On the optimal solutions of Lasserre relaxations

María López Quijorna
Advisor: Markus Schweighofer

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University of Konstanz

- $\mathbb{R}[X] := \mathbb{R}[x_1, ..., x_n]$, $X := (X_1, ..., X_n)$
- $\mathbb{R}[X]_d := \{p \in \mathbb{R}[X] : deg(p) \leq d\}$
- $\mathbb{R}[X]_{=d}$ vector space of real forms of degree d
- $\mathbb{R}[X]_d^* := \{L : \mathbb{R}[X]_d \to \mathbb{R}$ linear form$\}$
- $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$
- $|\alpha| := \alpha_1 + ... + \alpha_n$
- $X^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$
- Let $L \in \mathbb{R}[X]^*$, a quadrature rule for L is a tuple $(\lambda_1, a_1, ..., \lambda_r, a_r)$ with $\lambda_i \in \mathbb{R}_{\geq 0}$, with $\sum_{i=1}^r \lambda_i = 1$ and $a_i \in \mathbb{R}^n$ such that:

$$L(p) = \lambda_1 p(a_1) + \cdots + \lambda_r p(a_r)$$

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min $f(x)$ *over:*

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- If L has a quadrature rule then $L(f) \geq P^*$
- If (R_{2d}) has an optimal solution L^*, and moreover L^* has a quadrature rule then $L^*(f) = P^*$

Observation

To find a quadrature rule for $L \in (R_{2d})$ is equivalent to find:

- A finite dimensional euclidean vector space V and commuting self-adjoint endomorphisms $M_1, \ldots, M_n$ of V and $a \in V$ such that $L(p) =< p(M_1, \ldots, M_n)a, a >$, for $p \in \mathbb{R}[X]_{\leq 2d}$.

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Let $L \in (R_{2d})$ we set:

- $U_L := \{p \in \mathbb{R}[X]_d : \forall q \in \mathbb{R}[X]_d : L(pq) = 0\}$ **GNS kernel**
- $V_L := \frac{\mathbb{R}[X]_d}{U_L}$ **GNS representation space of L**
- $< \overline{p}, \overline{q} >_L := L(pq)$ $(p, q \in \mathbb{R}[X]_d)$ **GNS scalar product**
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- $\Pi_L : V_L \to \{\overline{p} : p \in \mathbb{R}[X]_{d-1}\}$ **orthogonal projection**
- $M_{L,i} : \Pi_L V_L \to \Pi_L V_L : \overline{p} \to \Pi_L(\overline{X_i p})$ $(p \in \mathbb{R}[X]_{d-1})$ and $i \in \{1, ..., n\}$.

$M_{L,i}$, called **i-th truncated GNS multiplication operator**, is self-adjoint endomorphism of $\Pi_L V_L$

Theorem

Let $L \in \ (R_{2d})$ *and assume that* $M_{L,1}, \ldots, M_{L,n}$ *paarwise commute. Then* $L_{|\mathbb{R}[X]_{2d-1}}$ *has a quadrature rule.*

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Sketch of the proof:

- There exist ONB $v_1, \ldots, v_r$ of $\Pi_L(V_L)$ st. $M_{L,i} v_j = y_{j_i} v_j$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, r\}$, and st. $\overline{1} = \sum_{i=1}^{r} a_i v_j$ and set $\lambda_j := a_j^2$.

