Symmetric Tensor Decomposition and Algorithms

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Contents

Abstract

Castellano: El objetivo de este trabajo es el estudio de la descomposición de tensores simétricos de dimensión "n" y orden "d". Equivalentemente el estudio de la descomposición de polinomios homogéneos de grado "d" en "n" variables como suma de "r" potencias d-ésimas de formas lineales.

Este problema tiene una interpretación geométrica en términos de incidencia de variedades secantes de variedades de Veronese: Problema de Waring [12],[6]. Clásicamente, en el caso de formas binarias el resultado completo se debe a Sylvester. El principal objeto de estudio del trabajo es el algoritmo de descomposición de tensores simétricos, que es una generalización del teorema de Sylvester y ha sido tomado de [1]. Pero antes de enfrentarnos al algoritmo, introducimos las herramientas necesarias como son los operadores de Hankel y propiedades de las álgebras de Gorenstein.

English: The aim of this work is studing the decomposition of symmetric tensors, of dimension "n" and order "d". Equivalently, studying the decomposition of homogeneous polynomials of degree "d" in "n" variables as sum of "r" dth-powers of linear forms.

This problem has a geometric interpretation with the secant varieties to the Veronese variety: "Big Waring Problem" [12] and [6]. Classically, the binary case was given by Sylvester. The main object of study is the symmetric tensor decomposition algorithm, which is a generalization of Sylvester theorem and it has been taken from [1]. But, before facing to the algorithm we introduce several tools, for instance the Hankel Operators and several properties of the Gorenstein Algebras.

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Chapter 1

Introduction

A tensor is an element in the product of vector space $\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k}$. We shall say that a tensor is cubical if all its k dimensions are identical, i.e. $n_1 = ... = n_k = n$. A cubical tensor $x \in \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ is said to be symmetric if for any permutation σ of $\{1, ..., k\}$:

$$x_{i_1 \ldots i_k} = x_{i_{\sigma(1)} \ldots i_{\sigma(k)}} \ .$$

The aim of this work is studying the decomposition of a symmetric tensor into a minimal linear combination of a tensor of the form $v \otimes \cdots \otimes v$. The minimal number of sums in this decomposition will be the symmetric rank. This decomposition of a tensor was first introduced and studied by Frank L. Hitchcook in 1927, and then was rediscovered in 1970's by psychometricians.

The tensors are objects which appear in many contexts and different applications. The most common tensors are the matrices, where the problem of decomposition is related to the singular value decomposition (SVD). The extension of higher order tensors gives arise to problems in the field of Electrical Engineering, Telecommunications, Chemometrics and Antenna Array Processing. For instance, the observations of experiences or physical phenomena which have a lot of parameters are stored in tensors.

The bijection between symmetric tensors and homogeneous polynomials will allow us to reformulate the problem as the decomposition of a homogeneous polynomial f of degree d in $n+1$ variables as a sum of d-th powers of linear forms [1], i.e.:

$$f(\overline{x}) = \sum_{i=1}^{r} \lambda_i (k_{i0} x_0 + ... + k_{in} x_n)^d \tag{1.1}$$

The problem of decomposition in the binary case can be obtained directly by computing ranks of catalecticant matrices [13], as can be seen in Sylvestert's Theorem. But in higher dimension this is not so simple, however the team of Bernard Mourrain [1], using apolar duality on polynomials, get an extension of Sylvestert's algorithm, reducing the problem of the symmetric tensor decomposition to the decomposition of a linear form as a linear combination of evaluations at distinct points. Moreover, they give a necessary and sufficient condition for the existence of a decomposition of symmetric rank r, based on rank conditions of Hankel operators and commutation properties.

Therefore the main ingredients in this work will be: reformulation of the problem in a dual space, exploitation of the properties of multivariate Hankel operators and Gorenstein algebras, studying an effective method for solving the truncated Hankel problem and deduction of the decomposition by solving a generalized eigenvalue problem.

