Universität Konstanz
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Winter Term 2016/2017

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Real Algebraic Geometry I – Exercise Sheet 7

Exercise 1 (4P). For which algebraically closed fields C does there exist a real closed subfield R of C with $C = R(\mathbb{i})$?

Exercise 2 (Bonus 4BP).

(a) Show that the field of rational functions $\mathbb{Q}(X)$ has an Archimedean order.

(b) Is it true that all real closed fields R_1 and R_2 with $R_1(\mathbb{i}) \cong R_2(\mathbb{i})$ are isomorphic?

Exercise 3 (4P). Prove the following statement or provide a counterexample: Let $f \in \mathbb{Q}[X]$ with $f(x) \geq 0$ for all $x \in \mathbb{Q}$. Then $f(x) \geq_K 0$ for all ordered fields $(K, \leq_K)$ and all $x \in K$.

Exercise 4 (4P). Let R be real closed field. Show that the semialgebraic subsets of R are exactly the finite unions of sets of the following form:

$$\{a\} \text{ and } (b,c)_R \qquad (a \in R,\ b, c \in R \cup \{\pm\infty\})$$

Exercise 5 (4P). Let K be a Euclidean field, $n \in \mathbb{N}_0$, $(a_{ij})_{1 \leq i,j \leq n} \in SK^{nxn}$ and

$$q := \textstyle\sum_{i,j=1}^n a_{ij} X_i X_j \in K[X_1, \dots, X_n]$$

a quadratic form with of rank r. For $A_k := (a_{ij})_{1 \leq i,j \leq k} \in SK^{kxk}$, suppose

$$d_k := \det(A_k) \neq 0 \text{ for } k \in \{0, \dots, r\}$$

(in particular $d_0 = \det(\emptyset) = 1$). Show with the help of $1.6.1(f)$, that there exist $\lambda_1, \dots, \lambda_r \in K^\times$ and linear forms $\ell_1, \dots, \ell_r \in K[X_1, \dots, X_n]$ with $q = \sum_{k=1}^r \lambda_k \ell_k^2$ satisfying the following conditions:

(a) $\ell_k \in X_k + K[X_{k+1}, \dots, X_n]$ for $k \in \{1, \dots, r\}$

(b) $\operatorname{sgn}(\lambda_1 \cdots \lambda_k) = \operatorname{sgn}(d_k)$ for $k \in \{0, \dots, r\}$

Deduce

$$\operatorname{sg} q = r - 2\sigma\left(\sum_{i=0}^r d_i T^i\right)$$

where T is a variable so that $\sigma(\sum_{i=0}^r d_i T^i)$ is the *number of sign changes in the sequence* $d_0, \dots, d_r$. This result i sometimes referred to as *Jacobi's criterion* for the signature of a quadratic form.

Please submit until Thursday, December 15, 2016, 11:44 in the box named RAG I, Number 10, near to the room F411.