. Take $g \in G$ with $gF = h$. It follows from Remark 4.2.3 that, for any $i \in I_1$, $G \vDash \phi_i(g, a_i, b_i)$. Now let $i \in I_3$. Assume first that $i \in J$. Because $h \neq a_iF$, we have $ga_i^{-1} \notin F$. By definition of I_3, we have $b_i \in F$. It follows from the convexity of F that $b_i \precsim ga_i^{-1}$. Since $\phi_i(x, a_i, b_i)$ is of type 3 or 4, this implies $G \vDash \phi_i(g, a_i, b_i)$. Therefore, we have $G \vDash \phi_i(g, a_i, b_i)$ for all $i \in I_1 \cup J$. If for all $i \in I_3 \backslash J$, $G \vDash \phi_i(g, a_i, b_i)$, then it follows that $G \vDash \phi(g, \bar{a}, \bar{b})$, hence $g \in A$, hence $h \in \pi(A)$. Now assume there is $i \in I_3 \backslash J$ with $G \nvDash \phi_i(g, a_i, b_i)$. Then we have $ga_i^{-1} \precsim b_i \in F$, which by convexity of F implies $gF = a_iF$. Since $i \notin J$, it follows from the definition of J that $h = a_iF \in \pi(A)$. □

Proposition 4.4.30
Let $(G, \precsim)$ be a C-q.o. group and F a definable convex normal subgroup of $(G, \precsim)$. If G is weakly C-minimal, then so are F and G/F. If G is C-minimal, then F and G/F are also C-minimal.

Proof. Let $\phi(x, \bar{a})$ be a formula with parameter $\bar{a} \subseteq F$. Take a formula $\phi^F(x, \bar{y})$ as given by Remark 4.4.10. Since G is weakly C-minimal, there exists a boolean combination of balls $\psi(x, \bar{b})$ with parameter $\bar{b} \in G$ such that $G \vDash \forall x(\phi^F(x, \bar{a}) \Leftrightarrow \psi(x, \bar{b}))$. By Lemma 4.4.28, there exists a boolean combination of balls $\chi(x, \bar{c})$ with parameter $\bar{c} \subseteq F$, such that for all $f \in F$, $G \vDash \chi(f, \bar{c}) \Leftrightarrow \psi(f, \bar{b})$. It follows that $F \vDash \forall x(\phi(x, \bar{a}) \Leftrightarrow \chi(x, \bar{c}))$.

Now assume $\bar{a}F \subseteq H := G/F$. Take ϕ^H as in Remark 4.4.10. Then we have $\phi(H, \bar{a}F) = \pi(\phi^H(G, \bar{a}))$. Because $(G, \precsim)$ is weakly C-minimal, $\phi^H(G, \bar{a})$ is a boolean combination of balls in $(G,\precsim)$. It then follows from Lemma 4.4.29 that $\phi(H,\bar{a}F)$ is a boolean combination of balls in $(H,\precsim)$.

Now assume that G is C-minimal. Let G' be an ω-saturated elementary extension of G, ϕ a formula with parameters in G defining F and and let $F' := \phi(G')$. Because G is C-minimal, G' is also C-minimal. We just proved that this implies that F' and $H' := G'/F'$ are weakly C-minimal. Moreover, it follows from Lemma 4.4.9 and Remark 4.4.10 that F' (respecively, H') is an ω-saturated elementary extension of F (respectively, H). It then follows from Proposition 4.4.2 that F and H are C-minimal.

□

For the proof of the next lemma, we refer to the definition of T_g at the beginning of Section 4.3.3.

Lemma 4.4.31
For any $g \in G$, T_g is $\{g\}$-definable. Moreover, the relation $v(x) \leq v(y)$ is $\emptyset$-definable.

Proof. Assume $g \in \mathcal{V}$. Then T_g is definable by the following formula: $x \in \mathcal{V} \wedge (\forall y, (g \precsim y \precsim x \vee x \precsim y \precsim g) \Rightarrow y \notin \mathcal{O}^+)$. Now assume $g \in \mathcal{O}^+$. T_g^+ is defined by the formula: $x \in \mathcal{O}^+ \wedge (\forall y, (g \precsim y \precsim x \vee x \precsim y \precsim g) \Rightarrow y \in \mathcal{O}^+)$. Now define T_g by the formula $x \in T_g^+ \vee x^{-1} \in T_g^+$. Therefore, T_g is $\{g\}$-definable. It follows from the definition of v (see Proposition 4.3.30) that $v(x) \leq v(y) \Leftrightarrow T_x = T_y \vee (\forall z \in T_y \forall w \in T_x, z \precsim w)$. □

We can now state our main Proposition on weakly C-minimal groups.

Proposition 4.4.32
For every $\gamma \in \Gamma$, the following holds:

(1) T_γ, G_γ and G^γ are $\emptyset$-definable in $\mathcal{L}$.

(2) G^γ and G_γ are normal in G.

(3) $(G^\gamma,\precsim)$ and $(B_\gamma,\precsim_\gamma)$ are weakly C-minimal. If moreover G is C-minimal, then $(G^\gamma,\precsim)$ and $(B_\gamma,\precsim_\gamma)$ are C-minimal.

(4) If $\gamma \neq \max(\Gamma)$, then $\min(T_\gamma) \neq \emptyset$.

Proof. Write $\Gamma := \{1,\dots,m\}$. Note that a C-q.o. has the same type-valuation and the same fundamental components as its unwelding. Therefore, by Proposition 4.4.27, we can assume that $(G,\precsim)$ is welding-free. Then for all $\gamma \in \Gamma$, G^γ is convex in $(G,\precsim)$. Let us prove (1). Let $\gamma \in \Gamma$. then $T_\gamma = v^{-1}(\{\gamma\})$. We wan define T_γ as the γ-th type component, e.g. with the following formula: $\exists y_1,\dots,y_{\gamma-1}, (v(y_1) < v(y_2) < \dots < v(y_{\gamma-1}) < v(x) \wedge (\forall z, (v(y_1) < v(y_2) < \dots < v(y_{\gamma-1}) < v(z)) \Rightarrow v(x) \leq v(z))))$. It follows from Lemma 4.4.31 that this is an $\mathcal{L}$-formula without parameter. Now note that $G^\gamma = \{g \in G \mid \exists h \in T_\gamma, g \precsim h\}$ and $G_\gamma = G^\gamma \backslash T_\gamma$. It follows that G^γ and G_γ are also $\emptyset$-definable. Now let us prove (2). Take $\gamma \in \Gamma$, $g \in G^\gamma$ and $z \in G$. Assume that $g^z \notin G^\gamma$. Then $v(g^z) < v(g)$. By condition (iv) of Definition 2.2.3, it follows that for all $n \in \mathbb{N}$, $v(g^{z^{n+1}}) < v(g^{z^n})$. This contradicts the fact that Γ is finite. Now let us show (3).

Assume first that $\gamma = 1$. Because $(G, \precsim)$ is welding-free, G_γ is convex in $(G, \precsim)$. It then follows from Proposition 4.4.30 that G_γ and B_γ are weakly C-minimal. Because G_γ is weakly C-minimal, we can repeat the argument to show that $G_{\gamma+1}$ and $B_{\gamma+1}$ are weakly C-minimal. The proof of (3) is then done by induction on Γ. The proof in the case of full C-minimality is similar. Now let us show (4). Let $\gamma \in \Gamma$ with $\gamma \neq m$ and set $F := G_\gamma$. If $\gamma \in \mathcal{O}$, then it directly follows from Proposition 4.3.11 that $T_\gamma^- = \min(T_\gamma)$, hence $\min(T_\gamma) \neq \varnothing$. Assume that $\gamma \in \mathcal{V}$. Then it follows from Theorem 4.3.33(3) that $\gamma+1 \in \mathcal{O}$. Since we assumed that $(G, \precsim)$ is welding-free, we know that F is convex. By Remark 4.4.20, we can write $F = J_1 \cup \cdots \cup J_n$, where each J_i is an interval and $J_1 \precsim \cdots \precsim J_n$. Let a, b denote the extremities of J_n. Since $\gamma + 1 \in \mathcal{O}$, F has no maximum. It follows that $b \notin F$. However, we have $a \in F$ (otherwise $J_n \cap F = \varnothing$). If there is $c \precnsim b_i$ with $c \notin F$, then $c \in J_n$, which is a contradiction. Therefore, we must have $b \in \min(T_\gamma)$. □

A natural question that arises is whether a (weakly) C-minimal group is a direct product of its fundamental components. We are now going to show that, if G is abelian, then this is indeed the case. We recall the following well-known fact from group theory (see for example [Hal63, Theorem 13.3.1]):

Proposition 4.4.33
Let G be an abelian group and F a divisible subgroup of G. Then F is a direct factor of G.

We will also use the following:

Proposition 4.4.34
Let F be a normal convex subgroup of G and assume that F is a direct factor of G. Set $H := F/G$. Then $(G, \precsim)$ is isomorphic to the valuational product of the family $((H, \precsim), (F, \precsim))$.

Proof. It follows easily from the formula given in Proposition 4.2.2 and from the definition of the valuational product. □

In [MS96], Macpherson and Steinhorn showed that, in the context of dense C-minimal groups, if $\min(\Gamma)$ is o-type, then G is divisible abelian (see Theorem 4.4.7 (2) above). Using our results on definable subgroups from Section 4.4.3, we can generalize these results to general (i.e. not necessarily dense) C-relations:

Proposition 4.4.35
Assume that $\min(\Gamma) \in \mathcal{O}$. Then G is divisible abelian.

Proof. Let $\gamma := \min(\Gamma)$ and take $g \in T_\gamma$. Let $C_g := \{h \in G \mid gh = hg\}$. C_g is a definable subgroup of G which contains g. It then follows from Lemma 4.4.23 that $C_g = G^g = G$. Now let $h \in G$. We just proved that C_h contains g, and since $g \in \mathcal{O}$, it follows from Lemma 4.4.23 that $C_h = G^g = G$. This proves that G is abelian. Now let $n \in \mathbb{N}$ and set $G^n := \{h \in G \mid \exists f \in G, f^n = h\}$. Because G is abelian, G^n is a group. Moreover, G^n is definable, and it contains $g^n \in \mathcal{O}$. It then follows from Lemma 4.4.23 that $G^n = G$. □

Proposition 4.4.36
Assume that $(G, \precsim)$ is abelian. Then G is the direct product of the B_γ's. If $(G, \precsim)$ is welding-free, then $(G, \precsim)$ is isomorphic to the valuational product of the family $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$.

Proof. We first consider the welding-free case. Let $\gamma := \min(\Gamma)$. Let us first show that G_γ is divisible. If $\gamma \in \mathcal{V}$, then it follows from Theorem 4.3.33(3) that $\gamma+1 \in \mathcal{O}$. We know from Proposition 4.4.32 that G_γ is weakly C-minimal. It then follows from Proposition 4.4.35 that G_γ is divisible. Now assume that $\gamma \in \mathcal{O}$, take $g \in G_\gamma$ and $k \in \mathbb{N}$. It follows from Proposition 4.4.35 that G is divisible, so there is $h \in G$ with $h^k = g$. Because B_γ is order-type, it is torsion-free. It follows that, for any $f \notin G_\gamma$, $f^k \notin G_\gamma$. It follows that $h \in G_\gamma$. This shows that G_γ is divisible.

By Proposition 4.4.33, it follows that G_γ is a direct factor of G. Since $(G, \precsim)$ is welding-free, G_γ is convex in G. It then follows from Proposition 4.4.34 that $(G, \precsim)$ is isomorphic to the valuational product of the family $((B_\gamma, \precsim_\gamma), (G_\gamma, \precsim))$. Because G_γ is weakly C-minimal, we can repeat the same arguments to show that G_γ is isomorphic to the valuational product of the family $((B_{\gamma+1}, \precsim_{\gamma+1}), (G_{\gamma+1}, \precsim))$. The proposition then follows by induction on Γ.

If $(G, \precsim)$ is not welding-free, then consider $\precsim^*$ as in Proposition 4.4.27. We just proved that $(G, \precsim^*)$ is the valuational product of the family $(B_\gamma, \precsim_\gamma)_{\gamma\in\Gamma}$, which in particular means that G is the direct product of the B_γ's.

□

This allows us to state our main theorem on the structure of C-minimal groups:

Theorem 4.4.37 (structure of C-minimal groups)
Let $(G, \precsim)$ be a C-minimal group. The following holds:

(1) Γ is finite.

(2) Every fundamental component of $(G, \precsim)$ is C-minimal.

(3) If G is abelian, then G is the direct product of its fundamental components. If moreover $(G, \precsim)$ is welding-free, then $(G, \precsim)$ is isomorphic to the valuational product of its fundamental components.

(4) If G is not abelian, then $\min(\Gamma) \in \mathcal{V}$.

In particular, every abelian welding-free C-minimal group is isomorphic to a valuational product of C-minimal fundamental C-groups.

Proof. (1) is Proposition 4.4.24. (2) directly follows from Proposition. 4.4.32, and (3) follows from Proposition 4.4.36. Finally, (4) follows from 4.4.35. □

Remark 4.4.38: Theorem 4.4.37 does not say anything about welding. In particular, it would be interesting to know under which condition welding can exist in a C-minimal group. So far, we do not know any example of C-minimal group with welding.

4.4.5 Products of C-minimal groups

We now want to give an example of a C-minimal group whose C-relation is neither order-type nor valuational. For this, we show that the product of an o-minimal group with a finite valued group is C-minimal. We first need to show the following lemma:

Lemma 4.4.39
Let $(F, \precsim)$ be a C-q.o. group and $(H, \precsim)$ a valuationally quasi-ordered group such that $H\backslash\{1\}$ has a minimum m and set $G := H \times F$. Then for any boolean combination $\phi(x, \bar{a}_H)$ of balls with $\bar{a}_H \subseteq H$, there exists a boolean combination $\phi^*(x, \bar{a}, m)$ of balls with $\bar{a} \subseteq G$ such that for any $g = (h, f) \in G$, $G \vDash \phi^*(g, \bar{a}, m)$ if and only if $H \vDash \phi(h, \bar{a}_H)$.

Proof. It is sufficient to show the lemma in the case where ϕ is a ball. Assume then that $\phi(x) \equiv xa_H^{-1} \precsim b_H$. If $b_H \neq 1$, then by definition of the valuational product, $ha_H^{-1} \precsim b_H$ is equivalent to $g.(a_H^{-1}, 1) \precsim (b_H, 1)$ for any $g \in G$, so we can just set $\phi^*(x) \equiv x.(a_H^{-1}, 1) \precsim (b_H, 1)$. If $b_H = 1$, then $ha_H^{-1} \precsim b_H$ is true if and only if $g.(a_H^{-1}, 1) \in F$ which is equivalent to $g.(a_H^{-1}, 1) \precnsim m$, so we can set $\phi^*(x) \equiv x.(a_H^{-1}, 1) \precnsim m$. Assume now that $\phi(x) \equiv xa_H^{-1} \precnsim b_H$. If $b_H = 1$, then $\phi(x)$ is not satisfiable in H by (CQ_1), so we set $\phi^*(x) \equiv x \precnsim 1$. If $b_H \neq 1$, we can set $\phi^*(x) \equiv x.(a_H^{-1}, 1) \precnsim (b_H, 1)$. It follows from the definition of the valuational product that ϕ^* has the desired property. □

Lemma 4.4.40
Let $(G, \precsim)$ be a C-q.o. group. Assume that F is a $\emptyset$-definable subgroup of G and that F has a finite group complement in G. Let $\phi(x)$ be a formula defining F, $(G', \precsim)$ a C-q.o group with $(G, \precsim) \equiv (G', \precsim)$ and let $F' = \phi(G')$. Then F' has a finite group complement in G'.

Proof. Let H denote a finite group complement of F. Set $n := |H|$. Let ψ be the formula saying "there exists a finite subset X of G of size n such that X is a group, $X \cap \phi(G) = 1$ and for all z there exists x, y such that $z = yx$, $\phi(y)$ holds and $x \in X$". Then G satisfies ψ (with $X = H$). It follows that G' satisfies ψ. But ψ precisely expresses the fact that F' has a finite group complement of size n in G'. □

Proposition 4.4.41
Let $(F, \precsim)$ be C-minimal order-type C-q.o. group and $(H, \precsim)$ a finite valuationally quasi-ordered group. Then $G := H \times F$ is C-minimal.

Proof. Note that any finite C-q.o. group is C-minimal, so in particular H is C-minimal. Moreover, $H\backslash\{1\}$ admits a minimum which we will denote by m. We first show that $(G, \precsim)$ is weakly C-minimal. Let $\phi(x, \bar{a})$ be a formula of $\mathcal{L}$ with one free variable and parameters $\bar{a} = (\bar{a}_H, \bar{a}_F) \subseteq G$. Take $\phi_1^F(\bar{x}), \dots, \phi_n^F(\bar{x}), \phi_1^H(\bar{x}), \dots, \phi_n^H(\bar{x})$ as in Lemma 4.4.12. By Lemma 4.4.12, it is sufficient to show that for each $i \in \{1, \dots, n\}$, the set $A_i := \{(h, f) \in G \mid F \vDash \phi_i^F(f, \bar{a}_F) \wedge H \vDash \phi_i^H(h, \bar{a}_H)\}$ is a boolean combination of balls. Fix an $i \in \{1, \dots, n\}$. Since F and H are C-minimal, there are two formulas $\theta_F(x, \bar{b}_F), \theta_H(x, \bar{b}_H)$ (with parameters $\bar{b}_F \subseteq F$ and $\bar{b}_H \subseteq H$) which are boolean combinations of balls such that $F \vDash \forall x(\phi_i^F(x, \bar{a}_F) \Leftrightarrow \theta_F(x, \bar{b}_F))$ and $H \vDash \forall x(\phi_i^H(x, \bar{a}_H) \Leftrightarrow \theta_H(x, \bar{b}_H))$. For any $g =$

$(h, f) \in G$, we now have $g \in A_i$ if and only if $F \vDash \theta_F(f, \bar{b}_F)$ and $H \vDash \theta_H(h, \bar{b}_H)$. Now note that $F \vDash \theta_F(f, \bar{b}_F)$ if and only if $G \vDash \bigwedge_{h' \in H}(g.(h',1)^{-1} \precsim m \Rightarrow \theta_F(g.(h',1)^{-1}, \bar{b}_F))$ (this follows from the fact that $(1, f) = g.(h', 1)^{-1}$ if and only if $g.(h', 1)^{-1} \in F$ if and only if $g.(h', 1)^{-1} \precsim m$). Note also that the formula $\bigwedge_{h' \in H}(x.(h',1)^{-1} \precsim m \Rightarrow \theta_F(x.(h',1)^{-1}, \bar{b}_F))$ is still a boolean combination of balls. Finally, take $\theta^*_H(x, \bar{b}, m)$ as given by Lemma 4.4.39. The formula $\theta^*_H(x, \bar{b}, m) \wedge \bigwedge_{h' \in H}(x.(h',1)^{-1} \precsim m \Rightarrow \theta_F(x.(h',1)^{-1}, \bar{b}_F))$ is a boolean combination of balls which defines A_i. This proves that $(G, \precsim)$ is weakly C-minimal. Now we must show that the same is true for G_2, where G_2 is an arbitrary C-q.o. group with $G_2 \equiv G$. Note that F is a definable subgroup of G because it is the set of o-type elements of G. Now denote by F_2 the set of o-type elements of G_2 and $H_2 := G_2/F_2$. It follows from Proposition 4.4.11 that $F_2 \equiv F$ and $H_2 \equiv H$. By assumption, it follows that F_2 and H_2 are C-minimal. Moreover, it follows from Lemma 4.4.40 that H_2 is finite and that $G_2 \cong H_2 \times F_2$. By what we just proved, we know that $H_2 \times F_2$ is weakly C-minimal, so G_2 is weakly C-minimal. □

Remark 4.4.42: The condition of H being finite was essential in the proof of Proposition 4.4.41. Indeed, the assumption that F is C-minimal only tells us that the set B of all f's such that $g = (h, f) \in A_i$ is a boolean combination of balls, so it gives us a formula $\theta_F(f, \bar{b}_F)$ in which f appears. We then need to characterize the g's of G such that $f \in B$ with an appropriate formula, i.e we need to "lift" $\theta_F(f, \bar{b}_F)$ to a formula in which g appears instead of f. The problem is that f is in general not definable in G if H is chosen arbitrarily, so we cannot express "$f \in B$" with a formula. However, if we happen to know that the h's of all g's in A_i only take finitely many values in H (as is the case in Proposition 4.4.41), then we can express "$f \in B$" via a formula $\bigwedge_{h'}(g(h',1)^{-1} \precsim m \Rightarrow \theta_F(g(h',1)^{-1}, \bar{b}_F))$, where h ranges over all possible values of h for g in A_i.