Chapter 2

Preliminaires

We will work in $\mathbb{K}$ and algebraically closed field, such that $char(\mathbb{K}) = 0$. Let E be a vector space of dimension $n+1$ and we will denote $T^d(E) := E \otimes \cdots \otimes E$, the set of all tensors of order d and dimension $n+1$. A tensor of order d and dimension $n+1$ can be represented by an array $[a_{i_1,\ldots,i_d}]_{i_1=0,\ldots,i_d=0}^{n,\ldots\ldots\ldots,n} \in T^d(E)$ with $a_{i_1,\ldots,i_d} \in \mathbb{K}$ in a basis of $T^d(E)$, due to the universal property of the tensor product. The set of all symmetric tensors of order d and dimension $n+1$ forms an algebra, $S^d(E)$, and a tensor $[a_{i_1,\ldots,i_d}]_{i_1=0,\ldots,i_d=0}^{n,\ldots\ldots,n}$ will be symmetric if $a_{i_1,\ldots,i_d} = a_{i_{\sigma(1)},\ldots,i_{\sigma(d)}}$ for any permutation σ of $\{1,\ldots,d\}$. We will use $\alpha, \beta, \ldots.$ to denote a vector in $\mathbb{N}^{n+1}$ (multi-index), and $|\alpha| := \sum_{i=0}^{n} \alpha_0 + \ldots + \alpha_n$. And we will denote $\overline{x}^{\alpha} := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We will work in $R := \mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials, and R_d will be the ring of polynomials of degree at most d. For a set $B = \{b_1, \ldots, b_r\} \subset R$ we will denote by $\langle B \rangle$ (resp. (B)) the corresponding vector space (resp. ideal) generated by B. We will denote by $S_d := \mathbb{K}[x_0, \ldots, x_n]_d$ the vector space of homogeneous polynomials in $n+1$ variables of degree d.
The dual space E^*, of a $\mathbb{K}$-vector space is the set of $\mathbb{K}$-linear forms from E to $\mathbb{K}$. We have to take into account that R^* has a natural structure of R-module; for all $p \in R$ and $\Lambda \in R^*$:

$$\begin{array}{rccc} p * \Lambda : & R & \to & \mathbb{K} \\ & q & \longmapsto & \Lambda(pq) \end{array}$$

Typical elements of R^* are the linear forms $ev(\xi)$ for $\xi \in \mathbb{K}^n$, and $\overline{d}^{\alpha} := d_1^{\alpha_1} \cdots d_n^{\alpha_n}$, defined as follows: for all $p = \sum p_\beta \overline{x}^{\beta} \in R$:

$$\begin{array}{rccc} ev(\xi) : & R & \to & \mathbb{K} \\ & p & \longmapsto & p(\xi) = \sum p_\beta \xi^\beta \end{array}$$

$$\begin{array}{rccc} \overline{d}^{\alpha} : & R & \to & \mathbb{K} \\ & p & \longmapsto & p_\alpha \end{array}$$

Particularly;

$$x_i * \overline{d}^{\alpha} = \begin{cases} d_1^{\alpha_1} \ldots d_{i-1}^{\alpha_{i-1}} d_i^{\alpha_i - 1} d_{i+1}^{\alpha_{i+1}} \ldots d_n^{\alpha_n} & \text{if } \alpha_i > 0; \\ 0 & \text{in other case.} \end{cases} \tag{2.1}$$

Let V be a $(n+1)$-dimensional vector space over $\mathbb{K}$, we will be interested in the decomposition of a symmetric tensor $A = [a_{j_1 \cdots j_d}]_{j_1=0,\ldots,j_d=0}^{n,\ldots\ldots\ldots,n} \in S^d(V)$ into a minimal linear combination of symmetric outer products of vectors (i.e. of the form $\overbrace{v \otimes \cdots \otimes v}^{d)}$) such that:

$$A = \sum_{i=1}^{r} \lambda_i \overbrace{v \otimes \cdots \otimes v}^{d)} \tag{2.2}$$

