Universität Konstanz
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Real Algebraic Geometry I – Exercise Sheet 6

Exercise 1 (8P). Let R be a real closed field. Suppose $a_0, \dots, a_d \in R$, $a_d \neq 0$, the polynomial $\sum_{i=0}^{d} a_i X^i \in R[X]$ is real-rooted and $j \in \{0, \dots, d-2\}$ with $a_j = a_{j+1} = 0$. Show that $a_0 = \dots = a_{j-1} = 0$ in two ways:

(a) Use the rule of Descartes for real-rooted polynomials together with elementary combinatorics.

(b) Use the intermediate value theorem and the relation between the position and the multiplicities of the roots of a real-rooted polynomial and its derivative.

Exercise 2 (6P). Let R be a real closed field, $a, b \in R$ and $f := X^3 + aX + b \in R[X]$.

(a) Show with the Hermite-method that f is real-rooted if and only if
$$(\tfrac{a}{3})^3 + (\tfrac{b}{2})^2 \leq 0.$$

(b) Show $-4a^3 - 27b^2 = (a_1 - a_2)^2(a_1 - a_3)^2(a_2 - a_3)^2$ where a_1, a_2, a_3 are the roots of f. In the case $R = \mathbb{R}$ and $f \in \mathbb{Q}[X]$, compare the result of (a) with Exercise 1 on Sheet 3.

(c) Show that f has three distinct roots in R if and only if
$$(\tfrac{a}{3})^3 + (\tfrac{b}{2})^2 < 0.$$

(d) How can we find out if an arbitrary monic polynomial f of degree 3 has exactly 3 roots in R?

Exercise 3 (4P). Let R be a real closed field. Consider polynomials $f, g \in R[X]$, where f is monic and $r \in R$. Show that there is an invertible matrix $P \in R^{\deg(f) \times \deg(f)}$ such that $H(f, g) = P^T H(f(X + r), g(X + r))P$ where $f(X + r)$ and $g(X + r)$ arise from f and g by substituting X by $X + r$.

Please submit until Thursday, December 8, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.