We can now give an example of a C-minimal group which is neither ordered nor valued:

Example 4.4.43
Let $F := \mathbb{Q}$ with the usual order; it is known that this is an o-minimal structure. Set $H := (\mathbb{Z}/p^k\mathbb{Z}, v_p)$ with $k \in \mathbb{N}$, where v_p denotes the valuation induced on H by the p-adic valuation of $\mathbb{Z}$. Then by Proposition 4.4.41, $H \times F$ is C-minimal.

Chapter 5

Differential-valued fields and power series

Introduction

The object of this chapter is to introduce and characterize a notion of differential rank for differential-valued fields. We start by defining a general notion of ϕ-rank for a valued field endowed with an operator ϕ in Section 5.1. Our motivation for introducing this notion of ϕ-rank is to provide a common framework for the several already existing notions of rank, which later motivates our definition of the differential rank of a differential-valued field. Our notion of ϕ-rank generalizes the three notions of rank which we know: the classical rank of a valued field (without any operator), the exponential rank of an exponential ordered field studied in [Kuh00] and the notion of difference rank for a difference field introduced in [KMP17].

Section 5.2 is dedicated to the study of asymptotic couples. We introduce the notion of cut point (Definition 5.2.2), which allows us to describe the behavior of the map ψ of an asymptotic couple. A cut point c separates the group in two parts, on each of which the behavior of ψ is very different. Roughly speaking, ψ acts like a contraction on the set of elements which are infinitely bigger than c, whereas all elements between 0 and c have an image under ψ which is close to $\psi(c)$ (see Proposition 5.2.4). The notion of cut point is related to the notion of gap defined in [AvdD02a] (see Lemma 5.2.9). These results on the behavior of ψ have important consequences for the differential rank.

In Section 5.3, we use Section 5.1 to introduce the differential rank (Definition 5.3.1) and give several characterizations of it (see Theorems 5.3.3, 5.3.5 and 5.3.11). It turns out that the differential rank is too coarse to describe the whole structure of the differential-valued field. This problem is connected to the notion of cut point. Indeed, if c is a cut point, then the differential rank gives no information on the behavior of ψ on the set of elements "below" c. For this reason, we introduce a notion of "unfolded" differential rank in Section 5.3.2 (see Definition 5.3.14), which extends the differential rank. More precisely, the differential rank is a final segment of the unfolded differential rank (see Proposition 5.3.17). We then connect the unfolded differential rank to the notion of exponential rank (Corollary 5.3.22).

In Section 5.4, we tackle the problem of defining a derivation on generalized power series. In Section 5.4.1, we give a method to “lift” the map ψ of an asymptotic couple (G, ψ) to a derivation on a field of power series $k((G))$. Theorem 5.4.12 gives necessary and sufficient conditions on (G, ψ) and k for this method to work. In Sections 5.4.2 and 5.4.3, we show that the existence of a ψ on G is connected to the notion of shift (see Definition 5.4.15). Using Theorem 5.4.12, this allows us to construct derivations on fields of power series from a shift on its value chain Γ (see Theorems 5.4.19 and 5.4.22). Finally, in Section 5.4.4, we use Theorem 5.4.12 to realize a given pair of ordered sets as the differential rank and the unfolded differential rank of a field of power series endowed with a Hardy-type derivation (Theorem 5.4.28).

We refer to Section 2.3 for our notations and conventions on valued fields. All groups considered in this chapter are abelian and will thus be denoted additively.

5.1 The ϕ-rank of a valued field

In this section, we develop the general theory of the rank of a field endowed with an operator ϕ. For us, a **partial map** on a set A is a map from some subset B of A to A. The domain of a partial map ϕ is denoted by Domϕ. The identity map of a set will always be denoted by id.

The classical rank of a valued field $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ is characterized on three different levels: at the level of the field K itself, at the level of its value group G and at the level of the valuation chain Γ (see Proposition 2.3.2). This is why we now want to define three notions of ϕ-ranks: one for quasi-ordered sets, one for ordered groups and another one for valued fields. Using quasi-orders instead of orders at the level of the set will be useful to give a certain characterization of the differential rank, see Theorem 5.3.5.

We first define the notion of rank of a quasi-ordered set. Let $(A, \precsim)$ be a q.o. set.

Definition 5.1.1
The **rank of the q.o. set** $(A, \precsim)$ is the order type of the set of all final segments of $(A, \precsim)$, ordered by inclusion. A final segment B of $(A, \precsim)$ is called a **principal final segment** of B if there is $a \in A$ such that $B = \{b \in A \mid a \precsim b\}$. The **principal rank of the q.o. set** $(A, \precsim)$ is the order type of the set of principal final segments of $(A, \precsim)$.

Note that the rank of $(A, \precsim)$ is the same as the rank of $(A/\sim, \leq)$, where $\leq$ is the order induced by $\precsim$ on $A/\sim$. Note also that the principal rank of $(A, \precsim)$ is isomorphic to $(A/\sim, \leq^*)$, where $\leq^*$ is the reverse order of $\leq$ (this is given by the order-reversing bijection $a \mapsto \{b \in A \mid a \precsim b\}$ from $A/\sim$ to the set of principal final segments of A).

We now define three notions of ϕ-rank:

Definition 5.1.2 (ϕ-rank for q.o. sets)
Let $(A, \precsim)$ be a q.o. set and ϕ a partial map on a set A. We say that a subset B of A is **compatible with** ϕ, or **ϕ-compatible**, if for any $a \in A \cap \mathrm{Dom}\phi$, $a \in B \Leftrightarrow \phi(a) \in B$. If $(A, \precsim)$ is a quasi-ordered set, ϕ a partial map on A and $b \in A$, we say that $B \subseteq A$

is the **ϕ-principal final segment of $(A, \precsim)$ generated by** b if B is the smallest ϕ-compatible final segment of $(A, \precsim)$ containing b. We say that a final segment B of $(A, \precsim)$ is **ϕ-principal** if there is some $b \in A$ such that B is ϕ-principal generated by b. We then define the **ϕ-rank** (respectively, the **principal ϕ-rank**) of the quasi-ordered set $(A, \precsim)$ as the order type of the set of ϕ-compatible (respectively, ϕ-principal) final segments of $(A, \precsim)$, ordered by inclusion.

Definition 5.1.3 (ϕ-rank for ordered groups)
If $(G, \leq)$ is an ordered abelian group with a partial map ϕ, we say that H is **the ϕ-principal convex subgroup of G generated by g** if H is the smallest ϕ-compatible convex subgroup of G containing g. We define the **ϕ-rank** (respectively, the **principal ϕ-rank**) of the ordered group $(G, \leq)$ as the order type of the set of convex ϕ-compatible (respectively, ϕ-principal) non-trivial subgroups of $(G, \leq)$.

Definition 5.1.4 (ϕ-rank for valued fields)
If (K, v) is a valued field with a partial map ϕ, we say that a coarsening w of v is **ϕ-compatible** if $\mathcal{U}_w$ is ϕ-compatible as a set. We say that w is **the ϕ-principal coarsening of v generated by a** if $\mathcal{O}_w$ is the smallest overring of $\mathcal{O}_v$ containing a such that w is ϕ-compatible. We define the **ϕ-rank** (respectively, the **principal ϕ-rank**) of the valued field (K, v) as the order type of the set of ϕ-compatible (respectively, ϕ-principal) strict coarsenings of v.

Example 5.1.5 (a) The rank of a q.o. set $(A, \precsim)$ is equal to its id-rank.

(b) Let $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ be a valued field. Then the classical rank of (K, v) as a valued field is the id-rank of the valued field (K, v). It is known that it is also equal to the id-rank of the ordered group $(G, \leq)$ and to the id-rank of the ordered set $(\Gamma, \leq)$.

(c) Let $(K, \leq, \exp)$ be an ordered field endowed with a $(GA), (T_1)$-exponential as defined in [Kuh00]. Let ϕ be the logarithm $\exp^{-1}$ restricted to $K^{>0} \backslash \mathcal{O}_v$, where v is the natural valuation associated to $\leq$. One can easily check that our notion of compatibility coincides with the notion of compatibility defined in [Kuh00]. In particular, a coarsening w of v satisfies our definition of compatibility with ϕ if and only if the logarithm is compatible with w in the sense of [Kuh00]. It follows that the exponential rank of $(K, \leq, \exp)$ is the ϕ-rank of the valued field (K, v).

(d) Let (K, v, σ) be a valued difference field. Then one can check that a coarsening w of v satisfies our condition of σ-compatibility if and only if w is σ-compatible in the sense of [KMP17]. It follows that the σ-rank of (K, v, σ) defined in [KMP17] coincides with our definition of the σ-rank of the valued field (K, v).

(e) S. Kuhlmann showed in [Kuh00] that the exponential rank of $(K, \leq, \exp)$ is also equal to the χ-rank of $(G, \leq)$, where χ is the map induced by the logarithm on the value group G of (K, v). She also showed that it is equal to the ζ-rank of $(\Gamma, \leq)$, where ζ is the map induced by χ on Γ. She together with F. Point and M. Matusinski also showed similar results for difference fields in [KMP17] (see Section 2.3).

Remark 5.1.6: (i) Let $(A, \precsim)$ be a q.o. set and ϕ a partial map on A. By definition, the ϕ-principal rank of $(A, \precsim)$ is a totally ordered set $(\Gamma, \leq)$. It is easy to see that the ϕ-rank of $(A, \precsim)$ is then equal to the rank of $(\Gamma, \leq)$. This shows in particular that the principal ϕ-rank completely determines the ϕ-rank. This remark still applies if we replace $(A, \precsim)$ by an ordered group or by a valued field.

(ii) Let R denote the set of ϕ-compatible final segments of $(A, \precsim)$ and let $B \subseteq A$. Then B is a ϕ-principal final segment if and only if there exists $a \in A$ such that $B = \bigcap\limits_{a \in C \in R} C$.

Following Example 5.1.5 (e), we now want to show that the ϕ-rank of a valued field with an operator ϕ can be characterized at three different levels, as happens in the classical case. This can only be done if ϕ induces a map on the value group, which is why we need the following definition:

Definition 5.1.7
We say that a map ϕ is **consistent** with a valuation v if $v(a) = v(b) \Rightarrow v(\phi(a))) = v(\phi(b))$ for all $a, b \in \mathrm{Dom}(\phi)$.

Let $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ be a valued field and ϕ a partial map on K. If ϕ is consistent with v, then ϕ naturally induces a partial map on G defined by $\phi_G(v(a)) = v(\phi(a))$. If ϕ_G is consistent with v_G, then it induces a partial map ϕ_Γ on Γ defined by $\phi_\Gamma(v_G(g)) = v_G(\phi_G(g))$. We then have the following result:

Proposition 5.1.8
Let $K \xrightarrow{v} G \xrightarrow{v_G} \Gamma$ be a valued field and ϕ a partial map on K. Assume that ϕ is consistent with v. Then the ϕ-rank (respectively, the principal ϕ-rank) of the valued field (K, v) is equal to the ϕ_G-rank (respectively, the principal ϕ_G-rank) of the ordered group $(G, \leq)$, where ϕ_G is the partial map of G induced by ϕ. If moreover ϕ_G is consistent with v_G, then the ϕ-rank (respectively, the principal ϕ-rank) of the valued field (K, v) is also equal to the ϕ_Γ-rank (respectively, the principal ϕ_Γ-rank) of the ordered set $(\Gamma, \leq)$, where ϕ_Γ denotes the partial map of Γ induced by ϕ_G.

Proof. We already know (see [Kuh00, chapter 3, Section 1]) that there is an inclusion-preserving bijection $\Xi : w \mapsto G_w := v(\mathcal{U}_w)$ between the set of coarsenings of v and the set of convex subgroups of G. Now note that, for any $a \in K$ and $g := v(a)$, we have $a \in \mathrm{Dom}\phi \Leftrightarrow g \in \mathrm{Dom}\phi_G$, $a \in \mathcal{U}_w \Leftrightarrow g \in G_w$ and $\phi(a) \in \mathcal{U}_w \Leftrightarrow \phi_G(g) \in G_w$. It immediately follows that w is ϕ-compatible if and only if G_w is ϕ_G-compatible. This in turn implies that w is ϕ-principal generated by a if and only if G_w is ϕ_G-principal generated by $v(a)$. Therefore, the map Ξ restricted to the set of ϕ-compatible coarsenings of v gives us an order-preserving bijection from the ϕ-rank of the valued field (K, v) to the ϕ_G-rank of the ordered group $(G, \leq)$. Moreover, the map Ξ restricted to the set of ϕ-principal coarsenings of v gives us an order-preserving bijection from the principal ϕ-rank of the valued field (K, v) to the principal ϕ_G-rank of the ordered group $(G, \leq)$. This proves the first part of the Proposition. We could use an analogous proof to show that there is a bijection between ϕ_G-compatible convex subgroups of G and ϕ_Γ-compatible final segments of Γ (using the map $G_w \mapsto v_G(G_w\backslash\{0\})$). □

Example 5.1.9 (a) Consider an ordered exponential field $(K, \leq, \exp)$ satisfying axioms $(GA), (T_1)$ of [Kuh00] and denote by v its natural valuation. Define ϕ as the logarithm $\log := \exp^{-1}$ restricted to $K^{>0}\backslash\mathcal{O}_v$. One easily sees that ϕ is consistent with v. The induced map on G is the contraction map χ studied in [Kuh00]. It then follows from Proposition 5.1.8 that the exponential rank of K is equal to the χ-rank of G, as stated in [Kuh00].

(b) Applying Proposition 5.1.8 to the case where $\phi = \sigma$ is a field automorphism, we recover Theorem 2.3.5.

In [KMP17], the authors characterized the difference rank of a difference field in terms of an equivalence relation induced by σ (see Theorem 2.3.6 above). We will now give a proposition which will later allow us to prove similar results for the notion of differential rank. Assume then that $(A, \precsim)$ is a quasi-ordered set and that ϕ is an increasing map on A with $\mathrm{Dom}\phi = A$.

We associate the following relations to ϕ:
$a \precsim_\phi b \Leftrightarrow \exists n, k \in \mathbb{N}_0 \quad \phi^n(a) \precsim \phi^k(b)$.
$a \sim_\phi b \Leftrightarrow a \precsim_\phi b \wedge b \precsim_\phi a$.

Proposition 5.1.10
The relation $\precsim_\phi$ is a quasi-order on A and a coarsening of $\precsim$. Moreover, for any $\precsim$-convex subset B of A the following statements are equivalent:

(1) B is ϕ-compatible.

(2) B is $\precsim_\phi$-convex.

(3) B is $\sim_\phi$-closed, in the sense that for any $a, b \in A$ with $a \sim_\phi b$, $a \in B \Leftrightarrow b \in B$.

In particular, the ϕ-rank of $(A, \precsim)$ is equal to the rank of $(A, \precsim_\phi)$ and the principal ϕ-rank of $(A, \precsim)$ is equal to the principal rank of $(A, \precsim_\phi)$.

Proof. We start by showing the following:

Claim: If $a \precsim_\phi b$ and $b \precsim_\phi c$, then there are $j, k, n \in \mathbb{N}_0$ with $\phi^k(a) \precsim \phi^j(b) \precsim \phi^n(c)$.

Proof. By definition of $\precsim_\phi$ there are $k, l, m, n \in \mathbb{N}_0$ with $\phi^k(a) \precsim \phi^l(b)$ and $\phi^m(b) \precsim \phi^n(c)$. Assume that $l \leq m$. Since ϕ is increasing, $\phi^k(a) \precsim \phi^l(b)$ implies $\phi^{k+m-l}(a) \precsim \phi^m(b) \precsim \phi^n(c)$, hence the claim. If $m < l$, then $\phi^m(b) \precsim \phi^n(c)$ implies $\phi^k(a) \precsim \phi^l(b) \precsim \phi^{n+l-m}(c)$. □

Now we prove the proposition. Obviously, $\precsim_\phi$ is reflexive and total because $\precsim$ is, and thanks to the claim $\precsim_\phi$ is also transitive, so $\precsim_\phi$ is a quasi-order. Note that $\sim_\phi$ is the equivalence relation induced by the q.o. $\precsim_\phi$. Now let B be $\precsim$-convex. By definition of convexity, (2) implies (3). Since $a \sim_\phi \phi(a)$ is true for all a, (3) implies (1). Now let us prove that (1) implies (2), so assume B is ϕ-compatible. Let $a, c \in B$ and $b \in A$ such that $a \precsim_\phi b \precsim_\phi c$. By the claim, there are $j, k, n \in \mathbb{N}_0$ with $\phi^k(a) \precsim \phi^j(b) \precsim \phi^n(c)$. Since B is ϕ-compatible, we have $\phi^k(a), \phi^n(c) \in B$. Since B is $\precsim$-convex, $\phi^k(a) \precsim \phi^j(b) \precsim \phi^n(c)$ implies $\phi^j(b) \in B$. By ϕ-compatibility, this implies $b \in B$. This shows that B is $\precsim_\phi$-convex.

Now let us prove the last statement. We show that a subset B of A is a final segment of $(A, \precsim_\phi)$ if and only if it is a ϕ-compatible final segment of $(A, \precsim)$. Let B be a ϕ-compatible final segment of $(A, \precsim)$. Let $b \in B$ and $a \in A$ with $b \precsim_\phi a$. If $b \sim_\phi a$ then since (1) implies (3) we have $a \in B$. If $b \precnsim_\phi a$, then since $\precsim_\phi$ is a coarsening of $\precsim$ we must have $b \precnsim a$. Since B is a final segment of $(A, \precsim)$, it follows that $a \in B$. This shows that B is a final segment of $(A, \precsim_\phi)$. Conversely, assume that B is a final segment of $(A, \precsim_\phi)$. Since $\precsim_\phi$ is a coarsening of $\precsim$ then B must also be a final segment of $(A, \precsim)$. In particular, B is $\precsim$-convex and $\precsim_\phi$-convex. Since (2) implies (1), B must be ϕ-compatible. This shows that the ϕ-rank of $(A, \precsim)$ is equal to the rank of $(A, \precsim_\phi)$. It then follows from Remark 5.1.6(ii) that the principal ϕ-rank of $(A, \precsim)$ is equal to the principal rank of $(A, \precsim_\phi)$. □

Remark 5.1.11: By applying Proposition 5.1.10 to the case of difference fields, we recover Theorem 2.3.6. Our relation $\sim_\phi$ corresponds to $\sim_{\sigma_\Gamma}$ in [KMP17] and to $\sim_\zeta$ in [Kuh00].

5.2 Asymptotic couples and cut points

This section is dedicated to the study of asymptotic couple, and more precisely of the behavior of the map ψ. We introduce the notion of cut point (Definition 5.2.2), which plays a central role in the description of ψ. These results will be extremely important for the study of the differential rank in Section 5.3. We refer to Section 2.4 for the definition of asymptotic couple.

Notation
In all this section, (G, ψ) is an asymptotic couple. We will denote by $\leq$ the order of G, by v_G the archimedean valuation associated to $(G, \leq)$ and by $\precsim$ the quasi-order induced by v_G, i.e the q.o. defined by $g \precsim h \Leftrightarrow v_G(g) \geq v_G(h)$. For any $g, h \in G$, we write $g \asymp h$ if and only if $g \sim h$ and g has the same sign as h. Note that "$\asymp$" is an equivalence relation. We set $G^{\neq 0} := G \backslash \{0\}$, $G^{>0} := \{g \in G \mid 0 < g\}$ and $G^{<0} := \{g \in G \mid g < 0\}$. If $g \in G$ and $H \subseteq G$, we write $g < H$ to mean that $g < h$ for all $h \in H$. Note that if H is a convex subgroup this implies $h \precnsim g$ for all $h \in H$. We denote by Ψ the set $\psi(G^{\neq 0})$). If (G, ψ) happens to be H-type, then ψ is consistent with v_G, in which case we will denote by ω the map induced by ψ on Γ. Finally, D_G denotes the map $D_G : G^{\neq 0} \to G$, $g \mapsto g + \psi(g)$.

We recall the following result from [AvdD02a, Proposition 2.3]:

Lemma 5.2.1
Let (G, ψ) be an asymptotic couple and $g, h \in G^{\neq 0}$ with $g \neq h$. Then $\psi(g) - \psi(h) \precnsim g - h$.