Definition 2.1. *If $A = [a_{j_1 \dots j_d}]_{j_1=0,\dots,j_d=0}^{n,\dots\dots\dots,n} \in S^d(\mathbb{C}^{n+1})$, the symmetric tensor rank of A is:*

$$rank_S A := min\{r | A = \textstyle\sum_{i=1}^{r} \lambda_i y_i \otimes \cdots \otimes y_i : y_i \in \mathbb{C}^{n+1}\}$$

We will see that a decomposition of this form always exists for any symmetric tensor in 4.23, ([2] page 12). Therefore the definition of symmetric rank is not vacuous.

Remark 2.2. *Note that over $\mathbb{C}$, the coefficients λ_i appearing in the decomposition 2.2 may be set 1; it is legitimate since any complex number admits a d-th root in $\mathbb{C}$. Henceforth, we will adopt the following notation.*

$$y^{\otimes k} := \overbrace{y \otimes \cdots \otimes y}^{kcopies}$$

Example 2.3.

Let $A \in S^3(\mathbb{C}^2)$ be defined by:

$$A = \left(\begin{array}{cc|cc} a_{111} & a_{121} & a_{112} & a_{122} \\ a_{211} & a_{221} & a_{212} & a_{222} \end{array} \right) = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

It is of symmetric rank 2 over $\mathbb{C}$:

$$A = \frac{\sqrt{-1}}{2} \begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3} - \frac{\sqrt{-1}}{2} \begin{pmatrix} \sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3}$$

Indeed:

$$A = \frac{\sqrt{-1}}{2} \begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3} - \frac{\sqrt{-1}}{2} \begin{pmatrix} \sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3} =$$

$$= \frac{\sqrt{-1}}{2} \left[\begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix} \begin{pmatrix} -\sqrt{-1} & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix} \right] - \frac{\sqrt{-1}}{2} \left[\begin{pmatrix} -\sqrt{-1} \\ 2 \end{pmatrix} \begin{pmatrix} -\sqrt{-1} & 2 \end{pmatrix} \begin{pmatrix} \sqrt{-1} \\ 2 \end{pmatrix} \right] =$$

$$\frac{\sqrt{-1}}{2} \left(\begin{array}{cc|cc} \sqrt{-1} & -1 & -1 & -\sqrt{-1} \\ -1 & -\sqrt{-1} & -\sqrt{-1} & 1 \end{array} \right) - \frac{\sqrt{-1}}{2} \left(\begin{array}{cc|cc} -\sqrt{-1} & -1 & -1 & \sqrt{-1} \\ -1 & \sqrt{-1} & \sqrt{-1} & 1 \end{array} \right) =$$

$$\left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

2.1 Applications

Let Q be a $(n+1)\times(n+1)$ invertible matrix and let E be a vector space of dimension $n+1$. We define the following application:

$$\begin{array}{cccc} Q: & T^d(E) & \rightarrow & T^d(E) \\ & A=[a_{i_1,\dots,i_d}]_{i_1=0,\dots,i_d=0}^{n,\dots\dots\dots\dots,n} & \longmapsto & Q(A)=[A_{i_1,\dots,i_d}]_{i_1=0,\dots,i_d=0}^{n,\dots\dots\dots\dots,n} \end{array}$$

Where $A_{i_1,\dots,i_d}=\sum_{j_1,\dots,j_d} Q_{i_1j_1}\dots Q_{i_dj_d}a_{i_1,\dots,i_d}$. A tensor A is symmetric if $A_{\sigma(ij\dots k)}=A_{ij\dots k}$ for any permutation σ. This property is referred to as the multilinearity property of tensor.

