Truncated Moment Problem and Polynomial Optimization with the GNS construction

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Introduction

This is part of my ongoing Ph.D. project directed by Prof. M. Schweighofer. This poster presents results concerning the truncated moment problem and polynomial optimization problems. The new tool we are using to understand these problems is the Gelfand-Neimark-Segal truncated construction.

Truncated Moment Problem (Bayer and Teichmann [1],[2])

Given a linear form $L \in \mathbb{R}[\underline{X}]_{2d}^*$ such that $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$ (where $\underline{X} := (X_1, ..., X_n)$): Do there exist $a_1, ..., a_N$ points in $\mathbb{R}^n$, and $\lambda_1, ..., \lambda_N > 0$ weights such that $L(p) = \sum_{i=1}^N \lambda_i p(a_i)$ for all $p \in \mathbb{R}[\underline{X}]_k$ for some $k \leq 2d$? I.e., is there a quadrature formula representation of L?

New formulation of the Truncated Moment Problem [7]

Given a linear form $L \in \mathbb{R}[\underline{X}]_{2d}^*$ such that $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$ we would like to find a:

- Finite dimensional euclidean vector space **V**, commuting self-adjoint endomorphisms $M_1, ..., M_n$ of **V** and a$\in$ **V** such that $L(p) =< p(M_1, ..., M_n)a, a >$.

Example: GNS (infinite dimensional case)

If we had $L(p^2) > 0$ for all $p \neq 0$:

- **V** $:= \mathbb{R}[\underline{X}]$
- $< p, q >:= L(pq)$
- $M_i : \mathbb{R}[\underline{X}] \to \mathbb{R}[\underline{X}] : p \mapsto X_i p,\ i \in \{1, ..., n\}$
- $a := 1 \in \mathbb{R}[\underline{X}]$

GNS-Truncated-Construction

The construction [7]

Given a linear form $L \in \mathbb{R}[\underline{X}]_{2d}^*$ such that $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$ we build:

- $U_L := \{p \in \mathbb{R}[\underline{X}]_d : \forall q \in \mathbb{R}[\underline{X}]_d : L(pq) = 0\}$ **GNS kernel**
- $V_L := \frac{\mathbb{R}[\underline{X}]_d}{U_L}$ **GNS representation space of L**
- $< \overline{p}, \overline{q} >_L := L(pq)$ $(p, q \in \mathbb{R}[\underline{X}]_d)$ **GNS scalar product**
- $\Pi_L : V_L \to \{\overline{p} : p \in \mathbb{R}[\underline{X}]_{d-1}\}$ **orthogonal projection**
- $M_{L,i} : \Pi_L(V_L) \to \Pi_L(V_L) : \overline{p} \to \Pi_L(\overline{X_i p})$ $(p \in \mathbb{R}[\underline{X}]_{d-1})$ and $i \in \{1, ..., n\}$ **GNS-truncated-multiplication operators**
- We built $(V_L, <,>_L)$ an Euclidean Vector Space and $M_{L,i}$ self-adjoint endomorphisms

Truncated Moment Problem with the GNS construction

Let $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$, we say L is flat if $\mathbb{R}[\underline{X}]_{d-1} + U_L = \mathbb{R}[\underline{X}]_d$.
Assume $L \in \mathbb{R}[\underline{X}]_{2d}^*$ with $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$, then we get the following results:

Result 1

If L is flat then the GNS-truncated-multiplication operators commute.

Result 2

If the GNS-truncated operators commute then $L_{|\mathbb{R}[\underline{X}]_{2d-1}+U_L}$ has a quadrature formula.

Proof:

- The coordinates of the nodes of the quadrature formula are the eigenvalues of the GNS-truncated-multiplication operators.

Result 3 (**Curto and Fialkow** [6],[4],[2],[7])

If L is flat then L has a quadratura formula.