We also define the following notion:

Definition 5.2.2
Let $c \in G$. we say that c is a **cut point** for ψ if $\psi(g) \precnsim g \Leftrightarrow c \precnsim g$ for any $g \in G^{\neq 0}$. A cut point c for ψ is called **regular** if $c = 0$ or if $c \neq 0$ has the same sign as $\psi(c)$.

The notion of cut point will play an essential role in the study of the differential rank, which is why the next results are crucial. Note that, for any $g, h \in G$, $g \precnsim h$ implies $g + h \asymp h$. Note also that $g - h \precnsim h$ implies $g \asymp h$.

Lemma 5.2.3
The following holds:

(i) For any $g \in G^{\neq 0}$ with $g \precsim \psi(g)$, we have $h \precsim \psi(g) \Rightarrow \psi(h) - \psi(g) \precnsim \psi(g)$, and in particular $h \precsim \psi(g) \Rightarrow \psi(g) \asymp \psi(h)$.

(ii) For any $c \in G^{\neq 0}$, c is a cut point for ψ if and only if $\psi(c) \sim c$, and c is a regular cut point if and only if $c \asymp \psi(c)$.

(iii) For any $c, g \in G$, if c is a cut point for ψ, then g is a cut point for ψ if and only if $g \sim c$.

(iv) For any $g \in G^{\neq 0}$, if $g \precsim \psi(g)$, then $\psi(g)$ is a regular cut point for ψ.

(v) If (G, ψ) is H-type and $c \in G^{\neq 0}$ is a cut point for ψ, then $\psi(c)$ is a fixpoint of ψ.

Proof. For (i): By Lemma 5.2.1, we have $\psi(h) - \psi(g) \precnsim h - g$. If $h \precsim g$, then $h - g \precsim g$, hence $\psi(h) - \psi(g) \precnsim \psi(g)$. This implies in particular that $\psi(h) \asymp \psi(g)$. For (ii): Assume c is a cut point. By Definition 5.2.2, we have $c \precsim \psi(c)$, hence by (i): $\psi(\psi(c)) \sim \psi(c)$. If $c \precnsim \psi(c)$ were true, then Definition 5.2.2 would imply $\psi(\psi(c)) \precnsim \psi(c)$, which is impossible. Thus, we must have $\psi(c) \sim c$. Conversely, assume $\psi(c) \sim c$ and let $h \in G$ with $c \precnsim h$. If $h \precsim \psi(h)$ were true, then (i) would imply $\psi(c) \sim \psi(h)$, hence $h \precsim c$, a contradiction. Therefore, $\psi(h) \precnsim h$ must hold. Now take $h \precsim c$. Then we have $h \precsim \psi(c)$, which by (i) implies $\psi(h) \sim \psi(c)$, hence $h \precsim \psi(h)$. This proves that c is a cut point. (iii) follows directly from Definition 5.2.2. For (iv): Let $0 \neq g \precsim \psi(g)$. Then by (i), we have $\psi(\psi(g)) \asymp \psi(g)$. By (ii), this means that $\psi(g)$ is a cut point. For (v): by (ii), we have $c \sim \psi(c)$. Since (G, ψ) is H-type, ψ is constant on archimedean classes of G, so $c \sim \psi(c)$ implies $\psi(c) = \psi(\psi(c))$.

□

We can now show that every asymptotic couple admits a cut point, and describe the behaviors of the maps ψ and D_G on G:

Proposition 5.2.4
Let (G, ψ) be an asymptotic couple. Then G admits a regular cut point for ψ. Moreover, for any regular cut point c and any $g \neq 0$, the following holds:

(1) If $g \precsim c \neq 0$, then $\psi(g) - \psi(c) \precnsim c$ and in particular $\psi(g) \asymp c$.

(2) If $g \precnsim c \neq 0$, then $D_G(g) \asymp c$.

(3) If $0 \neq g \sim c$, then $D_G(g) \precsim c$

(4) If $c \precnsim g$, then $\psi(g) \precnsim g$ and $D_G(g) \asymp g$.

Proof. Assume 0 is not a cut point. This means there is $g \in G^{\neq 0}$ with $g \precsim \psi(g)$, which by Lemma 5.2.3 (iv) means that $\psi(g)$ is a regular cut point for ψ. Now let c be a regular cut point for ψ and $0 \neq g$. (1) follows from Lemma 5.2.3 (i) and (ii). For (2): By (1), we have $g \precnsim \psi(g)$. It immediately follows that $g + \psi(g) \asymp \psi(g)$, which by (1) implies $g + \psi(g) \asymp c$. For (3): By (1), we have $\psi(g) \sim g$, so the ultrametric inequality implies $\psi(g) + g \precsim g$. For (4): $\psi(g) \precnsim g$ follows from Definition 5.2.2. It then follows that $\psi(g) + g \asymp g$. □

We now want to show that cut points can be used to characterize compatibility of convex subgroups of G with ψ. This will be given by Proposition 5.2.7.

Lemma 5.2.5
If $c \neq 0$ is a cut point and $g \in G^{\neq 0}$ is such that $c \precnsim \psi(g)$, then $\psi(g)$ is negative. In particular, $\psi(g) > 0$ implies that $\psi^2(g)$ is a non-zero cut point for ψ.

Proof. By Lemma 5.2.3(ii) and the ultrametric inequality, we have $\psi(c) + |c| \precsim c$. If $\psi(g) > 0$, then $c \precnsim \psi(g)$ implies $\psi(c) + |c| < \psi(g)$ which contradicts (AC3). Thus, $\psi(g) > 0$ implies $\psi(g) \precsim c$, which by Proposition 5.2.4(1) implies $\psi^2(g) \sim c$. Since $c \neq 0$, it follows that $\psi^2(g) \neq 0$. By Lemma 5.2.3(iii), $\psi^2(g)$ is then a non-zero cut-point for ψ. □

Lemma 5.2.6
Assume $c \in H$. Then $g \in H \Rightarrow \psi(g) \in H$.

Proof. Assume $g \in H$. If $g \precsim c$, then Proposition 5.2.4(1) implies $\psi(g) \sim c$. Since $c \in H$, it follows from the convexity of H that $\psi(g) \in H$. If $c \precnsim g$, then it follows from Proposition 5.2.4(4) that $\psi(g) \precnsim g$. Since $g \in H$, it follows from the convexity of H that $\psi(g) \in H$. □

Proposition 5.2.7
Let c be a cut point for ψ and H a non-trivial convex subgroup of G. Then H is compatible with ψ if and only if the following two conditions hold:

(1) $c \in H$.

(2) For any $g \in G$ with $c \precnsim g$, $\psi(g) \in H \Rightarrow g \in H$.

Moreover, if H is compatible with ψ, then $\psi(g) < H$ for any $g \in G \backslash H$.

Proof. Assume that H is compatible with ψ. Then clearly (2) is true. Towards a contradiction, assume that $c \notin H$ and take $g \in H$. By convexity of H, we have $g \precsim c$. By Proposition 5.2.4(1), this implies $\psi(g) \sim c$. It follows from the convexity of H that $\psi(g) \notin H$, which contradicts the fact that H is ψ-compatible. This proves that (1) must hold. Conversely, assume (1) and (2) and let us prove that H is compatible with ψ. It follows from condition (1) and from Lemma 5.2.6 that $g \in H \Rightarrow \psi(g) \in H$. Now assume that $\psi(g) \in H$ and let us show $g \in H$. If $c \precsim g$, this follows from condition (2). If $g \precsim c$, then since $c \in H$ it follows from convexity of H that $g \in H$. This proves that H is compatible with ψ. Now let $g \in G \backslash H$. Then $\psi(g) \notin H$, and since $c \in H$, the convexity of H implies $c \precsim \psi(g)$. It follows from Lemma 5.2.5 that $\psi(g)$ is negative. By convexity of H, we then have $\psi(g) < H$.

□

Proposition 5.2.4 shows that the behavior of ψ is particularly simple if 0 is the cut point for ψ (note that if 0 is a cut point for ψ then it is the only cut point). Indeed, in that case, ψ acts like a contraction map on $G^{<0}$ (see Section 2.3 for the definition of contraction). Therefore, it can be practical to transform a given map ψ into another map ψ' which has 0 as a cut point. One way of doing that is to translate ψ by a gap or by the maximum of Ψ if it exists. We recall that a **gap** for ψ is an element $g \in G$ such that $\Psi < g < D_G(G^{>0})$. Aschenbrenner and v.d.Dries showed that the existence of a gap or of a maximum of Ψ is connected to the existence of asymptotic integration in (G, ψ). We recall the following results (see [AvdD02a, Proposition 2.2 and Theorem 2.6]):

Proposition 5.2.8
Let (G, ψ) be an asymptotic couple. The following holds:

(1) ψ has at most one gap.

(2) $G \backslash D_G(G^{\neq 0})$ has at most one element.

(3) If g is a gap for ψ or the maximum of Ψ then $G \backslash D_G(G^{\neq 0}) = \{g\}$. In particular, if (G, ψ) has asymptotic integration, then Ψ has no maximum and ψ has no gap.

Gaps and maximum of Ψ are connected to our notion of cut point; more precisely, we have the following:

Lemma 5.2.9 (i) If 0 is a cut point for ψ, then $0 = \sup \Psi$ and (G, ψ) does not have asymptotic integration.

(ii) If g is a gap for ψ or the maximum of Ψ, then g is a regular cut point for ψ.

Proof. Let us prove (i). Since 0 is the cut point, Proposition 5.2.4(4) implies that $D_G(h) \sim h$ for all $h \in G^{\neq 0}$. This implies in particular that $0 \notin D_G(G^{\neq 0})$, which proves that (G, ψ) does not have asymptotic integration. Assume there is g with $\psi(g) > 0$ and take $h \in G^{\neq 0}$. By (AC3), we have $0 < \psi(g) < D_G(|h|)$, which implies $\psi(g) \precsim D_G(|h|)$. Since $D_G(|h|) \sim h$, we thus have $\psi(g) \precsim h$ for all $h \neq 0$. It follows that $\psi(\psi(g)) = 0$ (otherwise, taking $h := \psi(\psi(g))$, we would have $\psi(g) \precsim \psi(\psi(g))$, which contradicts the fact the 0 is the cut point). But by (AC3), we have $\psi(g) < \psi(\psi(g)) + \psi(g)$, hence $\psi(g) < \psi(g)$, a contradiction. Thus, $\sup \Psi \leq 0$. For any $g < 0$, since 0 is a cut point we have $g < \psi(g) < 0$, hence $\sup \Psi = 0$.

Now let us prove (ii). Let c be a regular cut point for ψ. If $c = 0$, then by (i) we must have $g = 0 = c$. Assume $c \neq 0$. Assume first that $v_G(c) \neq \max \Gamma$. Then there is $h \in G^{\neq 0}$ such that $h \precnsim c$. By Proposition 5.2.4(2) and Lemma 5.2.3(ii), $D_G(|h|) \asymp c \asymp \psi(c)$. By assumption on g, we have $\psi(c) \leq g < D_G(|h|)$. It follows that $g \asymp c$. By Proposition 5.2.4(1), $\psi(g) \asymp c$. It follows that $\psi(g) \asymp g$, which by Lemma 5.2.3(ii) means that g is a regular cut point. Now assume that $v_G(c) = \max(\Gamma)$. It follows from Proposition 5.2.4(1) that $\psi(h) = \psi(c)$ for any h with $h \sim c$. Take h such that $0 < h \leq |\psi(c)|$. Then $h \sim c$, and $D_G(h) = h + \psi(h) = h + \psi(c)$ has the same sign as $\psi(c)$. By assumption on g, we have $\psi(c) \leq g < D_G(h)$. It follows that $g \sim c$, hence $\psi(g) = \psi(c)$. In particular, g has the same sign as $\psi(g)$ and $g \sim \psi(g)$. This shows that g is a regular cut point.

□

As we mentioned above, if a gap or a maximum of Ψ exists, then we can transform ψ into a map which has 0 as a cut point. This will become important in Section 5.3.2:

Lemma 5.2.10
Assume that $c \in G$ is either a gap for ψ or the maximum of Ψ. Define $\psi'(g) := \psi(g) - c$. Then 0 is a cut point for ψ'.

Proof. Just note that 0 is a gap for ψ' or a maximum of Ψ', so the claim follows from Lemma 5.2.9(ii). □

We now want to focus on the case where (G, ψ) is H-type. In that case, there is a canonical choice for a regular cut point, namely the fixpoint of ψ:

Lemma 5.2.11
Assume (G, ψ) is H-type. Then 0 is a cut point for ψ if and only if ψ has no fixpoint. If ψ has a fixpoint, then it is unique and it is a regular cut point for ψ. Moreover, if ψ has a negative fixpoint, then ψ only takes negative values.

Proof. If 0 is a cut point then by definition $\psi(g) \precnsim g$ holds for every $g \neq 0$ so ψ has no fixpoint. If c is a cut point with $c \neq 0$, then by Lemma 5.2.3(v) $\psi(c)$ is a fixpoint of ψ. If d is another fixpoint then by Lemma 5.2.3(ii) it must be a cut point, so by Lemma 5.2.3(iii) $d \sim \psi(c)$, but since (G, ψ) is H-type this implies $\psi(\psi(c)) = \psi(d)$ hence $\psi(c) = d$. This proves the uniqueness of the fixpoint. If c is the fixpoint of G, then we have in particular $c \asymp \psi(c)$ which by Lemma 5.2.3(ii) implies that c is a regular cut point. Finally, assume $c < 0$. Since $c = \psi(c)$ is a cut point, we have $\psi(g) < 0$ for any $g \precnsim c$ by Proposition 5.2.4(1). Now take g with $c \precnsim g$. Since (G, ψ) is H-type, we have $\psi(g) \leq \psi(c) = c < 0$. Thus, $\psi(g)$ is negative for every g. □

Proposition 5.2.4(1) and (4) allows us to describe the behavior of the map ω:

Lemma 5.2.12
Let (G, ψ) be a H-type asymptotic couple and denote by ω the map induced by ψ on Γ. Let c be a cut point for ψ and $\alpha := v_G(c)$. Then we have $\omega(\gamma) > \gamma$ for all γ with $\gamma < \alpha$ and $\omega(\gamma) = \alpha$ for all $\gamma \in \Gamma$ with $\alpha \leq \gamma$.

Proof. It follows directly from Proposition 5.2.4. □

In the H-type case, the existence of a gap is part of a trichotomy (see [Geh17, Lemma 2.4]):

Proposition 5.2.13
Let (G, ψ) be a H-type asymptotic couple. Then exactly one of the following holds:

(1) ψ has a gap.

(2) Ψ has a maximum.

(3) (G, ψ) admits asymptotic integration.

Sometimes, it is practical to work with an asymptotic couple where ψ only takes negative values. The next lemma will be useful in that regard, especially for the proof of Theorem 5.3.5:

Lemma 5.2.14
Let (G, ψ) be a H-type asymptotic couple. There exists $x \in G^{\leq 0}$ such that, if $\hat{\psi}$ denotes the map $\hat{\psi} := \psi + x$, $\alpha := v_G(x)$ and $\hat{\omega}$ is the map induced by $\hat{\psi}$ on Γ, the following holds:

(i) The ψ-rank (respectively, the principal ψ-rank) of $(G, \leq)$ is equal to the $\hat{\psi}$-rank (respectively, the principal $\hat{\psi}$-rank) of $(G, \leq)$.

(ii) $\hat{\psi}(g) < 0$ for all $g \in G^{\neq 0}$.

(iii) for all $\gamma \in \Gamma$, $\hat{\omega}(\gamma) = \min(\alpha, \omega(\gamma))$.

(iv) For any $\gamma, \delta \in \Gamma$, $\alpha \leq \gamma \wedge \alpha \leq \delta \Rightarrow \alpha \leq \omega(\gamma) = \omega(\delta)$.

Proof. If $\psi(g) < 0$ for all $g \neq 0$ then we can take $x = 0$, so assume that there is g with $\psi(g) \geq 0$. We distinguish two cases:

(1) $0 = \max \Psi$. In that case set $c = 0$ and choose $x < 0$ with $\psi(x) = 0$.

(2) there exists g with $\psi(g) > 0$. In that case define c as the fixpoint of ψ (which exists and is positive by Lemma 5.2.11) and set $x := -2c$.

Note that in both cases, c is a cut-point for ψ, x is a cut point for $\hat{\psi}$ (because $\hat{\psi}(x) \sim x$), $\psi(x) = c$ and $c \precsim x$. Let us show (i). We use Proposition 5.2.7. Assume then that H is ψ-compatible. Then by Proposition 5.2.7 $c \in H$. Since $\psi(x) = c$, it follows by ψ-compatibility of H that $x \in H$. Now take g with $\hat{\psi}(g) \in H$. We have $\psi(g) = \hat{\psi}(g) - x$, hence $\psi(g) \in H$. By ψ-compatibility, this implies $g \in H$. By Proposition

5.2.7, this proves that H is $\hat{\psi}$-compatible. Conversely, assume that H is $\hat{\psi}$-compatible. By Proposition 5.2.7, $x \in H$, and since $c \lesssim x$ it follows by convexity that $c \in H$. Now take $g \in G$ with $\psi(g) \in H$. Then $\hat{\psi}(g) = \psi(g) + x \in H$, which by $\hat{\psi}$-compatibility implies $g \in H$. It follows by Proposition 5.2.7 that H is ψ-compatible. We showed that H is ψ-compatible if and only if it is $\hat{\psi}$-compatible. It then follows from Remark 5.1.6(ii) that H is ψ-principal if and only if it is $\hat{\psi}$-principal.

Now let us show (ii),(iii),(iv). We first consider case (1). In that case, x is the fixpoint of $\hat{\psi}$, so (ii) follows from Lemma 5.2.11. Clearly, by definition of $\hat{\psi}$, $x \lnsim \psi(g) \Rightarrow \psi(g) \sim \hat{\psi}(g)$. If $\psi(g) \lnsim x$, then by definition of $\hat{\psi}$ we have $\hat{\psi}(g) \sim x$. This proves (iii). If $g \lnsim x$, (ACH) implies $\psi(g) \geq \psi(x) = 0$, and since $0 = \max \Psi$ it follows that $\psi(g) = 0$, hence (iv). Now let us consider case (2). By (AC3), we have $\psi(g) < c + \psi(c) = 2c$ for all $g \in G^{\neq 0}$, hence $\hat{\psi}(g) < 0$, hence (ii). Clearly, by definition of $\hat{\psi}$, $x \lnsim \psi(g) \Rightarrow \psi(g) \sim \hat{\psi}(g)$. Now assume $\psi(g) \lesssim x$. By Proposition 5.2.4(1), $\psi(g) - c \lnsim c$, hence $\hat{\psi}(g) = \psi(g) - 2c \sim c \sim x$. This proves (iii). Now take g, h with $g \lesssim x$ and $h \lnsim x$. Since $x \sim c$, Proposition 5.2.4(1) implies $\psi(g) \sim c \sim x \sim \psi(h)$, hence (iv). $\square$

5.3 The differential rank

This section introduces the notion of differential rank, and gives several characterizations of it in the spirit of the work done in [KMP17] for difference field. We also introduce the notion of "unfolded" differential rank. In all this section, (K, v, D) will be a pre-differential-valued field whose field of constants is $\mathcal{C}$ and (G, ψ) its asymptotic couple. We use the same notations as in Section 5.2 for (G, ψ).

5.3.1 Characterization of the differential rank

Applying Section 5.1 to the special case of pre-differential-valued fields, we introduce the notion of differential rank:

Definition 5.3.1
The **differential rank** (respectively, the **principal differential rank**) of the pre-differential-valued field (K, v, D) is the ϕ-rank (respectively, the principal ϕ-rank) of the valued field (K, v), where ϕ is the map defined on $K^{\neq 0} \backslash \mathcal{U}_v$ by $\phi(a) = \frac{D(a)}{a}$.

Example 5.3.2
Consider the Hardy field K generated by $\mathbb{R}$ (all constants functions) and by the maps $x \mapsto x, x \mapsto \log(x)$ and $x \mapsto e^x$. Then the differential rank of K is 1. If we add all the iterates of the exponential ($e^{e^x}, e^{e^{e^x}}, \dots$), then the differential rank remains 1. Now let K' be the Hardy field generated by T over K, where T is a transexponential function. We know that such a field exists thanks to [Bos86]. Then K' has differential rank 2.