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- $L(X^\alpha) = \langle M^\alpha(\overline{1}), \overline{1} \rangle$ for $|\alpha| \leq 2d - 1$

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- $\langle M^\alpha(\bar{1}), \bar{1} \rangle = \sum_{j=1}^{r} \lambda_i {y_j}^\alpha$ for $|\alpha| \leq 2d - 1$

Lemma

Let $L \in (R_{2d})$ and assume $M_{L,1}, \ldots, M_{L,n}$ paarwise commute. Then there exist $y_1, \ldots, y_n \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^{r} \lambda_i = 1$ and $L_\infty \in \mathbb{R}[X]_{2d}^*$ such that:

$$L = L_0 + L_\infty \text{ on } W$$

where:

$$W := \{\sum_{i=1}^{m} (s_i)^2 + p | s_i \in \mathbb{R}[X]_{=2d},\, p \in \mathbb{R}[X]_{2d-1}\}$$

$$\text{and } L_0 := \sum_{i=1}^{r} \lambda_i ev_{y_i}$$

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Sketch of the proof: It is enough to show that: $L(p) \geq L_0(p)$ for all $p \in W$ with equality for $p \in \mathbb{R}[X]_{2d-1}$

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- $L(p) = L_0(p)$ for all $p \in \mathbb{R}[X]_{2d-1}$. (We apply the previous theorem)

To show $L(p) \geq L_0(p)$ for all $p \in W$ with equality for $p \in \mathbb{R}[X]_{2d-1}$:

- $L(p) = L_0(p)$ for all $p \in \mathbb{R}[X]_{2d-1}$. (We apply the previous theorem)
- $L(p) \geq L_0(p)$ for all $p = \sum s^2$ with $s_i \in \mathbb{R}[X]_{=d}$. Indeed, if $s = \sum_{|\alpha|=d} a_\alpha X^\alpha$, then $L(s^2) = L(ss) = \langle \bar{s}, \bar{s} \rangle = \left\langle \overline{\sum_{|\alpha|=d} a_\alpha X^\alpha}, \overline{\sum_{|\alpha|=d} a_\alpha X^\alpha} \right\rangle = \left\langle \overline{\sum_{|\alpha|=d} X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}}, \overline{\sum_{|\alpha|=d} X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}} \right\rangle \geq \left\langle \Pi(\overline{\sum_{|\alpha|=d} X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}}), \Pi(\overline{\sum_{|\alpha|=d} X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}}) \right\rangle = \left\langle \sum_{|\alpha|=d} \Pi(\overline{X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}}), \sum_{|\alpha|=d} \Pi(\overline{X_{j_\alpha} a_\alpha X^{\alpha - e_{j_\alpha}}}) \right\rangle = \left\langle \sum_{|\alpha|=d} M_{L,j_\alpha}(\overline{a_\alpha X^{\alpha - e_{j_\alpha}}}), \sum_{|\alpha|=d} M_{L,j_\alpha}(\overline{a_\alpha X^{\alpha - e_{j_\alpha}}}) \right\rangle = \ldots = \langle s(M_{L,1}, \ldots, M_{L,n}), s(M_{L,1}, \ldots, M_{L,n}) \rangle = \sum_{j=1}^r \lambda_j s^2(y_j) = L_0(s^2)$

Corollary

Let $L \in (R_{2d})$ be an optimal solution and $f \in W$, and assume $M_{L,1}, ..., M_{L,n}$ commute. Then there exist $y_1, ... y_r \in \mathbb{R}^n$ and $\lambda_1, ..., \lambda r \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^{r} \lambda_i = 1$ such that $L = \sum_{i=1}^{r} \lambda_i ev_{y_i}$. And for $i = 1, ..., r$, y_i is a minimizer of (P).

Proof:

- Since $f \in W$ we have $L_\infty(f) \geq 0$
- Since L is an optimal solution and $L = L_0 + L_\infty$ on W

$$L(f) = L_0(f) + L_\infty(f) \leq f^* \leq \sum_{i=1}^{r} \lambda_i f(y_i) = L_0(f)$$

  Therefore $L_\infty(f) \leq 0$.

Then we have $L_\infty(f) = 0$. And then $L(f) = f^* = L_0(f) = \lambda_1 f(y_1) + ... + \lambda_r f(y_r)$, and we conclude $y_1, ... y_r$ are minimizers of the original polynomial optimization problem (P).

Danke schon!