Proof:

- Since in this case $\mathbb{R}[\underline{X}]_{d-1} + U_L = \mathbb{R}[\underline{X}]_d$, we have this result as a collorary of the result 2.

Result 4

L having a quadratura formulae does not imply in general that the GNS-truncated-multiplication operators commute.

Example: Take the linear form:

- $L : \mathbb{R}[X_1, X_2]_4 \to \mathbb{R}, p \mapsto \frac{1}{4}(p(-1, 0) + p(1, 0) + p(0, -1) + p(0, 1))$

Result 5

If the GNS-truncated-multiplication operators commute we do not have in general L flat.

Example: Take the linear forms:

- $L : \mathbb{R}[X_1, X_2]_4 \to \mathbb{R}, p \mapsto \frac{1}{4}(p(0, 0) + p(1, 0) + p(-1, 0) + p(0, 1))$
- $L : \mathbb{R}[X_1, X_2, X_3]_4 \to \mathbb{R}, p \mapsto \frac{1}{4}(p(1, 0, 1) + p(1, 1, -2) + p(1, -1, 0) + p(1, 2, -3))$

That is to say, to be flat is stronger condition than the conmutativity condition in the GNS-truncated-multiplication operators associated to the linear form.

Polynomial Optimization

Polynomial optimization problem (POP)

We are interested in the following problem:

$$\min p(x)$$
$$\text{such that } x \in \mathbb{R}^n$$
$$\text{and } x \in S := \{x \in \mathbb{R}^n : g_1(x) \geq 0, ..., g_m(x) \geq 0\}$$

where $p, g_1, ..., g_m \in \mathbb{R}[\underline{X}]_k$

Lasserre relaxation of order k (P_k) ([3],[2],[5])

Solving this semidefinite program we get a lower bound of the original POP:

$$\min L(p)$$
$$\text{such that } L \in \mathbb{R}[\underline{X}]_k^*$$
$$\text{and } L(\sum \mathbb{R}[\underline{X}]^2 \cap \mathbb{R}[\underline{X}]_k + \sum \mathbb{R}[\underline{X}]^2 g_1 \cap \mathbb{R}[\underline{X}]_k + \ldots + \sum \mathbb{R}[\underline{X}]^2 g_m \cap \mathbb{R}[\underline{X}]_k) \subseteq \mathbb{R}_{\geq 0}$$
$$\text{and } L(1) = 1$$

Polynomial Optimization properties with the GNS construction

Extracting global minimizers

If L is an optimal solution of P_k such that $L_{|\mathbb{R}[\underline{X}]_l}$ has a quadratura formula representation with $l \geq \deg p$ and nodes over S then:

- If the GNS operators commute we can extract with simultaneaus diagonalization the nodes. And these will be global minimizers of the original POP.

On the optimal solutions of Lasserre relaxations

Let $d \in \mathbb{N}_0$ and $L \in \mathbb{R}[\underline{X}]_{2d}^*$ and $L(\sum \mathbb{R}[\underline{X}]_d^2) \subseteq \mathbb{R}_{\geq 0}$ be an optimal solution of Lasserre relaxation of degree d with f as objective function , $f \in W$, where

$$W := \{\sigma + p | \sigma \in \sum \mathbb{R}[\underline{X}]_{=d}^2,\ p \in \mathbb{R}[\underline{X}]_{2d-1}\}$$

and assume $M_{L,1}, ..., M_{L,n}$ commute. Then there exist $y_1, ...y_r \in \mathbb{R}^n$ and $\lambda_1, ..., \lambda_r \in \mathbb{R}_{\geq 0}$ with $\sum_{i=1}^r \lambda_i = 1$ such that $L(f) = \sum_{i=1}^r \lambda_i f(y_i)$. And for $i = 1, ..., r$, y_i are minimizers of the original polynomial optimization problem.

References

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[7] http://www.math.uni-konstanz.de/$\sim$schweigh/presentations/polopt-kirchberg.pdf