Proposition 5.1.8 then allows us to characterize the differential rank at three different levels:

Theorem 5.3.3
Let (K, v, D) be a pre-differential-valued field with asymptotic couple (G, ψ). Then the differential rank (respectively, the principal differential rank) of (K, v, D) is equal to the ψ-rank (respectively, the principal ψ-rank) of the ordered abelian group G. Moreover, if (G, ψ) happens to be H-type, then the differential rank (respectively, the principal differential rank) of (K, v, D) is also equal to the ω-rank (respectively, the principal ω-rank) of Γ, where Γ is the value chain of G and ω is the map induced by ψ on Γ.

We now want to express the differential rank as the rank of some quasi-order. This will give us a differential analog of Theorem 2.3.6. We mentioned in Remark 5.1.11 that applying our Proposition 5.1.10 recovers Theorem 2.3.6, so our idea is to apply Proposition 5.1.10 to the differential case to obtain similar results. One difficulty here is that, even assuming that (G, ψ) is H-type, the maps ϕ, ψ and ω are not increasing on their domains, so we cannot directly apply Proposition 5.1.10. We still managed to obtain similar results to those in [KMP17], as Theorem 5.3.5 below shows. The idea is to remark that, if ψ is H-type and only takes negative values, then we can apply Proposition 5.1.10 to the ordered set $(G^{<0}, <)$. If ψ takes non-negative values, one can use Lemma 5.2.14 which brings back to the case where ψ only takes negative values.

Similarly to what was done in [KMP17], we define the set $P_K := K \backslash \mathcal{O}_v$. We now introduce three binary relations $\precsim_\phi, \precsim_\psi, \precsim_\omega$ respectively defined on $P_K, G^{<0}$ and Γ as follows:

$$a \precsim_\phi b \Leftrightarrow \exists n, k \in \mathbb{N}_0, v(\phi^n(a)) \leq v(\phi^k(b))$$

$$g \precsim_\psi h \Leftrightarrow \exists n, k \in \mathbb{N}_0, \psi^n(g) \leq \psi^k(h)$$

$$\gamma \precsim_\omega \delta \Leftrightarrow \exists n, k \in \mathbb{N}_0, \omega^n(\gamma) \leq \omega^k(\delta)$$

Three important remarks are in order: first, it is not obvious from their definitions that these relations are quasi-orders, but we will show it in Theorem 5.3.5. Secondly, note that it can happen that $\phi(a) \notin P_K$ when $a \in P_K$, which is also a reason why we cannot apply Proposition 5.1.10 directly (ϕ was assumed to be a map from A to itself in Section 5.1). Thirdly, it can happen that the term $\phi^n(a)$ is not well-defined for a certain $n \in \mathbb{N}$: indeed, remember that the domain of ϕ is $K^{\neq 0} \backslash \mathcal{U}_v$. Thus, if $v(\phi(a)) = 0$, $\phi^2(a)$ is not well-defined. Therefore, when we write $\phi^n(g)$ it is always implicitly assumed that this expression is defined, i.e expressions like "$\phi^n(g) \leq \phi^k(h)$" should be read as "$\phi^n(g), \phi^k(g)$ both exist and $\phi^n(g) \leq \phi^k(h)$ holds". Note that if we allowed the domain of ϕ to be $K^{\neq 0}$ then $\precsim_\phi$ would not be a quasi-order, and Theorem 5.3.5 would fail. Similar remarks apply to ψ and ω: $G^{<0}$ is not necessarily stable under ψ, Γ is not necessarily stable under ω (we can have $\omega(\gamma) = \infty$), $\psi^2(g)$ is not well-defined if $\psi(g) = 0$ and $\omega^2(\gamma)$ is not well-defined if $\omega(\gamma) = \infty$. Note however that for any $a \in \mathrm{Dom}\phi$, $\phi^n(a)$ is well-defined if and only if $\psi^n(v(a))$ is, and that we then have $v(\phi^n(a)) = \psi^n(v(a))$. Similarly, for any $g \in G^{<0}$, $\psi^n(g)$ is well-defined if and only if $\omega^n(v_G(g))$ is, in which case $v_G(\psi^n(g)) = \omega^n(v_G(g))$ holds. As a consequence, we have the following lemma:

Lemma 5.3.4
Assume that (G, ψ) is H-type. For all $a, b \in P_K$, $a \precsim_\phi b \Leftrightarrow v(a) \precsim_\psi v(b)$. For all $g, h \in G^{<0}$, $g \precsim_\psi h \Leftrightarrow v_G(g) \precsim_\omega v_G(h)$.

Proof. The first statement follows directly from the definitions of $\precsim_\psi$ and $\precsim_\phi$. Note however that the image of ψ may contain positive elements, and that v_G reverses the order on $G^{>0}$, so the second statement is not trivial. Let us now prove the second statement. Let c be a regular cut point for ψ (which exists thanks to Proposition 5.2.4). Take $g, h \in G^{<0}$ and set $\gamma := v_G(g)$ and $\delta := v_G(h)$. We first show that $g \precsim_\psi h \Rightarrow \gamma \precsim_\omega \delta$. Assume $g \precsim_\psi h$ and let $n, k \in \mathbb{N}_0$ with $\psi^n(g) \leq \psi^k(h)$. If $\psi^k(h) \leq 0$, then this implies $v_G(\psi^n(g)) \leq v_G(\psi^k(h))$, hence $\omega^n(\gamma) \leq \omega^k(\delta)$, hence $\gamma \precsim_\omega \delta$. Assume that $0 < \psi^k(h)$. Then by Lemma 5.2.5, $\psi^{k+1}(h)$ is a non-zero cut point for ψ. By Lemma 5.2.3(iii), this implies $\psi^{k+1}(h) \sim c$. It follows that $c \neq 0$ and $v_G(c) = \omega^{k+1}(\delta)$. If $\psi^n(g) = 0$, then in particular $n \neq 0$. Moreover, it then follows from Proposition 5.2.4(1) that $c \precnsim \psi^{n-1}(g)$, hence $\omega^{n-1}(\gamma) \leq v_G(c) = \omega^{k+1}(\delta)$. If $c \precnsim \psi^n(g)$, then $v_G(\psi^n(g)) \leq v_G(c)$, hence $\omega^n(\gamma) \leq \omega^{k+1}(\delta)$. If $0 \neq \psi^n(g) \precsim c$, then Proposition 5.2.4(1) implies $\psi^{n+1}(g) \sim c$, hence $\omega^{n+1}(\gamma) = \omega^{k+1}(\delta)$. In any case, we have $\gamma \precsim_\omega \delta$. This proves $g \precsim_\psi h \Rightarrow \gamma \precsim_\omega \delta$, let us now prove the converse. Assume that $\gamma \precsim_\omega \delta$ holds and take $n, k \in \mathbb{N}_0$ with $\omega^n(\gamma) \leq \omega^k(\delta)$. This implies $\psi^k(h) \precsim \psi^n(g)$. If $c \precnsim \psi^k(h)$, then by Lemma 5.2.5 $\psi^n(g)$ and $\psi^k(h)$ are both negative. Moreover, it follows from Definition 5.2.2 that $\psi^{k+1}(h) \precnsim \psi^k(h)$, so we have $\psi^{k+1}(h) \precnsim \psi^n(g)$. Since $\psi^n(g) < 0$, this implies $\psi^n(g) < \psi^{k+1}(h)$, hence $g \precsim_\psi h$. Now assume that $\psi^k(h) \precsim c$. If $c \precnsim g$, then $\psi^k(h) \precnsim g$, and since $g < 0$ this implies $g < \psi^k(h)$, so $g \precsim_\psi h$. Similarly, if $\psi^k(h) \geq 0$, then $g < \psi^k(h)$, so $g \precsim_\psi h$. Assume then that $\psi^k(h) < 0$ and $g \precsim c$. By Proposition 5.2.4(1), $\psi^k(h) \precsim c$ and $g \precsim c$ imply $\psi^{k+1}(h) \sim c \sim \psi(g)$. If $c = 0$, then this implies $\psi^{k+1}(h) = 0 = \psi(g)$, hence $g \precsim_\psi h$. If $c \neq 0$, then because (G, ψ) is H-type, the relations $\psi^{k+1}(h) \sim c \sim \psi(g)$ imply $\psi^{k+2}(h) = \psi(c) = \psi^2(g)$, hence $g \precsim_\psi h$. In any case, we have $g \precsim_\psi h$. □

Theorem 5.3.5
Then the differential rank (respectively, the principal differential rank) of (K, v, D) is equal to the rank (respectively the principal rank) of each one of these q.o. sets:

(1) The q.o. set $(P_K, \precsim_\phi)$.

(2) The q.o. set $(G^{<0}, \precsim_\psi)$.

(3) The q.o. set $(\Gamma, \precsim_\omega)$.

Proof. Assume that (3) has been proved. In particular, $\precsim_\omega$ is a q.o. By Lemma 5.3.4, it follows that $\precsim_\psi$ is a q.o on $G^{<0}$. Consider the map: $S \mapsto v_G(S)$ from the set of final segments of $(G^{<0}, \precsim_\psi)$ to the power set of Γ. It follows from Lemma 5.3.4 that this map gives a bijection between final segments of $(G^{<0}, \precsim_\psi)$ and final segments of $(\Gamma, \precsim_\omega)$, which means that $(G^{<0}, \precsim_\psi)$ and $(\Gamma, \precsim_\omega)$ have the same rank. Moreover, one easily sees that if $S \subseteq G^{<0}$ is the principal final segment of $(G^{<0}, \precsim_\psi)$ generated by g, then $v_G(S)$ is the principal final segment of $(\Gamma, \precsim_\omega)$ generated by $v_G(g)$ and that this gives a bijection between the principal rank of $(G^{<0}, \precsim_\psi)$ and the principal rank of $(\Gamma, \precsim_\omega)$. This shows that (3) implies (2), and using Lemma 5.3.4 again one can do a similar proof that (2) implies (1). Therefore, it is sufficient to prove (3).

Now let us show (3). By Theorem 5.3.3, the (principal) differential rank of (K, v, D) is equal to the (principal) ω-rank of $(\Gamma, \leq)$. Therefore, we just have to show that $\precsim_\omega$ is a q.o. and that the (principal) rank of $(\Gamma, \precsim_\omega)$ is equal to the (principal) ω-rank of $(\Gamma, \leq)$. We first assume that we have $\Psi < 0$. In that case, we have $\infty \notin \omega(\Gamma)$. Moreover, ω is increasing: indeed, if $v_G(g), v_G(h) \in \Gamma$ are such that $v_G(g) \leq v_G(h)$, then since (G, ψ) is H-type we have $\psi(g) \leq \psi(h)$, and since $\psi(h) < 0$ this implies $v_G(\psi(g)) \leq v_G(\psi(h))$, hence $\omega(v_G(g)) \leq \omega(v_G(h))$. Therefore, we can apply Proposition 5.1.10 with $A := \Gamma$ and $\phi := \omega$, which states that the (principal) ω-rank of Γ is equal to the (principal) rank of the q.o. set $(\Gamma, \precsim_\omega)$, so (3) holds. Now assume that the condition $\Psi < 0$ is not satisfied. Let $x, \alpha, \hat{\psi}, \hat{\omega}$ be as in Lemma 5.2.14. We know from Lemma 5.2.14 that $\hat{\Psi} < 0$, so we know by what we just proved that the (principal) $\hat{\omega}$-rank of Γ is equal to the (principal) rank of the q.o. set $(\Gamma, \precsim_{\hat{\omega}})$. We also know by Lemma 5.2.14 that the (principal) $\hat{\psi}$-rank of the group $(G, \leq)$ is equal to the (principal) ψ-rank of the group $(G, \leq)$. By Theorem 5.3.3, this implies that the (principal) $\hat{\omega}$-rank of $(\Gamma, \leq)$ is equal to the (principal) ω-rank of $(\Gamma, \leq)$. Therefore, the (principal) ω-rank of $(\Gamma, \leq)$ is equal to the (principal) rank of $(\Gamma, \precsim_{\hat{\omega}})$. All that remains to show is that $\precsim_{\hat{\omega}}$ and $\precsim_\omega$ define the same relation on Γ. We use the following claim which follows directly from Lemma 5.2.14(iii) and (iv):

Claim: Let $\gamma \in \Gamma$ and $n \in \mathbb{N}_0$. If $\hat{\omega}^n(\gamma) < \alpha$ or $\omega^n(\gamma) < \alpha$, then $\omega^l(\gamma) = \hat{\omega}^l(\gamma) < \alpha$ for every $l \leq n$. If $\omega^n(\gamma) \geq \alpha$, then $\hat{\omega}^l(\gamma) = \alpha$ for all $l \geq n$.

Now let us show that $\gamma \precsim_\omega \delta \Leftrightarrow \gamma \precsim_{\hat{\omega}} \delta$. Assume $\gamma \precsim_\omega \delta$ and take $n, k \in \mathbb{N}_0$ with $\omega^n(\gamma) \leq \omega^k(\delta)$. If $\omega^k(\delta) < \alpha$, then also $\omega^n(\gamma) < \alpha$ and it follows from the claim that $\hat{\omega}^n(\gamma) = \omega^n(\gamma) \leq \omega^k(\delta) = \hat{\omega}^k(\delta)$. Assume $\alpha \leq \omega^k(\delta)$. It then follows from the claim that $\alpha = \hat{\omega}^k(\delta)$. Since $\alpha = \max(\hat{\omega}(\Gamma))$, we have $\hat{\omega}(\gamma) \leq \alpha$, hence $\hat{\omega}(\gamma) \leq \hat{\omega}^k(\delta)$. This proves $\gamma \precsim_{\hat{\omega}} \delta$. Conversely, assume that $\gamma \precsim_{\hat{\omega}} \delta$ holds, $\hat{\omega}^n(\gamma) \leq \hat{\omega}^k(\delta)$. Assume first that $\hat{\omega}^k(\delta) = \alpha$. Then by the claim, there must be $l \leq k$ with $\alpha \leq \omega^l(\delta)$. If $\gamma \leq \alpha$ or $\omega^l(\delta) = \infty$, then clearly $\gamma < \omega^l(\delta)$. If $\alpha \leq \gamma$ and $\omega^l(\delta) \neq \infty$, then Lemma 5.2.14(iv) implies $\omega(\gamma) = \omega^{l+1}(\delta)$. Now assume $\hat{\omega}^k(\delta) < \alpha$. Then also $\hat{\omega}^n(\gamma) < \alpha$, and it follows from the claim that $\omega^n(\gamma) = \hat{\omega}^n(\gamma) \leq \hat{\omega}^k(\delta) = \omega^k(\delta)$. This proves $\gamma \precsim_\omega \delta$ and concludes the proof of the Theorem. $\square$

In the case of valued difference fields, Theorem 2.3.4 above gives several characterizations of the compatibility of σ with v. We now want to explore the possibility of a similar characterization for differential-valued fields. We say that D **induces a derivation** $\bar{D}$ **on** Kw if $D(\mathcal{O}_w) \subseteq \mathcal{O}_w$ and $D(\mathcal{M}_w) \subseteq \mathcal{M}_w$ for all $a, b \in \mathcal{O}_w$. The derivation $\bar{D}$ is then defined by $\bar{D}(a + \mathcal{M}_w) := D(a) + \mathcal{M}_w$ (the fact that $\bar{D}$ is a derivation follows directly from its definition).

We want to characterize the coarsenings w of v such that D induces a derivation on Kw. The notion of cut point developed above for asymptotic couples plays here an important role, so we extend this notion to fields: If (K, v, D) is a pre-differential-valued field with asymptotic couple (G, ψ), we say that $y \in K$ is a **cut point** (respectively, a **regular cut point**) for (K, v, D) if $v(y)$ is a cut point (respectively, a regular cut point) for (G, ψ). Such an element always exists thanks to Proposition 5.2.4. We recall that $\frac{v}{w}$ denotes the valuation induced by v on Kw (see Section 2.3).

Proposition 5.3.6
Let (K, v, D) be a pre-differential-valued field, y a regular cut point for K and w a strict coarsening of v. The following holds:

(1) If $y \notin \mathcal{O}_w$, then $D(\mathcal{O}_w) \nsubseteq \mathcal{O}_w$, so D does not induce a map on Kw.

(2) If $y \in \mathcal{M}_w$, and if D induces a derivation on Kw, then D induces the constant map 0 on Kw.

(3) If $y \in \mathcal{U}_w$, then D induces a non-trivial derivation on Kw making $(Kw, \frac{v}{w}, \bar{D})$ a pre-differential-valued field. Moreover, if (K, v, D) is a differential-valued field , then $(Kw, \frac{v}{w}, \bar{D})$ is also a differential-valued field.

Proof. Remember that, for any $a \in K$ with $v(a) \neq 0$, $v(D(a)) = D_G(v(a))$. Set $G_w := v(\mathcal{U}_w)$ and $c := v(y)$. Let us prove (1). Take $a \in \mathcal{U}_w \backslash \mathcal{U}_v$, so $v(a) \in G_w \backslash \{0\}$. By assumption, we have $c < G_w$, hence $v(a) \precsim c$. By Proposition 5.2.4(2), this implies $D_G(v(a)) \asymp c$. It follows that $D_G(v(a)) < G_w$, hence $D(a) \notin \mathcal{O}_w$. Now let us prove (2). By assumption, we have $G_w < c$. Take $a \in \mathcal{U}_w$, and let us show that $D(a) \in \mathcal{M}_w$. Assume first that $a \in \mathcal{O}_v$. Then (DV2) implies $D_G(v(a)) > \psi(c)$. By Lemma 5.2.3(ii), we have $\psi(c) \asymp c$, hence $G_w < \psi(c)$. It follows that $G_w < D_G(v(a))$, hence $D(a) \in \mathcal{M}_w$. Now assume $a \in \mathcal{U}_w \backslash \mathcal{O}_v$, so in particular $v(a) \neq 0$. Since $a \in \mathcal{U}_w$, we have $v(a) \in G_w$. By convexity, $v(a) \precsim c$. Proposition 5.2.4(2) then implies $D_G(v(a)) \asymp c$, hence $G_w < D_G(v(a))$. This implies $D(a) \in \mathcal{M}_w$. Now let us prove (3). By assumption, we have $c \in G_w$. Let $a \in \mathcal{O}_w$ and let us show that $D(a) \in \mathcal{O}_w$. Assume first that $a \in \mathcal{O}_v$. If $c = 0$, then by Lemma 5.2.9(i) we have $0 = \sup \Psi$. It then follows from (DV2) that $0 \leq v(D(a))$,which implies $D(a) \in \mathcal{O}_w$. If $c \neq 0$, then (DV2) implies $v(D(a)) > \psi(c)$. By Lemma 5.2.3(ii), $\psi(c) \asymp c$, hence $\psi(c) \in G_w$ by convexity of G_w. It follows that either $v(D(a)) \in G_w$ or $G_w < v(D(a))$ holds, which implies $D(a) \in \mathcal{O}_w$. Now assume $a \in \mathcal{O}_w \backslash \mathcal{O}_v$ and set $g := v(a)$. Note that $g \neq 0$. By Proposition 5.2.4, we either have $D_G(g) \asymp g$ or $D_G(g) \precsim c$. Since $c \in G_w$ and $g \in G_w$, it follows from the convexity of G_w that $D_G(g) \in G_w$, hence $D(a) \in \mathcal{U}_w$. Now assume $a \in \mathcal{M}_w$ and let us show $D(a) \in \mathcal{M}_w$. We have so $G_w < g$. By convexity of G_w, we then have $c \precsim g$, which by Proposition 5.2.4(4) implies $D_G(g) \asymp g$, hence $G_w < D_G(g)$ hence $D(a) \in \mathcal{M}_w$. This shows that D induces a derivation on Kw. Note that we showed that $D(\mathcal{O}_w \backslash \mathcal{O}_v) \subseteq \mathcal{U}_w$, which proves that $\bar{D}$ is non-trivial. The fact that $\bar{D}$ is a derivation satisfying (DV2) follows directly from the definition of D and $\frac{v}{w}$, so $(Kw, \frac{v}{w}, \bar{D})$ is a pre-differential-valued field. Moreover, the condition $\mathcal{O}_v = \mathcal{C} + \mathcal{M}_v$ clearly implies $\mathcal{O}_{\frac{v}{w}} = \mathcal{C}_{\frac{v}{w}} + \mathcal{M}_{\frac{v}{w}}$, where $\mathcal{C}_{\frac{v}{w}} = \{c + \mathcal{M}_w \mid c \in \mathcal{C}\}$. □

In case we start with a pre-H-field, then the induced derivation in Proposition 5.3.6(3) will also be a pre-H-field:

Proposition 5.3.7
Let $(K, v, \leq, D)$ be a pre-H-field and $y \in K$ a regular cut point. If w is a coarsening of v

with $y \in \mathcal{U}_w$, then $(Kw, \frac{v}{w}, \leq_w, \bar{D})$ is a pre-H-field. If $(K, \leq, D)$ is a H-field, then so is $(Kw, \leq_w, \bar{D})$.