Symmetric tensors form an important class of tensors and examples where they arise include multivariate moments and cumulants of random vectors, since the set of cumulants of order d of a multichannel real random variable X of dimension $n+1$ form a symmetric tensor of order d and dimension $n+1$. The same holds true for moments, due to the fact that symmetric tensors satisfy the multilinearity property [7]:

For a vector-valued random variable $X=(X_0,\dots,X_n)$ we obtain three tensors of order d:

- The *dth* non-central moment $s_{i_1,\dots,i_d}$ $(1\le i_j\le n\ j\in\{1,\dots,d\})$ of X is:

  $$s_{i_1,\dots,i_d}:=E(X_{i_1}X_{i_2}\dots X_{i_d})$$

  and the set of non-central moments of X can be identified with the following tensor of order d and dimension $n+1$:

  $$S_d(X)=[E(X_{i_1}X_{i_2}\dots X_{i_d})]_{i_1=0,\dots,i_d=0}^{n,\dots\dots\dots\dots,n}$$

- The *dth* central moment of X is the following tensor:

  $$M_d(X)=S_d(X-E[X])$$

- The *dth* cumulant $k_{i_1\dots i_d}$ $(1\le i_j\le n\ j\in\{1,\dots,d\})$ is:

  $$k_{i_1\dots i_d}:=(-1)^{q-1}(q-1)!s_{P_1}\dots s_{P_q}$$

  where $P_1\cup\dots\cup P_q=\{i_1,\dots,i_d\}$ are the partitions of the index set. And the set of cumulants of X can be identified with the following tensor of order d and dimension $n+1$:

  $$K_d(X)=[\textstyle\sum_P(-1)^{q-1}(q-1)!s_{P_1}\dots s_{P_q}]_{i_1=0,\dots,i_d=0}^{n,\dots\dots\dots,n}$$

  where the sum is over all the partitions $P=P_1\cup\dots\cup P_q$ of the index set.

This cumulant tensors have been used in array processing. And the symmetric outer product decomposition is also important in areas such as: mobile communications, machine learning, biomedical engineering, psychometrics and chemometrics [2].

2.2 From symmetric tensor to homogeneous polynomials

It can be pointed out that there exists a bijective relation between the space of tensors of dimension $n+1$ and order d, $S^d(\mathbb{C}^{n+1})$, and the space of homogeneous polynomials of degree d in $n+1$ variables, S_d. A symmetric tensor $[t_{j_1,\ldots,j_d}]_{j_1=0,\ldots,j_d=0}^{n,\ldots\ldots\ldots,n}$ of order d an dimension $n+1$, can be written with a homogeneous polynomial $f(\overline{x}) \in S_d$:

$$[t_{j_1,\ldots,j_d}] \longrightarrow f(\overline{x}) = \sum_{j_1=0,\ldots,j_d=0}^{n,\ldots\ldots,n} t_{j_1,\ldots,j_d} x_{j_1}..x_{j_d}$$

The correspondence between symmetric tensors and homogeneous polynomials is bijective:

$$S^d(\mathbb{C}^n) \cong \mathbb{C}[x_o, \ldots, x_n]_d$$

Example 2.4.

Alternatively, the tensor of the first example 2.3:

$$A = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

is associated with the homogeneous polynomial in two variables:

$$p(x,y) = 3xy^2 - x^3$$

which can be decomposed over $\mathbb{C}$ into:

$$p(x,y) = \tfrac{\sqrt{-1}}{2}(-\sqrt{-1}x + y)^3 - \tfrac{\sqrt{-1}}{2}(\sqrt{-1}x + y)^3$$

Therefore in the following formulations of the problem we will work with homogeneous polynomials instead of symmetric tensors.