Proof. We know from Proposition 5.3.6(3) that $(Kw, \frac{v}{w}, \bar{D})$ is a pre-differential-valued field. It follows from the definitions of $\frac{v}{w}$ and $\leq_w$ that axioms (PH2),(PH3) and (H2) are preserved when going to the residue field Kw, hence the claim. □

Proposition 5.3.6 kills our hope of establishing an exact analog of Theorem 2.3.4. Indeed, there can be coarsenings w of v which are not compatible with ϕ but still contain y, so that D will induce a derivation on Kw. Consider the following example:

Example 5.3.8
Let K be the Hardy field generated by all constant real functions and the maps $x \mapsto x$, $x \mapsto e^x$ and $x \mapsto e^{e^x}$. Note that a regular cut point of K is the function $x \mapsto \frac{1}{x}$. Now let $\mathcal{O}_w$ be the convex hull of $\mathbb{R}(e^x)$ (i.e. w is the principal coarsening of v generated by $x \mapsto e^x$). We have $\frac{1}{x} \in \mathcal{O}_w$, so by Proposition 5.3.6 D induces a derivation on Kw. However, w is not ϕ-compatible, because $e^{e^x} \notin \mathcal{O}_w$ but $\phi(e^{e^x}) = e^x \in \mathcal{O}_w$.

Another idea to characterize the differential rank is look at the valued field (K, w), where w is a coarsening of v. One can then wonder if (K, w, D) is still a pre-differential-valued field. The following proposition gives us an answer:

Proposition 5.3.9
Let (K, v, D) be a pre-differential-valued field (respectively, a pre-H-field), y a regular cut point for (K, v, D) and w a strict coarsening of v such that $y \in \mathcal{U}_w$. Then $\mathcal{O}_w$ is in the differential rank of (K, v, D) if and only if (K, w, D) is a pre-differential-valued field (respectively, a pre-H-field).

Proof. Set $c := v(y)$. We want to show that (K, w, D) satisfies (DV2) if and only if w is compatible with ϕ. Let $a \in \mathcal{O}_w$ and $b \in \mathcal{M}_w, b \neq 0$. Since $b \in \mathcal{M}_w$, we have $D(b) \neq 0$, so $w(\phi(b)) \neq \infty$. If $D(a) = 0$ then obviously $w(D(a)) > w(\phi(b))$. If $D(a) \neq 0$, then the inequality $w(D(a)) > w(\phi(b))$ is equivalent to $\frac{D(a)}{\phi(b)} \in \mathcal{M}_w$. Therefore, (K, w, D) satisfies (DV2) if and only if for all $a \in \mathcal{O}_w$ with $D(a) \neq 0$ and all $b \in \mathcal{M}_w$, $\frac{D(a)}{\phi(b)} \in \mathcal{M}_w$. Assume w is not ϕ-compatible. Since $y \in \mathcal{U}_w$, it follows from Proposition 5.2.4(1) and (4) that for all $b \in K$, $b \in \mathcal{U}_w \Rightarrow \phi(b) \in \mathcal{U}_w$. Therefore, there must exist $b \notin \mathcal{U}_w$ with $\phi(b) \in \mathcal{U}_w$, without loss of generality $b \in \mathcal{M}_w$ (otherwise, take b^{-1}). Take an $a \in \mathcal{U}_w \backslash \mathcal{U}_v$, so $v(a) \in G_w^{\neq 0}$. By Proposition 5.2.4(2), (3) and (4), either $D_G(v(a)) \sim v(a)$ or $D_G(v(a)) \precsim c$ is true. By assumption, we have $c \in G_w$ and $v(a) \in G_w$. It follows from the the convexity of G_w that $D_G(v(a)) \in G_w$, hence $D(a) \in \mathcal{U}_w$. We thus have $D(a), \frac{1}{\phi(b)} \in \mathcal{O}_w \backslash \mathcal{M}_w$, and since $\mathcal{M}_w$ is a prime ideal of $\mathcal{O}_w$ this implies $\frac{D(a)}{\phi(b)} \notin \mathcal{M}_w$, which contradicts (DV2) for w. Assume now that w is ϕ-compatible. Take $a \in \mathcal{O}_w$ with $D(a) \neq 0$ and $b \in \mathcal{M}_w$. By Proposition 5.3.6(3), we know that $D(a) \in \mathcal{O}_w$. Since $v(b) \notin G_w$ and since G_w is ψ-compatible, we know by Proposition 5.2.7 that $\psi(v(b)) < G_w$, hence $\phi(b) \notin \mathcal{O}_w$, so $\frac{1}{\phi(b)} \in \mathcal{M}_w$, hence $\frac{D(a)}{\phi(b)} \in \mathcal{M}_w$. This proves (DV2). Now assume that $(K, v, \leq, D)$ is a pre-H-field. Since

$\mathcal{O}_v$ is $\leq$-convex and w is a coarsening of v, $\mathcal{O}_w$ is also $\leq$-convex, so $(K, w, \leq, D)$ satisfies (PH2). Since $\mathcal{O}_v \subseteq \mathcal{O}_w$, it also satisfies (PH3). □

Remark 5.3.10: If $w \neq v$, (K, w, D) cannot satisfy (DV1), so it is not a differential-valued field.

Finally, Proposition 5.3.6 and 5.3.9 allow us to give a full characterization of coarsenings w of v which belong to the differential rank of (K, v, D) by looking simultaneously at the valued field (K, w) and at the residue field Kw:

Theorem 5.3.11
Let (K, v, D) be a pre-differential-valued field (respectively, a pre-H-field) and w a strict coarsening of v. Then $\mathcal{O}_w$ is in the differential rank of (K, v, D) if and only if the two following conditions are satisfied:

(1) D induces a non-trivial derivation on Kw.

(2) (K, w, D) is a pre-differential-valued field (respectively, a pre-H-field).

Proof. Take a regular cut point y for (K, v, D) and set $c := v(y)$. If w is ϕ-compatible, then by Proposition 5.2.7 we must have $c \in G_w$ hence $y \in \mathcal{U}_w$, which by Proposition 5.3.6 implies that D induces a non-trivial derivation on Kw. We can then apply Proposition 5.3.9 and we get that (K, w, D) is a pre-differential-valued field (respectively, a pre-H-field). Conversely, assume (1) and (2) hold. By Proposition 5.3.6, (1) implies $y \in \mathcal{U}_w$, so we can apply 5.3.9 and we get that, since (2) holds, $\mathcal{O}_w$ must be in the differential rank of (K, v, D). □

5.3.2 The unfolded differential rank

Our definition of the differential rank is not quite satisfactory if the cut point for ψ is not 0. Indeed, by Proposition 5.2.7, we see that the ψ-rank of G does not give any information on what happens for "small" elements, i.e elements g with $0 \precsim g \precsim c$ where c is a cut point for ψ. Consider the following example:

Example 5.3.12
Let K, T and K' be as in Example 5.3.2, and let L be the compositional inverse of T. L is a function that grows to infinity more slowly than any iterate of log. However, adding L to K or K' has no effect on the differential rank. This is because $v(L) \precsim c$, where $c = v(\frac{1}{x})$ is a cut point for ψ.

We need to "unfold" the map ψ around 0 to get the information on "small" elements, i.e we need to translate ψ in order to obtain a new map whose cut point is closer to 0 than c is, thus gaining information on the behavior of ψ around 0. Ideally, this translate of ψ should have 0 as a cut point. If ψ happens to have a gap or a maximum g, then we can do this by considering the map $\psi' := \psi - g$, since this map has 0 as the cut point. However, things are more complicated in general; in particular if (G, ψ) has asymptotic integration, then we cannot obtain a map with cut point 0 by simply translating ψ. Instead, we need to consider an infinite family of translates of ψ whose cut points approach 0, and then take the union of their ranks.

Now let us denote by R the set of non-trivial ψ-compatible convex subgroups of G and by P the set of non-trivial ψ-principal convex subgroups of G. For any $g \in G^{\neq 0}$, let us denote by ψ_g the map $G^{\neq 0} \to G$, $h \mapsto \psi(h) - \psi(g)$. Note that (G, ψ_g) is also an asymptotic couple. For every g we choose a cut point c_g for ψ_g. We denote by S_g set of non-trivial ψ_g-compatible convex subgroups of G and by Q_g set of non-trivial ψ_g-principal convex subgroups of G.

Lemma 5.3.13
Let $h \in G^{\neq 0}$. For any cut point c_h for ψ_h, we have $c_h \precsim h$.

Proof. If $c_h = 0$ this is clear, so assume $c_h \neq 0$. By Lemma 5.2.3(ii), we have $c_h \sim \psi_h(c_h)$. By Lemma 5.2.1, we have $\psi_h(c_h) = \psi(c_h) - \psi(h) \precnsim c_h - h$. It follows that $c_h \precnsim c_h - h$, which implies $c_h \precsim h$. $\square$

Lemma 5.3.13 shows in particular that we can choose g so that c_g is arbitrarily small, which means that the family $\{\psi_g\}_{g \in G}$ is well-suited for our purpose. This motivates the following definition:

Definition 5.3.14
The **unfolded ψ-rank** of the asymptotic couple (G, ψ) is the order-type of the totally ordered set $S := \bigcup_{0 \neq g \in G} S_g$. The **principal unfolded ψ-rank** of the asymptotic couple (G, ψ) is the order-type of the totally ordered set $Q := \bigcup_{0 \neq g \in G} Q_g$. If (K, v, D) is a pre-differential-valued field, we define its **unfolded differential rank** (respectively, its **principal unfolded differential rank**) as the unfolded ψ-rank of G (respectively the principal unfolded ψ-rank of G), where (G, ψ) is the asymptotic couple associated to (K, v, D).

In order to justify Definition 5.3.14, we still need to check that S and P satisfy the conditions that we want. We want to show that S contains the ψ-rank of G, and that the only new subgroups that were added in the process are subgroups contained in $\{g \in G \mid g \precsim c\}$, where c is a cut point for ψ.

Lemma 5.3.15
Let $g \in G^{\neq 0}$. The following holds:

(i) We have $S_g = \{H \in S \mid g \in H\}$. In particular, S_g is a final segment of S.

(ii) For any convex subgroup H of G, $H \in Q$ if and only if there is $h \in G$ such that $H = \bigcap_{F \in S_h} F$.

(iii) We have $Q_g = \{H \in Q \mid g \in H\}$. In particular, Q_g is a final segment of Q.

Proof. Let us prove (i). If $H \in S_g$, then obviously $H \in S$. Moreover, we have $\psi_g(g) = 0 \in H$, so if $H \in S_g$ it follows from ψ_g-compatibility that $g \in H$. This proves $S_g \subseteq \{H \in$

$S \mid g \in H\}$. Now assume that $g \in H \in S$ holds. By definition of S, there is $h \in G$ such that $H \in S_h$. Let us show that $H \in S_g$. Let $f \in G^{\neq 0}$. We already showed that $H \in S_h$ implies $h \in H$. By definition of S_h, we have $f \in H \Leftrightarrow \psi_h(f) \in H$. Moreover, we have $\psi_h(f) = \psi_g(f) + \psi(g) - \psi(h)$. By Lemma 5.2.1, we have $\psi(g) - \psi(h) \precsim g - h$. Since $g, h \in H$, it follows from convexity of H that $\psi(g) - \psi(h) \in H$. It follows that $\psi_g(f) \in H$ if and only if $\psi_h(f) \in H$. It follows that $f \in H \Leftrightarrow \psi_g(f) \in H$. This shows $H \in S_g$. Let us show (ii). By definition of Q, $H \in Q$ if and only if there exists $h, f \in G$ such that $H = \bigcap_{f \in F \in S_h} F$. Now note that, for any $F \in S$, it follows from (i) that $f \in F \in S_h \Leftrightarrow f, h \in F \in S \Leftrightarrow h \in F \in S_f$, so $H = \bigcap_{f \in F \in S_h} F$ if and only if $H = \bigcap_{h \in F \in S_f} F$. Therefore, we can assume without loss of generality that $f \precsim h$. It follows from (i) that $f \in F \in S_h \Rightarrow h \in F \in S_h$, and since $f \precsim h$ it follows from convexity that $h \in F \in S_h \Rightarrow f \in F \in S_h$. It follows that $H = \bigcap_{f \in F \in S_h} F$ holds if and only if $H = \bigcap_{h \in F \in S_h} F$. By (i), we have $\bigcap_{h \in F \in S_h} F = \bigcap_{F \in S_h} F$. This shows (ii). Now let us show (iii). Assume $H \in Q_g$. Then in particular, $H \in S_g$, hence $g \in H$ by (i). Conversely, assume $g \in H \in Q$. By (ii), there is h such that $H = \bigcap_{F \in S_h} F$. Because $g \in H$, we have $H = \bigcap_{g \in F \in S_h} F$. It follows from (i) that, for any $F \in S$, $g \in F \in S_h \Leftrightarrow g, h \in F \in S \Leftrightarrow h \in F \in S_g$. It follows that $H = \bigcap_{h \in F \in S_g} F$. This means that H is the smallest element of S_g with $h \in H$, hence $H \in Q_g$.

□

The principal unfolded differential rank is related to the unfolded differential rank the same way that principal ranks are usually related to the corresponding rank:

Proposition 5.3.16
If H is a convex subgroup of G, then $H \in Q$ if and only if there exists $h \in G$ such that H is the smallest element of S containing h.

Proof. Let $H \in Q$. It follows from Lemma 5.3.15(ii) that $H = \bigcap_{F \in S_g} F$ for some $g \in G$. It then follows from Lemma 5.3.15(i) that $H = \bigcap_{g \in F \in S} F$, i.e H is the smallest element of S containing g. Conversely, assume that H is the smallest element of S containing g. It follows from Lemma 5.3.15(i) that H is then the smallest element of S_g containing g, hence $H \in Q_g \subseteq Q$.

□

Now we can describe the connection between the (principal) ψ-rank and the (principal) unfolded ψ-rank:

Proposition 5.3.17
Let c be a cut point for ψ. The following holds:

(1) We have $R = \{H \in S \mid c \in H\}$. In particular, R is a final segment of S.

(2) We have $P = \{H \in Q \mid c \in H\}$. In particular, P is a final segment of Q.

(3) Assume $c \neq 0$ and let G_c be the ψ-principal subgroup of G generated by c. Then R (respectively, P) is the principal final segment of S (respectively, of Q) generated by G_c.

(4) Assume $c = 0$. Then $R = S$ and $P = Q$.

(5) Assume g is either the maximum of Ψ or a gap for ψ. Then S (respectively, Q) is equal to the ψ'-rank of G (respectively, the principal ψ'-rank of G), where $\psi' := \psi - g$.

Proof. Let us prove (1). It follows from Proposition 5.2.7 that $H \in R \Rightarrow c \in H$. Now we just have to show that, for any non-trivial convex subgroup H of G containing c, $H \in R \Leftrightarrow H \in S$. Let H be a convex subgroup of G containing c. Take $g \in H^{\neq 0}$. It follows from Lemma 5.3.15(i) that $H \in S$ if and only if $H \in S_g$. By Lemma 5.2.6, we have $\psi(g) \in H$. For any $f \in G^{\neq 0}$, we have $\psi_g(f) = \psi(f) - \psi(g)$, which implies $\psi_g(f) \in H \Leftrightarrow \psi(f) \in H$. It then follows from the definition of R and S_g that $H \in R$ if and only if $H \in S_g$. This shows that $H \in R$ if and only if $H \in S$. Now let us show (2). Assume $H \in P$. Then there exists $g \in G$ such that H is the smallest element of R containing g. It follows from (1) that H is the smallest element of S containing c and g. It then follows from convexity that H is the smallest element of S containing h, where we set $h := c$ if $g \precsim c$ and $h := g$ is $c \precsim g$. By Proposition 5.3.16, this implies $H \in Q$. Conversely, assume $c \in H \in Q$. Then by Proposition 5.3.16, there is $g \in H$ such that H is the smallest element of S containing g. Since $c \in H$, it follows from (1) that H is the smallest element of R containing g, hence $H \in P$. This shows (2). (3) and (4) then follow directly from (1) and (2). Let us prove (5). Note that $\psi_g(h) = \psi'(h) - \psi'(g) = \psi'_g(h)$ for any g, h. It follows that the (principal) unfolded ψ-rank of G is equal to the (principal) unfolded ψ'-rank of G. By lemma 5.2.10, 0 is a cut point for ψ'. It then follows from (4) that the (principal) unfolded ψ-rank of G is equal to the (principal) ψ'-rank of G. □

Remark 5.3.18: Proposition 5.3.17 shows that the unfolded differential rank has the desired properties. Indeed, (1) and (2) show that the (principal) differential rank is contained in the (principal) unfolded differential rank and that the only subgroups which were added in the process are groups which do not contain c. Moreover, (5) shows that taking the unfolded differential rank generalizes the idea of translating ψ by a gap.

For a pre-differential-valued field (K, v, D) with asymptotic couple (G, ψ), we say that a coarsening w of v lies in the unfolded differential rank of (K, v, D) if G_w lies in unfolded ψ-rank of G. We can give an analog of Theorem 5.3.11 for the unfolded differential rank, which characterizes the convex subrings of K lying in the unfolded differential rank:

Theorem 5.3.19
Let (K, v, D) be a pre-differential-valued field and w a coarsening of v. Then w is in the

unfolded differential rank of (K, v, D) if and only if (K, w, D) is a pre-differential-valued field.

Proof. Assume w is in the unfolded differential rank of (K, v, D). This means that there is $g \in G^{\neq 0}$ such that G_w is in the ψ_g-rank of G. Now take $a \in K$ with $v(a) = \psi(g)$. (K, v, aD) is a pre-differential-valued field with asymptotic couple (G, ψ_g) and $\mathcal{O}_w$ is in the differential rank of (K, v, aD). By Theorem 5.3.11, it follows that (K, w, aD) is a pre-differential valued field, so (K, w, D) is also a pre-differential-valued field. Conversely, assume (K, w, D) is a pre-differential-valued field, $w \neq v$, and take $a \in \mathcal{U}_w \backslash \mathcal{O}_v$. Set $g := v(a)$. By Lemma 5.3.13, we have $c_g \precsim g$, so by convexity $c_g \in G_w$. By Proposition 5.3.6(3) and Theorem 5.3.11, it follows that $\mathcal{O}_w$ is in the differential rank of (K, v, aD), so G_w is in the ψ_g-rank of G, so it is in the unfolded ψ-rank of G. □

Finally, we want to connect the unfolded differential rank with the exponential rank of exponential ordered fields. The connection will be given by the following proposition:

Proposition 5.3.20
Assume there exists a map $\chi : G^{\neq 0} \to G^{\neq 0}$ such that for any $g \in G^{<0}$, $\chi(g) + \psi(\chi(g)) = \psi(g)$ and $\chi(-g) = -\chi(g)$. Then the unfolded ψ-rank (respectively, the principal unfolded ψ-rank) of G coincides with the χ-rank (respectively, the principal χ-rank) of G.

Proof. We start by showing the following claim:

Claim: For any $g, h \in G^{\neq 0}$, if $g \precsim \chi(h)$ then $\psi_g(h) \sim \chi(h)$.

Proof. Because $\chi(-h) = -\chi(h)$ and $\psi_g(h) = \psi_g(-h)$, it is sufficient to consider the case $h < 0$. By assumption, we have $\chi(h) + \psi(\chi(h)) = \psi(h)$ for any $h < 0$, hence $\chi(h) = \psi(h) - \psi(\chi(h)) = \psi_g(h) + \psi(g) - \psi(\chi(h))$ hence $\chi(h) - \psi_g(h) = \psi(g) - \psi(\chi(h))$. By Lemma 5.2.1, we have $\psi(g) - \psi(\chi(h)) \precnsim g - \chi(h)$, so if $g \precsim \chi(h)$ we have $\chi(h) - \psi_g(h) = \psi(g) - \psi(\chi(h)) \precnsim \chi(h)$ which implies $\chi(h) \sim \psi_g(h)$. □

Now let us prove the proposition. Let H be a convex subgroup of G. Assume H is χ-compatible and take $g \in H$, $g \neq 0$. By Lemma 5.3.13, $c_g \precsim g$, hence $c_g \in H$ by convexity. Now take $h \in G$ such that $\psi_g(h) \in H$. If $\chi(h) \precsim g$, then by convexity $\chi(h) \in H$. If $g \precsim \chi(h)$, then the claim implies $\chi(h) \sim \psi_g(h)$, hence by convexity $\chi(h) \in H$. In any case we have $\chi(h) \in H$, and since H is χ-compatible it follows that $h \in H$. We proved $c_g \in H$ and $\psi_g(h) \in H \Rightarrow h \in H$, so by Proposition 5.2.7 H is ψ_g-compatible, which means that H is in the unfolded ψ-rank of G. Conversely, assume $H \in S$ and take $h \in G$. Since $H \in S$, there is $g \in G$ with $H \in S_g$. By Lemma 5.3.15(i), we have $g \in H$. If $\chi(h) \precnsim g$, then it follows from convexity of H that $\chi(h) \in H$, which by Lemma 5.3.15(i) implies $H \in S_{\chi(h)}$. Therefore, we can always assume that $g \precsim \chi(h)$. By the claim, we then have $\psi_g(h) \in H \Leftrightarrow \chi(h) \in H$. Since $H \in S_g$, this implies $h \in H \Leftrightarrow \chi(h) \in H$, so H is χ-compatible. This proves that the unfolded ψ-rank of G is equal to the χ-rank of G. It then immediately follows from Proposition 5.3.16 that the principal unfolded ψ-rank is equal to the principal χ-rank. □

Remark 5.3.21: In [Asc03], the author explained how to obtain such a map χ as in Proposition 5.3.20. Assume that (G, ψ) has asymptotic integration; then we can define $\int g := D_G^{-1}(g)$ for any $g \in G$ (note that D_G is injective because it is strictly increasing). We can then define the map: $\chi(g) : G^{<0} \to G, g \mapsto \int \psi(g)$. We extend this map to G by setting $\chi(0) := 0$ and $\chi(g) = -\chi(-g)$ for every $g > 0$. The map χ satisfies the conditions of Proposition 5.3.20. Moreover, χ is a precontraction (see [Asc03, Section 5]).

Proposition 5.3.20 yields an immediate corollary for exponential fields. We refer to [Kuh00] for the definition of v-compatible $(GA), (T_1)$-exponential:

Corollary 5.3.22
Let $(K, \leq, D)$ be a H-field, let v be the natural valuation associated to $\leq$ and assume that there exists a v-compatible $(GA), (T_1)$-exponential exp on K such that $\phi(\exp(a)) = D(a)$ for any $a \in K$. Then the exponential rank of $(K, \leq, \exp)$ is equal to the unfolded differential rank of $(K, \leq, D)$.

Proof. We know from [Kuh00] that $\log = \exp^{-1}$ induces a map χ on G and that the exponential rank of K is equal to the χ-rank of G (note that this also follows from our Proposition 5.1.8). With our assumption, it is easy to check that χ satisfies the assumptions of Proposition 5.3.20. The claim then follows from Proposition 5.3.20. □

5.4 Derivations on power series

The goal of this Section is to answer the following question:

Question 1: Given an ordered abelian group G and a field k of characteristic 0, under which conditions on k and G can we define a strongly linear derivation on the field $k((G))$ of generalized power series so that $k((G))$ is a differential-valued field? a H-field?

We do this by first answering the following question:

Question 2: Let a H-type asymptotic couple (G, ψ) and a field k of characteristic 0 be given. Under which condition on (G, ψ) and k can we define a strongly linear derivation D on $K := k((G))$ such that (K, v, D) is a differential-valued field (or a H-field) whose associated asymptotic couple is (G, ψ)?

We then apply our results to construct a derivation on a field of generalized power series so that we obtain a H-field of given principal differential rank and principal unfolded differential rank. All the asymptotic couples appearing in this section are H-type. Section 5.4.1 answers Question 2 and Section 5.4.2 answers a variant of Question 1 where D is required to be a H-derivation. Section 5.4.3 answers another variant of Question 1 where we require D to be of Hardy type, which is connected to the work done in [KM12] and [KM11].

Throughout this section, k will denote a field, G an ordered abelian group and $K := k((G))$ the field of generalized power series with exponents in G and coefficients in k. We denote by v the usual valuation on K, i.e. $v(\sum_{g \in G} a_g t^g) = \min \operatorname{supp}((a_g)_{g \in G})$.

If k is ordered, then we also consider K as an ordered field in the usual sense, i.e K is endowed with the order defined in Section 2.5. We will always denote by $(\Gamma, (B_\gamma)_{\gamma\in\Gamma})$ the skeleton of the valued group (G, v_G), where v_G is the archimedean valuation on G.

5.4.1 Defining a derivation of given asymptotic couple

Let (G, ψ) be a H-type asymptotic couple. In this section, we want to define a strongly linear derivation D on $K := k((G))$ such that (K, v, D) is a differential-valued field whose associated asymptotic couple is (G, ψ). If k happens to be ordered, then we want $(K, \leq, D)$ to be a H-field.

Proposition 5.4.1
Assume D is a strongly linear derivation on K such that, for every $a \in K$ with $a \neq 0$ and $v(a) \neq 0$, $v(D(a)) = v(a) + \psi(v(a))$. The following holds:

(1) (K, v, D) is a differential-valued field with asymptotic couple (G, ψ).

(2) Assume k is ordered. Assume also that for every $a \notin \mathcal{O}_v$, the leading coefficient of a has the same sign as the leading coefficient of $D(a)$. Then $(K, \leq, D)$ is a H-field with asymptotic couple (G, ψ).

Proof. Let us show (1). We first show that $\mathcal{C} = k.t^0$. We have $D(t^0) = 0$ and it follows from strong linearity that $D(a.t^0) = 0$ for all $a \in k$. This proves $k.t^0 \subseteq \mathcal{C}$. Now let $a \in \mathcal{C}$, $a \neq 0$. If $v(a) \neq 0$, then it follows from the assumption on D that $D(a) \neq 0$. Therefore, we must have $v(a) = 0$. Write $a = a_0t^0 + b$ with $v(b) > 0$. By linearity, $D(a) = D(b)$. Since $v(b) \neq 0$, $b \neq 0$ would imply $D(b) \neq 0$, which is a contradiction. Therefore, we have $b = 0$, i.e. $a = a_0t^0$. This shows $\mathcal{C} = k.t^0$. Moreover, we have $\mathcal{O}_v = k((G^{\geq 0}))$. We thus have $\mathcal{O}_v = k.t^0 \bigoplus k((G^{>0})) = \mathcal{C} \bigoplus \mathcal{M}_v$, which shows (DV1). Now let $a \in \mathcal{O}_v$ and $b \in \mathcal{M}_v$ with $b \neq 0$. We then have $D(b) \neq 0$. If $D(a) = 0$ then clearly $v(D(a)) > v(\frac{D(b)}{b})$. Assume $D(a) \neq 0$. Then by assumption, we have $v(D(a)) = v(a) + \psi(v(a))$ and $v(\frac{D(b)}{b}) = \psi(v(b))$. Because $v(a) > 0$, axiom (AC3) implies $\psi(v(b)) < v(a) + \psi(v(a))$, hence $v(\frac{D(b)}{b}) < v(D(a))$. This proves that (K, v, D) is a differential-valued field. The fact that (G, ψ) is the asymptotic couple associated to (K, v, D) clearly follows from the assumption. Now let us prove (2). Clearly, $\mathcal{O}_v$ is the convex hull of $\mathcal{C}$. Let $a \in K$ with $\mathcal{C} < a$. This means that $a \notin \mathcal{O}_v$ and that the leading coefficient of a is positive. It then follows from the assumption that $D(a) > 0$, which proves that $(K, \leq, D)$ is a H-field. □

We now aim at defining a derivation satisfying the conditions of Proposition 5.4.1. The key idea to our approach is to note that ψ is a valuation on G. Indeed, we can extend ψ to G by declaring $\psi(0) := \infty$. It then follows from (AC1) and (AC2) that ψ is a $\mathbb{Z}$-module valuation on G. It follows from (ACH) that ψ is compatible with $\leq$ (in the sense of Definition 2.7.17). We denote by $(\Psi, (C_\lambda)_{\lambda\in\Psi})$ the skeleton of (G, ψ). For any $\lambda \in \Psi$, we denote by $\widehat{C}_\lambda$ the divisible hull of C_λ. It follows from Corollary 3.2.3 that for each $\lambda \in \Psi$, $\leq$ induces an order on C_λ. This order can be extended to an order on

$\widehat{C}_\lambda$ in a unique way. Now denote by $(H, \leq_H)$ the lexicographic product of the family of ordered groups $(\widehat{C}_\lambda, \leq_\lambda)_{\lambda\in\Psi}$. We will denote elements of H as formal sums $g = \sum_{\lambda\in\Psi} g_\lambda$, where $g_\lambda \in \widehat{C}_\lambda$. We view each $\widehat{C}_\lambda$ as a subgroup of H. We can state a variant of Hahn's embedding theorem for asymptotic couples:

Theorem 5.4.2 (Hahn's embedding for asymptotic couples)
There exists a map $\psi_H : H\backslash\{0\} \to H$ such that the following holds:

(1) (H, ψ_H) is a H-type asymptotic couple.

(2) There is an embedding of asymptotic couples: $\iota : (G, \leq, \psi) \hookrightarrow (H, \leq_H, \psi_H)$.

Proof. Denote by w the valuation of H defined by $w(h) = \min \operatorname{supp}(h)$. Note that $w(H^{\neq 0}) = \Psi \subseteq G$. Let $\iota : (G, \psi) \hookrightarrow (H, w)$ be an embedding of valued groups as in Theorem 2.2.9. By Proposition 3.2.6, we know that ι is an embedding of ordered groups. Now define ψ_H as follows: for $h = \sum_{\lambda\in\Psi} h_\lambda \in H$, set $\psi_H(h) := \iota(w(h))$. By condition (1) of Theorem 2.2.9, we have $\psi(g) = w(\iota(g))$ for all $g \in G^{\neq 0}$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd} G^{\neq 0} \arrow[r, hook, "\iota"] \arrow[d, "\psi"] & H^{\neq 0} \arrow[d, "\psi_H"] \arrow[dl, "w"'] \\ \Psi \arrow[r, hook, "\iota"] & \iota(\Psi) \end{tikzcd} \tag{D}$$

All we have to show now is that (H, ψ_H) is a H-type asymptotic couple. Let $g, h \in H\backslash\{0\}$ and $n \in \mathbb{Z}\backslash\{0\}$. Because w is a $\mathbb{Z}$-module valuation, we have $w(ng) = w(g)$, hence $\psi_H(ng) = \iota(w(g)) = \psi_H(g)$. This proves (AC2). We also have $w(g+h) \geq \min(w(g), w(h))$. Since ι preserves the order, it follows that $\min(\psi_H(g), \psi_H(h)) \leq_H \psi_H(g + h)$, hence (AC1). Assume that $g \leq_H h <_H 0$. Because w is compatible with $\leq_H$, we then have $w(g) \leq w(h)$, hence (since ι preserves the order) $\psi_H(g) \leq_H \psi_H(h)$. This proves (ACH). Now we just have to prove that ψ_H satisfies (AC3). We will use the following claim:

Claim: Let $h \in H^{>0}$ and $\mu := w(h)$. There exists $g \in G^{>0}$ with $\iota(g) \leq h$ and $\psi(g) = \mu$.

Proof. Take any $g \in G^{>0}$ with $\psi(g) = \mu$ (such a g exists because $\mu \in \Psi$). We write $h = \sum_{\lambda\in\Psi} h_\lambda$ and $\iota(g) = \sum_{\lambda\in\Psi} g_\lambda$. Set $\nu := w(\iota(g) - h)$. We distinguish two cases:

Case 1: $\nu > \mu$. Choose $f \in G^\nu$ with $0 \neq f + G_\nu < h_\nu - g_\nu$. Set $g' := g + f$ and write $\iota(f) = \sum_{\lambda\in\Psi} f_\lambda$. Because $\mu = \psi(g) < \nu = \psi(f)$, we have $\psi(g') = \mu$ and $g' > 0$. By condition (1) of Theorem 2.2.9, $w(\iota(f)) = \psi(f) = \nu$, hence $f_\lambda = 0$ for all $\lambda < \nu$. By condition (2) of Theorem 2.2.9, we have $f_\nu = f + G_\nu$. We have: $\iota(g') - h = \sum_\lambda (g_\lambda + f_\lambda - h_\lambda)$. By definition of ν, we have $g_\lambda + f_\lambda - h_\lambda = 0$ for all $\lambda < \nu$. Moreover, by choice of f, we have $f_\nu + g_\nu - h_\nu < 0$. It then follows from the definition of $\leq_H$ that $\iota(g') - h < 0$, hence $\iota(g') < h$.

Case 2: $\nu = \mu$. Take $f \in G^{\mu}$ such that $f + G_{\mu} = h_{\mu} - g_{\mu}$ and set $g' := g + f$. We then have $w(h - \iota(g')) > \mu$, which brings us back to Case 1.

□

Now let us finish the prove of the Theorem. Take $h, g \in H$. Because $w(g) \in \Psi$, there is $g' \in G$ with $w(g) = \psi(g')$. By the claim, there is $h' \in G$ with $0 <_H \iota(h') \leq_H |h|$ and $\psi(h') = w(h)$. By (AC3) on (G, ψ), we have $\psi(g') < \psi(h') + h'$. Because ι preserves the order, it follows that $\iota(\psi(g')) <_H \iota(\psi(h')) + \iota(h')$, hence $\psi_H(g) <_H \psi_H(h) + \iota(h')$. Because $\iota(h') \leq_H |h|$, it follows that $\psi_H(g) < \psi_H(h) + |h|$. This shows (AC3).

□

Now take ι, w and ψ_H as in Theorem 5.4.2. Keep in mind diagram (D) above. To simplify notation, we will now write $\leq$ instead of $\leq_H$. Set $L := k((H))$. Then L is a field extension of K:

Proposition 5.4.3
The map ι induces an embedding of valued fields $\rho : K \to L$ defined by $\rho(\sum_{g \in G} a_g t^g) := \sum_{g \in G} a_g t^{\iota(g)}$. If k is ordered, then ρ is even an embedding of ordered fields.

Proof. Clearly, $\rho(0) = 0$ and $\rho(1) = \rho(t^0) = t^0 = 1$. Let $a, b \in K$. $\rho(a+b) = \sum_{g \in G}(a_g + b_g)t^{\iota(g)} = \sum_{g \in G} a_g t^{\iota(g)} + \sum_{g \in G} b_g t^{\iota(g)}$ and $\rho(ab) = \rho(\sum_{g \in G} c_g t^g) = \sum_{g \in G} c_g t^{\iota(g)} = \sum_{g \in G} a_g t^{\iota(g)}.\sum_{g \in G} b_g t^{\iota(g)}$. We have $v(\rho(a)) = \iota(v(a))$. Since ι is an embedding of ordered groups, it follows that ρ is an embedding of valued fields. Now assume that k is ordered. Assume $a \leq b$ and set $h := v(a - b)$. Then $v(\rho(a - b)) = \iota(h)$. It follows that $a \leq b \Leftrightarrow a_h - b_h \leq 0 \Leftrightarrow \rho(a) \leq \rho(b)$. □

Therefore, we can view K as a subfield of L. In order to define a derivation on K, we will first define a derivation on L, whose restriction to K will be the derivation we want.

Assume that, for each $\lambda \in \Psi$, a homomorphism of groups $u_{\lambda} : (\widehat{C}_{\lambda}, +) \to (k, +)$ has been given. To define a derivation on L we proceed in three steps.

Step 1: Define D on the "fundamental monomials" of L, i.e define $D(t^{g_{\lambda}})$ for each $g_{\lambda} \in \widehat{C}_{\lambda}$ for every $\lambda \in \Psi$.

Because of the condition of Proposition 5.4.1, we want to define $D(t^{g_{\lambda}})$ as an element with valuation $g_{\lambda} + \iota(\lambda)$. We thus define:

$$D(t^{g_{\lambda}}) := u_{\lambda}(g_{\lambda}) t^{g_{\lambda} + \iota(\lambda)}.$$

Note that the presence of the coefficient $u_{\lambda}(g_{\lambda})$ is essential to ensure that the usual product rule of derivations is satisfied. Indeed, take $g_{\lambda}, h_{\lambda} \in \widehat{C}_{\lambda}$. We have $t^{g_{\lambda}} t^{h_{\lambda}} = t^{g_{\lambda} + h_{\lambda}}$. By our definition of D, we have $D(t^{g_{\lambda} + h_{\lambda}}) = u_{\lambda}(g_{\lambda}) t^{g_{\lambda} + h_{\lambda} + \iota(\lambda)} + u_{\lambda}(h_{\lambda}) t^{g_{\lambda} + h_{\lambda} + \iota(\lambda)} = t^{h_{\lambda}} D(t^{g_{\lambda}}) + t^{g_{\lambda}} D(t^{h_{\lambda}})$, which is what we want. However, $t^{g_{\lambda} + h_{\lambda} + \iota(\lambda)} + t^{g_{\lambda} + h_{\lambda} + \iota(\lambda)} \neq t^{g_{\lambda} + h_{\lambda} + \iota(\lambda)}$.

Step 2: Extend D to all monomials by using a "strong Leibniz rule".

Let $g = \sum_{\lambda \in \Psi} g_\lambda \in H$. We naively define $D(t^g)$ by assuming that the usual product rule of derivations also holds for infinite products, i.e. we set:

$$D(t^g) := \sum_{\lambda \in \Psi} D(t^{g_\lambda}) t^{g - g_\lambda} = t^g \sum_{\lambda \in \Psi} u_\lambda(g_\lambda) t^{\iota(\lambda)}.$$

Note that the support of the family $(u_\lambda(g_\lambda)t^{\iota(\lambda)})_{\lambda \in \Psi}$ is isomorphic to a subset of the support of g so it is well-ordered. Therefore, the family $(u_\lambda(g_\lambda)t^{\iota(\lambda)})_{\lambda \in \Psi}$ is summable, which proves that the above formula makes sense.

Step 3: Extend D to L by strong linearity.

Let $a = \sum_{g \in H} a_g t^g \in L$. We naively define $D(a)$ by assuming that D is strongly linear, so we set:

$$D(a) := \sum_{g \in H} a_g D(t^g) = \sum_{0 \neq g \in H} \sum_{\lambda \in \Psi} a_g u_\lambda(g_\lambda) t^{g + \iota(\lambda)}. \qquad (\dagger)$$

We need to check that formula $(\dagger)$ makes sense, i.e that the family $(a_g D(t^g))_{g \in \text{supp}(a)}$ is summable. In other words, we want to show:

1. The set $A := \bigcup_{g \in \text{supp}(a)} \text{supp}(D(t^g)) = \{g + \iota(\lambda) \mid g \in \text{supp}(a), \lambda \in \text{supp}(g)\}$ is well-ordered.
2. For any $g \in \text{supp}(a)$ and any $\lambda \in \text{supp}(g)$, the set $A_{g,\lambda} := \{(h, \mu) \mid h \in \text{supp}(a), \mu \in \text{supp}(h), g + \iota(\lambda) = h + \iota(\mu)\}$ is finite.

The key to summability is the following fact which was proved in [AvdD02a, Proposition 2.3(1)]:

Lemma 5.4.4
For any $\lambda, \mu \in \iota(\Psi)$ with $\mu \neq \lambda$, $\psi_H(\lambda - \mu) > \min(\lambda, \mu)$.

Proof. Since $\iota(\Psi) = \psi_H(H^{\neq 0})$, this is just a direct application of [AvdD02a, Proposition 2.3(1)]. □

The next two lemmas will help us prove the summability of $(a_g D(t^g))_{g \in \text{supp}(a)}$:

Lemma 5.4.5
Let $g, h \in H$ with $g \leq h$ and $\lambda, \mu \in \Psi$ such that $h + \iota(\mu) \leq g + \iota(\lambda)$. Then $\mu \in \text{supp}(g) \Leftrightarrow \mu \in \text{supp}(h)$.

Proof. The case $g = h$ is trivial so assume $g < h$. It then follows from $h + \iota(\mu) \leq g + \iota(\lambda)$ that $\mu < \lambda$. Moreover, $0 < h - g \leq \iota(\lambda) - \iota(\mu)$, hence (since ψ_H is H-type) $\psi_H(h - g) \geq \psi_H(\iota(\lambda) - \iota(\mu))$, hence by 5.4.4 $\psi_H(g - h) > \min(\iota(\lambda), \iota(\mu)) = \iota(\mu)$. Now if μ were in $\mathrm{supp}(g)$ but not in $\mathrm{supp}(h)$ then μ would also be in $\mathrm{supp}(g - h)$, hence $w(g - h) \leq \mu$ hence $\psi_H(g - h) \leq \iota(\mu)$, which is a contradiction. This proves the Lemma. □

Lemma 5.4.6
If $(g_n + \iota(\lambda_n))_n$ is a decreasing sequence in A, then $(g_n)_n$ cannot be strictly increasing. If $(g_n + \iota(\lambda_n))_n$ is strictly decreasing then $(g_n)_n$ cannot be constant.

Proof. Assume that $(g_n + \iota(\lambda_n))_n$ is decreasing and $(g_n)_n$ increasing. Then for all $n \in \mathbb{N}$ we have $g_0 \leq g_n$ and $g_n + \iota(\lambda_n) \leq g_0 + \iota(\lambda_0)$, so by Lemma 5.4.5 $\lambda_n \in \mathrm{supp}(g_0)$. Since $\mathrm{supp}(g_0)$ is well-ordered, it follows that $(\lambda_n)_n$ cannot be strictly decreasing. Since $(g_n + \iota(\lambda_n))_n$ is decreasing, it follows that $(g_n)_n$ cannot be strictly increasing. Moreover, if $(g_n)_n$ is constant, then $(g_n + \iota(\lambda_n))_n$ cannot be strictly decreasing. □

Proposition 5.4.7
$(a_g D(t^g))_{g \in \mathrm{supp}(a)}$ is a summable family.

Proof. Assume there exists a strictly decreasing sequence $(g_n + \iota(\lambda_n))_n$ in A. Without loss of generality we can assume that $(g_n)_n$ is either constant, strictly decreasing or strictly increasing. Since $g_n \in \mathrm{supp}(a)$ for all n, $(g_n)_n$ cannot be strictly decreasing, so without loss of generality $(g_n)_n$ is either constant or strictly increasing. This contradicts Lemma 5.4.6, and it follows that A is well-ordered. Now let $h + \iota(\mu) \in A$ and assume there is an infinite subset $\{(g_n, \lambda_n) \mid n \in \mathbb{N}\}$ of $A_{h,\mu}$, with $(g_n, \lambda_n) \neq (g_m, \lambda_m)$ for all $n \neq m$. Without loss of generality we can assume that $(g_n)_{n \in \mathbb{N}}$ is either constant, strictly decreasing or strictly increasing. Since the sequence $(g_n + \iota(\lambda_n))_{n \in \mathbb{N}}$ is constant, it is in particular decreasing, so $(g_n)_n$ cannot be strictly increasing by Lemma 5.4.6; since $\mathrm{supp}(a)$ is well-ordered, $(g_n)_n$ cannot be strictly decreasing. Therefore, $(g_n)_n$ is constant, but then λ_n must also be constant, which contradicts $(g_n, \lambda_n) \neq (g_m, \lambda_m)$ for all $n \neq m$. This proves that $A_{h,\mu}$ is finite. □

Thus, formula (†) defines a map on L, and it is easy to see from its definition that it is a derivation. It remains to see if (L, v, D) is a differential-valued field.

Proposition 5.4.8
Let $a = \sum_{g \in H} a_g t^g \in L$ with $v(a) \neq 0$, $g = v(a)$ and $\lambda = w(g)$. Then $v(D(a)) \geq g + \iota(\lambda)$, and the coefficient of $D(a)$ at $g + \iota(\lambda)$ is $a_g u_\lambda(g_\lambda)$. In particular, if $u_\lambda(g_\lambda) \neq 0$ then $v(D(a)) = v(a) + \psi_H(v(a))$.

Proof. Let $h + \iota(\mu) \in A$ with $h + \iota(\mu) \leq g + \iota(\lambda)$. Since $g = v(a)$ and $h \in \mathrm{supp}(a)$ we have $g \leq h$. By Lemma 5.4.5, we have $\mu \in \mathrm{supp}(g)$, and since $\lambda = w(g)$ it follows that $\lambda \leq \mu$, hence $g + \iota(\lambda) \leq h + \iota(\mu)$. This proves that $g + \iota(\lambda) = \min A$. Now note that if $g < h$ then $h + \iota(\mu) \leq g + \iota(\lambda)$ would imply $\mu < \lambda$ which would contradict $\mu \in \mathrm{supp}(g)$, so $h + \iota(\mu) = g + \iota(\lambda)$ implies $h = g$ and thus $\mu = \lambda$. It follows that $A_{g,\lambda} = \{(g, \lambda)\}$, and so by formula (†) the coefficient in front of $t^{g+\iota(\lambda)}$ is just $a_g u_\lambda(g_\lambda)$. □

If $u_\lambda(g_\lambda) = 0$, then $v(D(a))$ is not equal to $v(a) + \psi_H(v(a))$. It follows that in general, L endowed with the derivation D is not even a pre-differential valued field since there may be constants whose valuation is not trivial. However, (L, v, D) becomes a differential-valued field if we impose a condition on u_λ:

Proposition 5.4.9
Assume that, for every $\lambda \in \Psi$, u_λ is injective. The following holds:

(1) (L, v, D) is a differential-valued field with asymptotic couple (H, ψ_H).

(2) If moreover k is ordered, and if u_λ is order-reversing for every $\lambda \in \Psi$, then $(L, \leq, D)$ is a H-field with asymptotic couple (H, ψ_H).

Proof. It follows from Proposition 5.4.8 that D satisfies the conditions of Proposition 5.4.1. Therefore, by Proposition 5.4.1, (L, v, D) is a differential-valued field with asymptotic couple (H, ψ). □

Therefore, if each u_λ is injective, then Proposition 5.4.9 tells us that formula (†) gives us the derivation we want on L. Now let us define a derivation on K. For this, we use the embedding ρ given by Proposition 5.4.3. Because $\iota(G)$ is stable under ψ, it is clear from formula (†) that $D(\rho(K)) \subseteq \rho(K)$. Therefore, we can define D on K as follows:

For $a \in K$, set $D(a) := \rho^{-1}(D(\rho(a)))$.

This gives us the following formula for D on K:

$$D(a) = \sum_{g \in G} \sum_{\lambda \in \Psi} a_g u_\lambda(g_\lambda) t^{g+\lambda}, \text{ where } \iota(g) = \sum_{\lambda \in \Psi} g_\lambda \tag{‡}$$

Note that the definition of D on K depends on ι, because the definition of g_λ depends on ι. We have the following:

Proposition 5.4.10
Assume that, for every $\lambda \in \Psi$, u_λ is injective. The following holds:

(1) (K, v, D) is a differential-valued field with asymptotic couple (G, ψ).

(2) If moreover k is ordered, and if u_λ is order-reversing for every $\lambda \in \Psi$, then $(K, \leq, D)$ is a H-field with asymptotic couple (G, ψ).

Proof. The fact that D is a derivation on K follows directly from the fact that ρ is an embedding of fields. Now note that $v(D(a)) = v(\rho^{-1}(D(\rho(a))) = \iota^{-1}(v(D(\rho(a))) = \iota^{-1}(v(\rho(a)) + \psi_H(v(\rho(a)))) = \iota^{-1}(v(\rho(a))) + \iota^{-1}(\psi_H(v(\rho(a)))) = v(a) + \psi(v(a))$. Both claims then follow from Proposition 5.4.1. □

Thus, the method described above allows us to define a derivation on any field of generalized power series $k((G))$ where (G,ψ) is a given H-type asymptotic couple. However, if we want to have a differential-valued field, we saw that our method only works if each $\widehat{C}_\lambda$ $(\lambda \in \Psi)$ is embeddable into $(k,+)$ as a group. One can then wonder if we could find a method which does not need this condition; the next proposition proves that it is not possible:

Proposition 5.4.11
Let D be a derivation on K such that (K,v,D) is a differential-valued field with asymptotic couple (G,ψ). Then for each $\lambda \in \Psi$, there exists a group embedding u_λ from C_λ into $(k,+)$. If moreover k is ordered and $(K,\leq,D)$ is a H-field, then we can even choose u_λ so that it is order-reversing.

Proof. We start by showing the following claim:

Claim: Let $h,g \in G$ be such that $\psi(h)=\psi(g)$ and $\psi(g-h)>\psi(g)$. Then $D(t^g)$ and $D(t^h)$ have the same leading coefficient.

Proof. By the product rule, we have $D(t^h)=t^{h-g}D(t^g)+t^gD(t^{h-g})$. Moreover, we have $v(t^{h-g}D(t^g))=h-g+g+\psi(g)=h+\psi(g)$ and
$v(t^gD(t^{h-g}))=g+h-g+\psi(h-g)=h+\psi(h-g)$. Since $\psi(g-h)>\psi(g)$, we have $v(t^{h-g}D(t^g))<v(t^gD(t^{h-g}))$. It follows that $v(D(t^h))=v(t^{h-g}D(t^g))$. Therefore, the leading coefficient of $D(t^h)$ is the leading coefficient of $t^{h-g}D(t^g)$, which is equal to the leading coefficient of $D(t^g)$. $\square$

Now let $\lambda \in \Psi$. Set $u_\lambda(0):=0$. For any $g_\lambda \in C_\lambda$, take any $g \in G^\lambda$ such that $g+G_\lambda=g_\lambda$ and define $u_\lambda(g_\lambda)$ as the leading coefficient of $D(t^g)$. The claim makes sure that $u_\lambda(g_\lambda)$ does not depend on the choice of g, so this gives us a well-defined map $u_\lambda : C_\lambda \to k$. One can easily check that this is a group homomorphism. The fact that $\ker u_\lambda=\{0\}$ follows from the fact that $D(a)\neq 0$ when $v(a)\neq 0$. Now assume that k is ordered and that $(K,\leq,D)$ is a H-field. Let $\lambda \in \Psi$ and $h_\lambda,g_\lambda \in C_\lambda$ with $h_\lambda<g_\lambda$. Take $g,h \in G^\lambda$ with $g+G_\lambda=g_\lambda$ and $h+G_\lambda=h_\lambda$. We then have $h<g$. It follows that $v(t^{h-g})<0$, hence $t^{h-g}>\mathcal{C}$, which by (PH3) implies $D(t^{h-g})>0$, so the leading coefficient $u_\lambda(h_\lambda-g_\lambda)=u_\lambda(h_\lambda)-u_\lambda(g_\lambda)$ of $D(t^{h-g})$ is positive, hence $u_\lambda(h_\lambda)>u_\lambda(g_\lambda)$, so u_λ is order-reversing.

$\square$

We can now formulate our answer to Question 2:

Theorem 5.4.12
Let k be a field (respectively, an ordered field) and (G,ψ) a H-type asymptotic couple. Let $(\Psi,(C_\lambda)_{\lambda\in\Psi})$ be the skeleton of the valuation ψ. The following conditions are equivalent:

(1) There exists a derivation D on $k((G))$ making $(k((G)),v,D)$ a differential-valued field (respectively, a H-field) with asymptotic couple (G,ψ).

(2) For every $\lambda \in \Psi$, the group $(C_\lambda, +)$ is embeddable into the group $(k, +)$ (respectively, the ordered group $(C_\lambda, +, \leq)$ is embeddable into the ordered group $(k, +, \leq)$).

Moreover, if these conditions are satisfied, then we can choose D to be strongly linear by using formula (‡).

Proof. The implication (1)⇒(2) follows directly from Propositions 5.4.11 (Note that the existence of an order-reversing group embedding between two groups is equivalent to the existence of an order-preserving embedding between them). Now assume that (2) holds and let $u_\lambda : (C_\lambda, +) \to (k, +)$ be an embedding. Because k is a field, $(k, +)$ is a divisible group. It follows that u_λ can be extended in a unique way to $(\widehat{C}_\lambda, +)$. Then (1) follows from 5.4.10. □

Remark 5.4.13: This idea to define D in 3 steps as above using "strong Leibniz rule" and strong linearity comes from the work in [KM12]. However, we would like to point out three major differences between the present work and the one done in [KM12]:

1. In [KM12], Kuhlmann and Matusinski worked in a more restricted framework, where G is a Hahn product of copies of $\mathbb{R}$. Our approach allows us to define a derivation on K from any H-type asymptotic couple (G, ψ), without any further assumption on G.

2. The key idea to make our approach work was to use ψ instead of v_G as a valuation to decompose G into a Hahn product of the C_λ's. This allows us to write elements of G as sums $g = \sum_{\lambda \in \Psi} g_\lambda$ indexed by Ψ and not by Γ. This contrasts with the procedure of [KM12], where the group was decomposed into a Hahn product of its archimedean components. Note that our definition of D would fail to work if we used an archimedean decomposition, i.e if we wrote elements of G as sums $g = \sum_{\gamma \in \Gamma} g_\gamma$ indexed by Γ. Indeed, applying step 2 in this context would yield the formula: $D(t^g) = t^g \sum_{\gamma \in \Gamma} u_\gamma(g_\gamma) t^{\psi(g_\gamma)}$. For a fixed $\gamma \in \Gamma$, it could then happen that the set $\{\delta \in \Gamma \mid \psi(g_\gamma) = \psi(g_\delta)\}$ is infinite. This shows that the family $(u_\gamma(g_\gamma) t^{\psi(g_\gamma)})_{\gamma \in \Gamma}$ is not summable, which means that the formula for $D(t^g)$ is not well-defined. This shows the necessity of using ψ, and not v_G, as a valuation to decompose the group.

3. In [KM12], the authors did not define any explicit derivation on K. Instead, they assumed a derivation was already defined on their "fundamental monomials" and gave conditions for this derivation to be extendable to the whole field via a strong Leibniz rule and strong linearity.

Remark 5.4.14: The idea used in Step 1 to define the derivation on fundamental monomials was already used in [AvdD02b], but only in the case where G is divisible and admits a valuation basis (for the archimedean valuation), which is a strong restriction.

5.4.2 Fields of power series admitting a H-derivation

We now want to use Theorem 5.4.12 to answer Question 1. This means we need to characterize the ordered groups G which can be endowed with a map ψ satisfying the properties of Theorem 5.4.12. Note that if (G, ψ) is a H-type asymptotic couple, then ψ is consistent with v_G, which means that ψ naturally induces two maps on Γ:

1. the map $\hat{\psi} : \Gamma \to G$ defined by $\hat{\psi}(v_G(g)) := \psi(g)$.
2. the map $\omega : \Gamma \to \Gamma \cup \{\infty\}$ defined by $\omega(\gamma) := v_G(\hat{\psi}(\gamma))$.

The main idea to answer Question 1 is to characterize the maps on Γ which can be lifted to a map ψ satisfying the conditions of Theorem 5.4.12. This is connected to the notion of shift.

Definition 5.4.15
$\sigma : \Gamma \to \Gamma \cup \{\infty\}$ is called a **right-shift** if $\sigma(\gamma) > \gamma$ holds for every $\gamma \in \Gamma$.

The authors of [KM12] already found a connection between shifts on Γ and the existence of Hardy-type derivations on K. In particular, it was showed in [KM12] that a shift on Γ can be lifted to a derivation on $\mathbb{R}((G))$, where $G = \mathrm{H}_{\gamma\in\Gamma}\mathbb{R}$. We show here that any H-derivation comes from a shift on Γ (see Theorems 5.4.19 and 5.4.22).

We extend the notion of shift to maps from Γ to $G^{\leq 0}$: we say that a map $\sigma : \Gamma \to G^{\leq 0}$ is a **right-shift** if the map $v_G \circ \sigma : \Gamma \to \Gamma \cup \{\infty\}$ is a right-shift. The following two lemmas show the connection between asymptotic couples and shifts. The increasing right-shifts of Γ are exactly the maps induced by asymptotic couples of cut point 0:

Lemma 5.4.16
Let $\sigma : \Gamma \to G^{\leq 0}$ (respectively, $\omega : \Gamma \to \Gamma \cup \{\infty\}$) be a map. Then the following statements are equivalent:

(i) There exists a map $\psi : G^{\neq 0} \to G$ such that (G, ψ) is a H-type asymptotic couple with 0 as a cut point and $\sigma(v_G(g)) = \psi(g)$ (respectively, $\omega(v_G(g)) = v_G(\psi(g))$) for all $g \in G^{\neq 0}$.

(ii) σ (respectively, ω) is an increasing right-shift.

Proof. Assume that (i) holds. Then by (ACH), σ must be increasing and since 0 is a cut point then σ must be a right-shift. Moreover, by Lemma 5.2.9(i) we have $\sigma(\Gamma) \subseteq G^{\leq 0}$. It then follows that $\omega = v_G \circ \sigma$ is an increasing right-shift. Conversely, assume σ is an increasing right-shift. Define $\psi(g) := \sigma(v_G(g))$ for any $g \in G^{\neq 0}$. (AC2) is obviously satisfied. Let $g, h \in G^{\neq 0}$. We have $v_G(g - h) \geq \min(v_G(g), v_G(h))$, and since σ is increasing it follows that $\sigma(v_G(g - h)) \geq \min(\sigma(v_G(g)), \sigma(v_G(h)))$ which proves (AC1). Now assume $0 < h$. Since σ is a right-shift, we have $v_G(\psi(h)) > v_G(h)$, so $h + \psi(h)$ must be positive. Since $\sigma(\Gamma) \subseteq G^{\leq 0}$, $\psi(g)$ is not positive, so (AC3) holds. Finally, (ACH) follows directly from the fact that σ is increasing. Now let c be a cut point for ψ. If $c \neq 0$, then Lemma 5.2.3(ii) implies $v_G(c) = v_G(\psi(c))$, hence $v_G(\sigma(v_G(c))) = v_G(c)$, which contradicts the fact that σ is a right-shift. Therefore, we must have $c = 0$, and ψ satisfies the desired conditions. If we are only given an increasing right-shift $\omega : \Gamma \to \Gamma \cup \{\infty\}$, we can choose a $g_\delta \in G^{<0}$ with $v_G(g_\delta) = \delta$ for each $\delta \in \omega(\Gamma)$. We can then set $\sigma(\gamma) := g_{\omega(\gamma)}$ for every $\gamma \in \Gamma$. This gives us an increasing right-shift $\sigma : \Gamma \to G^{\leq 0}$ with $\omega = v_G \circ \sigma$. We already proves that there exists ψ with 0 as a cut point and such that $\sigma(v_G(g)) = \psi(g)$. This equality then implies $\omega(v_G(g)) = v_G(\psi(g))$. $\square$

Moreover, there is a natural way of associating a shift to any H-type asymptotic couple:

Lemma 5.4.17
Let (G, ψ) be a H-type asymptotic couple. There exists an increasing right-shift $\sigma : \Gamma \to G^{\leq 0}$ such that for any $g, h \in G^{\neq 0}$, $\sigma(v_G(g)) \leq \sigma(v_G(h)) \Leftrightarrow \psi(g) \leq \psi(h)$.

Proof. We make a case distinction following Proposition 5.2.13. Assume first that ψ has a gap or a maximum c. Set $\psi' := \psi - c$ and let $\sigma(v_G(g)) := \psi'(g)$. By Lemma 5.2.10, 0 is a cut point for ψ', so it follows from Lemma 5.4.16 that σ is an increasing right-shift. It is clear that $\sigma(v_G(g)) \leq \sigma(v_G(h)) \Leftrightarrow \psi'(g) \leq \psi'(h)$ holds. Since $\psi'(g) \leq \psi'(h) \Leftrightarrow \psi(g) \leq \psi(h)$, it follows that σ has the desired properties. Now we just have to consider the case where (G, ψ) has asymptotic integration. In that case, let χ be the same map as in Remark 5.3.21. Now define σ by $\sigma(v_G(g)) := \chi(g)$. Since χ is a centripetal precontraction map, it follows that σ is an increasing right-shift. It is also clear from the definition of σ that $\sigma(v_G(g)) \leq \sigma(v_G(h)) \Leftrightarrow \chi(g) \leq \chi(h)$. Moreover, we know from [AvdD02a, Proposition 2.3] that the map D_G is strictly increasing, so $\int$ is also strictly increasing, so $\chi(g) = \chi(h) \Leftrightarrow \psi(g) = \psi(h)$ holds, which proves that σ satisfies the desired properties. $\square$

For any increasing map $\sigma : \Gamma \to G^{\leq 0}$ and any $f \in \sigma(\Gamma)$, we now set $G_\sigma(f, +) := \{g \in G \mid \sigma(v_G(g)) \geq f\} \cup \{0\}$ and $G_\sigma(f, -) := \{g \in G \mid \sigma(v_G(g)) > f\} \cup \{0\}$. We have the following:

Lemma 5.4.18
Let $\sigma : \Gamma \to G^{\leq 0}$ be an increasing map. Then for any $f \in \sigma(\Gamma)$, the sets $G_\sigma(f, +)$ and $G_\sigma(f, -)$ are convex subgroups of G.

Proof. Just note that $\sigma \circ v_G$ is a valuation and a coarsening of v_G. The fact that $G_\sigma(f, +)$ and $G_\sigma(f, -)$ are subgroups then follows from the fact that $\sigma \circ v_G$ is a valuation, and the fact that they are convex follows from Corollary 3.2.4. $\square$

Lemma 5.4.18 allows us to define the group $H_\sigma(f) := G_\sigma(f, +)/G_\sigma(f, -)$. Since $G_\sigma(f, -)$ is convex, $H_\sigma(f)$ is naturally an ordered group. This allows us to partially answer Question 1:

Theorem 5.4.19
Let G be an ordered abelian group and k a field (respectively, an ordered field). Let Γ denote the value chain of G. Then there exists a H-derivation D on $K := k((G))$ making (K, v, D) a differential-valued field (respectively, a H-field) if and only if there exists an increasing map $\sigma : \Gamma \to G^{\leq 0}$ such that the following holds:

(1) The map $v_G \circ \sigma$ is a right-shift.

(2) For any $f \in \sigma(\Gamma)$, $H_\sigma(f)$ is embeddable into $(k, +)$ (respectively, $(k, +, \leq)$).

Proof. Assume that (K, v, D) is a differential-valued field having a H-type asymptotic couple (G, ψ). By Theorem 5.4.12, it follows that C_λ is embeddable in $(k, +)$ for every $\lambda \in \Psi$. Now take σ as in Lemma 5.4.17. Let $f \in \sigma(\Gamma)$, $f = \sigma(v_G(g))$ for some $g \in G$. We have $\sigma(v_G(h)) \geq f \Leftrightarrow \sigma(v_G(h)) \geq \sigma(v_G(g)) \Leftrightarrow \psi(h) \geq \psi(g)$. It follows that $G_\sigma(f, +) = G^\lambda$, where $\lambda = \psi(g)$. Similarly, $G_\sigma(f, -) = G_\lambda$, hence $H_\sigma(f) = C_\lambda$, so $H_\sigma(f)$ is embeddable into $(k, +)$. If $(K, \leq, D)$ is a H-field then by Theorem 5.4.12, $H_\sigma(f)$ must be embeddable as an ordered group in $(k, +, \leq)$. This proves one direction of the theorem, let us prove the converse. Assume that such a σ as in the theorem exists. By Lemma 5.4.16, there exists ψ on G making (G, ψ) a H-type asymptotic couple and such that σ is induced by ψ. It follows that, for each $\lambda \in \Psi$, we have $\lambda \in \sigma(\Gamma)$, $G^\lambda = G_\sigma(\lambda, +)$ and $G_\lambda = G_\sigma(\lambda, -)$. By assumption, it follows that each C_λ is embeddable into $(k, +)$, and the existence of D then follows from Theorem 5.4.12. If each $H_\sigma(f)$ is embeddable into $(k, +, \leq)$, then so is each C_λ, and we conclude by Theorem 5.4.12 again. □

Remark 5.4.20: If we are given a σ as in Theorem 5.4.19, we can explicitly construct D. We first define ψ on G by $\psi(g) := \sigma(v_G(g))$. This gives us a H-type asymptotic couple (G, ψ). We then define D with formula (‡) above.

5.4.3 Hardy-type derivations

The goal of this section is to characterize fields of power series which can be endowed with a Hardy-type derivation as defined in [KM12]. If (G, ψ) is an asymptotic couple, we say that ψ is **of Hardy type** if (G, ψ) is H-type and $\psi(g) = \psi(h) \Rightarrow v_G(g) = v_G(h)$ for all $g, h \in G^{\neq 0}$. We say that a derivation D on an ordered field $(K, \leq)$ is of Hardy type if $(K, \leq, D)$ is a H-field with asymptotic couple (G, ψ) such that ψ is of Hardy type. This coincides with the notion of Hardy-type derivation defined in [KM12]. The natural derivation of a Hardy field is an example of a Hardy-type derivation. Note that if ψ is Hardy-type, we have $v_G(g) = v_G(h) \Leftrightarrow \psi(g) = \psi(h)$, which means that ψ and v_G are equivalent as valuations. In particular, the valued groups (G, v_G) and (G, ψ) have the same components. We will denote by $(\Gamma, (B_\gamma)_{\gamma \in \Gamma})$ the skeleton of the valued group (G, v_G).

In general, H-derivations are not necessarily Hardy-type derivations. However, the two notions coincide in a field of power series if the field of coefficients is archimedean:

Proposition 5.4.21
Let k be an archimedean ordered field, G an ordered abelian group, $K := k((G))$ and D a derivation on K. Then $(K, \leq, D)$ is a H-field if and only if D is of Hardy-type.

Proof. Assume $(K, \leq, D)$ is a H-field with asymptotic couple (G, ψ). By Theorem 5.4.12, each component of the valued group (G, ψ) is embeddable as an ordered group into $(k, +, \leq)$. It follows that each component of the valued group (G, ψ) is archimedean, and it follows that ψ must be of Hardy type. Indeed, if ψ is not of Hardy type, then there are $g, h \in G$ with $v_G(g) > v_G(h)$ and $\psi(g) = \psi(h)$. It follows that g, h are not archimedean-equivalent. Now set $\lambda := \psi(h)$. Then $g + G_\lambda$, $h + G_\lambda$ are two non-zero elements of C_λ, but they are not archimedean-equivalent, which is a contradiction. □

We can now answer a variant of Question 1, where D is required to be of Hardy type. The criterion for the existence of D in this case is simpler than the criterion for the existence of a H-derivation given by Theorem 5.4.19:

Theorem 5.4.22
Let G be an ordered abelian group and k an ordered field. Let Γ denote the value chain of G. Then there exists a Hardy-type derivation D on $K := k((G))$ if and only if the following conditions are satisfied:

(1) Each B_γ is embeddable as an ordered group into $(k, +, \leq)$.

(2) There exists an increasing right-shift $\tau : \Gamma \to \Gamma \cup \{\infty\}$ such that for any $\delta \in \tau(\Gamma)$, $\tau^{-1}(\delta)$ is embeddable as an ordered set into $v_G^{-1}(\delta) \cap G^{\leq 0}$.

Proof. Assume there exists a Hardy-type derivation D on K, denote by ψ the map induced by the logarithmic derivative on G. By Theorem 5.4.12, each C_λ is embeddable into $(k, +, \leq)$. Since D is Hardy-type, we have that, for every $\gamma \in \Gamma$, there exists $\lambda \in \Psi$ such that $B_\gamma = C_\lambda$. This shows (1). By Theorem 5.4.19, there exists $\sigma : \Gamma \to G^{\leq 0}$ such that $\tau := v_G \circ \sigma$ is an increasing right-shift. Now let $\delta \in \tau(\Gamma)$. For any $\gamma \in \tau^{-1}(\delta)$, set $\phi_\delta(\gamma) := \sigma(\gamma)$. Then ϕ_δ is an order-preserving map from $\tau^{-1}(\delta)$ to $v_G^{-1}(\delta) \cap G^{\leq 0}$.

Conversely, assume that (1) and (2) hold. Denote by ϕ_δ the embedding from $\tau^{-1}(\delta)$ to $v_G^{-1}(\delta) \cap G^{<0}$ for every $\delta \in \tau(\Gamma)$. For any $\gamma \in \Gamma$, define $\sigma(\gamma) := \phi_\delta(\gamma)$, where $\delta := \tau(\gamma)$. σ is clearly an order-preserving right-shift, so by Lemma 5.4.16 there is ψ on G such that σ is induced by ψ. Note that σ is moreover injective, which implies that ψ is Hardy-type. It then follows from (1) that C_λ is embeddable into $(k, +, \leq)$ for every $\lambda \in \Psi$. The existence of D is then given by Theorem 5.4.12.

□

Remark 5.4.23: The proof of Theorem 5.4.22 gives an explicit construction of ψ from τ. Together with formula (‡), this gives us an explicit construction of D from τ.

In the case where k contains $\mathbb{R}$, we can overlook condition (1) of Theorem 5.4.22:

Corollary 5.4.24
Let k be an ordered field containing $\mathbb{R}$ and G an ordered abelian group. Let Γ denote the value chain of G. Then there exists a Hardy-type derivation D on $K := k((G))$ if and only if there exists an increasing right-shift $\tau : \Gamma \to \Gamma \cup \{\infty\}$ such that for any $\delta \in \tau(\Gamma)$, $\tau^{-1}(\delta)$ is embeddable as an ordered set into $v_G^{-1}(\delta) \cap G^{\leq 0}$.

Proof. Since each B_γ is archimedean, and since k contains $\mathbb{R}$, condition (1) of Theorem 5.4.22 is always satisfied, so D exists if and only if condition (2) of Theorem 5.4.22 is satisfied. □

Finally, if k is archimedean, we can simplify Theorem 5.4.19 to give a criterion for the existence of a H-derivation on K:

Corollary 5.4.25
Let k be an archimedean ordered field and G an ordered abelian group with value chain Γ. There exists a derivation D on $K := k((G))$ making $(K, \leq, D)$ a H-field if and only if if and only if the following conditions are satisfied:

(1) Each B_γ is embeddable as an ordered group into $(k, +, \leq)$.

(2) There exists an increasing right-shift $\tau : \Gamma \to \Gamma \cup \{\infty\}$ such that for any $\delta \in \tau(\Gamma)$, $\tau^{-1}(\delta)$ is embeddable as an ordered set into $v_G^{-1}(\delta) \cap G^{\leq 0}$.

Proof. By Proposition 5.4.21, there exists a H-derivation on K if and only if there exists a Hardy-type derivation on K. The claim then follows directly from Theorem 5.4.22. □

5.4.4 Realizing a linearly ordered set as a principal differential rank

Assume (K, v, D) is a pre-differential valued field with principal differential rank P and principal unfolded differential rank Q. We know by Proposition 5.3.17 that P is either a principal final segment of Q or equal to Q. The goal of this section is to show a converse statement, i.e that any pair (P, Q) of totally ordered sets, where P is a principal final segment of Q or $Q = P$, can be realized as the pair "(principal differential rank, principal unfolded differential rank)" of a certain field of power series endowed with a Hardy-type derivation.

The construction is done in three steps: we first show that any totally ordered set Q can be realized as the principal ω-rank of a certain ordered set. We actually give an explicit example. We then show that there exists an asymptotic couple (G, ψ) whose principal ψ-rank is P and whose principal unfolded ψ-rank is Q, where P is any principal final segment of Q or Q itself. Finally, we use Theorem 5.4.12 to obtain the desired field.

Example 5.4.26
We want to give an explicit example of an ordered set $(\Gamma, \leq)$ with arbitrary principal ω-rank.

(a) We first construct an example of principal ω-rank 1. Take $\Gamma_1 := \mathbb{Z}$ ordered as usual an define $\omega_1(n) := n + 1$. Then it is easy to check that ω_1 is an order-preserving right-shift and that the principal ω_1-rank of $(\Gamma_1, \leq)$ is 1.

(b) We now construct an example which has principal ω-rank $(Q, \leq)$, where $(Q, \leq)$ is an arbitrary totally ordered set. Define $\Gamma := Q \times \Gamma_1$ and order Γ as follows: $(a, \gamma) \leq (b, \delta) \Leftrightarrow (a <^* b) \vee (a = b \wedge \gamma \leq \delta)$, where $<^*$ denotes the reverse order of $<$. Now define $\omega(a, n) := (a, \omega_1(n))$. One easily sees that ω is an increasing right-shift. Note also that ω is injective. Now consider the map: $a \mapsto \{(b, n) \in \Gamma \mid a \leq b, n \in \mathbb{Z}\}$ from Q to the set of final segments of $(\Gamma, \leq)$. One can see that this is an order-preserving bijection from $(Q, \leq)$ to the principal ω-rank of $(\Gamma, \leq)$, so $(Q, \leq)$ is the principal ω-rank of $(\Gamma, \leq)$.

Proposition 5.4.27
Let Q be a totally ordered set and P a final segment of Q such that P is either equal to Q or a principal final segment of Q. Then there exists a H-type asymptotic couple (G, ψ) whose principal ψ-rank is P and whose principal unfolded ψ-rank is Q. Moreover, we can choose (G, ψ) so that ψ is of Hardy type.

Proof. Let $(\Gamma, \leq)$ be an ordered set with an injective increasing shift $\omega : \Gamma \to \Gamma$ such that the principal ω-rank of $(\Gamma, \leq)$ is Q (take for instance Example 5.4.26(b)). Let $G := \underset{\gamma\in\Gamma}{\mathrm{H}} \mathbb{R}$. We know from Lemma 5.4.16 that there exists a map ψ_0 on $G^{\neq 0}$ such that (G, ψ_0) is a H-type asymptotic couple with cut point 0 and such that ω is the map induced by ψ_0 on Γ. Since ω is injective, it follows that ψ_0 is of Hardy type. By proposition 5.3.17(4), the principal unfolded ψ_0-rank of (G, ψ_0) is equal to its principal ψ_0-rank. Moreover, by Theorem 5.3.3, the principal ψ_0-rank of (G, ψ_0) is equal to the principal ω-rank of $(\Gamma, \leq)$, which by construction is equal to Q. We now assimilate Q with the set of non-trivial ψ_0-principal convex subgroups of G.

If $Q = P$, then set $\psi := \psi_0$, and then (G, ψ) satisfies the condition we wanted. Now assume $P \neq Q$. We know that P is a principal final segment of Q, which means that there is a ψ_0-principal convex subgroup H of G such that P is the set of ψ_0-principal convex subgroups of G containing H. Now let $c \in H$ be such that H is ψ_0-principal generated by c and set $\psi(g) := \psi_0(g) + c$. Obviously, ψ is Hardy-type. Now note that c is either a gap or a maximum for ψ: indeed, by Lemma 5.2.9(i), we have for all $g > 0$: $\psi_0(g) \leq 0 < \psi_0(g) + g$, hence $\psi_0(g) + c \leq c < \psi_0(g) + c + g$ i.e. $\psi(g) \leq c < \psi(g) + g$. It then follows from Proposition 5.3.17(5) that Q is the principal unfolded ψ-rank of G. Moreover, it follows from Lemma 5.2.9(ii) that c is a cut point for ψ. It then follows from Proposition 5.3.17(2) that the principal ψ-rank of G is isomorphic to the set of elements of Q containing c. By choice of c, this set is equal to P. □

We can now state our Theorem:

Theorem 5.4.28
Let Q be a totally ordered set and P a final segment of Q such that P is either equal to Q or a principal final segment of Q. Then there exists an ordered field k, an ordered abelian group G and a Hardy-type derivation D on $K := k((G))$ such that $(K, \leq, D)$ is a H-field of principal differential rank P and of principal unfolded differential rank Q.

Proof. By Proposition 5.4.27, there exists a H-type asymptotic couple (G, ψ) which has principal ψ-rank P and principal unfolded ψ-rank Q, and such that ψ is Hardy-type. Now set $K := \mathbb{R}((G))$. Since ψ is Hardy-type, each C_λ is archimedean, so it is embeddable into $(\mathbb{R}, +, \leq)$. By theorem 5.4.12, there exists a derivation D on K making $(K, \leq, D)$ a H-field with asymptotic couple (G, ψ). By Theorem 5.3.3, the principal differential rank of $(K, \leq, D)$ is equal to P and by definition the principal unfolded differential rank of $(K, \leq, D)$ is equal to Q.

□

Remark 5.4.29: It would be interesting to improve Theorem 5.4.28 by requiring K to have asymptotic integration. However, our construction in Proposition 5.4.27 gives us an asymptotic couple with a gap. In [Geh17, Section 4], Gehret gave methods to extend a given asymptotic couple with a gap into an asymptotic couple with asymptotic integration. However, this construction changes the unfolded differential rank, so we cannot use it in Theorem 5.4.28. It is unknown if Theorem 5.4.28 remains true if we require the field to have asymptotic integration.

Bibliography

[AN96] S.A. Adeleke and P.M. Neumann. Primitive permutation groups with primitive jordan sets. *Journal of the London Mathematical Society 53(2)*, April 1996.

[AN98] S.A. Adeleke and P.M. Neumann. Relations related to betweenness: their structure and their automorphisms. *Memoirs of the American Mathematical Society 623*, January 1998.

[Asc03] M. Aschenbrenner. Some remarks about asymptotic couples. *Valuation theory and its applications, Vol.II (Saskatoon, SK, 1999), Fields Inst. Commun., vol. 33, Amer. Math. Soc., Providence, RI, pp. 7–18*, 2003.

[AvdD02a] M. Aschenbrenner and L. van den Dries. H-fields and their liouville extensions. *Math. Z. 242, 543–588*, 2002.

[AvdD02b] M. Aschenbrenner and L. van den Dries. Liouville-closed H-fields. *Journal of Pure and Applied Algebra 197, 83 – 139*, 2002.

[Bos86] M. Boschernitzan. Hardy fields and existence of transexponential functions. *Aequationes mathematicae 30, 258-280*, 1986.

[Del11] F. Delon. C-minimal structures without the density assumption. *Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry.Cambridge University Press, Berlin*, 2011.

[DS17] F. Delon and P. Simonetta. Abelian C-minimal valued groups. *Annals of pure and Applied logic*, 2017.

[EP05] A.J. Engler and A. Prestel. *Valued Fields*. Springer Monographs in Mathematics.Springer-Verlag, Berlin, Zbl 1128.12009. MR 2183496. 286, 2005.

[Fak87] S.M. Fakhruddin. Quasi-ordered fields. *Journal of Pure and Applied Algebra 45, 207-210*, 1987.

[Fuc63] L. Fuchs. *Partially ordered algebraic systems*. Pergamon Press, 1963.

[Geh17] A. Gehret. The asymptotic couple of the field of logarithmic transseries. *J. Algebra 470, 1-36*, 2017.

[Gla99] A. Glass. *Partially ordered groups.* World Scientific, 1999.

[Hal63] M. Hall. *The theory of groups.* The Macmillan company, 1963.

[Hol95] J. Holly. Canonical forms for definable subsets of algebraically closed and real closed valued fields. *the Journal of Symbolic Logic Volume 60, Number 3*, September 1995.

[KM11] S. Kuhlmann and M. Matusinski. Hardy type derivations on fields of exponential logarithmic series. *Journal of Algebra 345, 171-189*, 2011.

[KM12] S. Kuhlmann and M. Matusinski. Hardy type derivations on generalized series fields. *Journal of Algebra 351 , 185-203*, 2012.

[KMP17] S. Kuhlmann, M. Matusinski, and F. Point. The valuation difference rank of a quasi-ordered difference field. *Groups, Modules and Model Theory-Surveys and Recent Developments,in Memory of Rüdiger Göbel, Springer Verlag, p399-414*, 2017.

[Kuh94] F-V. Kuhlmann. Abelian groups with contractions i. In *Proceedings of the Oberwolfach Conference on Abelian Groups 1993, Amer. Math. Soc. ContemporaryMathematics 171,217 241*, 1994.

[Kuh00] S. Kuhlmann. *Ordered exponential fields.* The Fields Institute Monograph Series, vol 12, 2000.

[MAv17] L. v.d.Dries M. Aschenbrenner and J. v.d.Hoeven. *Asymptotic Differential Algebra and Model Theory of Transseries.* Princeton University Press, 2017.

[MS96] D. Macpherson and C. Steinhorn. On variants of o-minimality. *Annals of Pure and Applied Logic 79, 165-209*, 1996.

[PC83] S. Priess-Crampe. *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen.* Springer-Verlag, 1983.

[Pre84] A. Prestel. *Lectures on Formally Real Fields.* Springer Lecture Notes 1093, Springer-Verlag, 1984.

[Ros80] M. Rosenlicht. Differential valuations. *Pacific J. Math. 86, 301–319*, 1980.

[Sca00] T. Scanlon. A model complete theory of valued D-fields. *The Journal of Symbolic Logic, Vol. 65, No. 4, pp. 1758-1784*, 2000.

[Sim03] P. Simonetta. On non-abelian C-minimal groups. *Annals of pure and applied logic, 122, 263 – 287*, 2003.