Chapter 3

The binary case

The present survey is a generalization of Sylvester's algorithm devised to decompose homogeneous polynomials in two variables into a sum of powers of linear forms, extracted from [1]. First of all we recall this theorem, ([3] page: 102):

Theorem 3.1. *A binary form $f(x_1, x_2) = \sum_{i=o}^{d} \binom{d}{i} c_i x_1^i x_2^{d-i}$ can be written as a sum of dth powers of r distinct linear forms in $\mathbb{C}$ as:*

$$f(x_1, x_2) = \sum_{j=1}^{r} \lambda_j (\alpha_j x_1 + \beta_j x_2)^d$$

if and only if:

- *there exist a vector $\overline{q} = (q_l)_{l=0}^{r}$ such that:*

$$\begin{bmatrix} c_0 & c_1 & \dots & c_r \\ \vdots & \dots & \dots & \vdots \\ c_{d-r} & \dots & \dots & c_d \end{bmatrix} [\overline{q}] = [\overline{0}]$$

- *the polynomial $q(x_1, x_2){=}\sum_{l=0}^{r} q_l x_1^l x_2^{r-l}$ admits r distint roots, i.e. it can be written as $q(x_1, x_2){=}\prod_{j=1}^{r} (\beta_j x_1 - \alpha_j x_2)$.*

We will see a partial proof of this theorem in 4.3. The proof of this theorem is constructive and yields the following algorithm: let $p(x_0, x_1)$ be a binary form of degree d and coefficients $a_i{=}\binom{d}{i} c_i$, $0 \leq i \leq d$, the algorithm builds the Hankel Matrix ($H[r]$) of dimension $d - r + 1 \times r + 1$ whose entries are:

$$H[r]_{ij}{=}c_{i+j-2}$$

and then compute its kernel.

Algorithm 3.2. *Binary form decomposition*

Input:A binary polynomial $p(x_1, x_2)$ of degree d with coefficients $a_i = \binom{d}{i} c_i$, s.t. $0 \leq i \leq d$
Output: A decomposition of $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j k_j^d(\overline{x})$ with minimal r

- 1. Initialize $r = 0$
- 2. Increment $r := r + 1$

- 3. If the matrix $H[r]$ has $ker(H[r]) = \overline{0}$ go to step 2
- 4. Else compute a basis $k_1, .., k_l$ of the $ker(H[r])$
- 5. Specialization:
  - Take any vector in the kernel, e.g. $\overline{k}$
  - Compute the roots of the associated polynomial $k(x_1, x_2) = \sum_{l=0}^{r} k_l x_1^l x_2^{d-l}$
  - If the roots are not distinct in $\mathbb{P}_2$, try another specialization. If cannot be obtained , go to step 2.
  - Else if $k(x_1, x_2)$ admits r distinct roots,$(\alpha_j : \beta_j)$ for $j = 1, ..., r$, then compute coefficients λ_j $1 \leq j \leq r$
    $$\begin{bmatrix} \alpha_1^d & \cdots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \cdots & \alpha_r^{d-1}\beta_r \\ \vdots & \cdots & \vdots \\ \beta_1^r & \cdots & \beta_r^d \end{bmatrix} \overline{\lambda} = \begin{bmatrix} a_0 \\ \cdot \\ \cdot \\ a_d \end{bmatrix}$$
- 6. The decomposition is $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j (\alpha_j x_1 + \beta_j x_2)^d$

Example 3.3. *Let apply the Sylvester algorithm to the polynomial:*

$$p(x_1, x_2) = 17x_1^4 + 48x_1^3 x_2 + 120x_2^2 x_1^2 + 264x_1 x_2^3 + 257x_2^4$$

for $r = 1$, we have the following Hankel matrix:

$$\begin{bmatrix} c_0 & c_1 \\ c_1 & c_2 \\ c_2 & c_3 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 17 & 12 \\ 12 & 20 \\ 20 & 66 \\ 66 & 257 \end{bmatrix}$$

This matrix has full column rank. Therefore, we build the Hankel matrix for $r = 2$:

$$\begin{bmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 17 & 12 & 20 \\ 12 & 20 & 66 \\ 20 & 66 & 257 \end{bmatrix}$$

This matrix has rank equal to 2, therefore we compute a basis of the kernel, to do this we use the singular value decomposition and the help of "Matlab" and we get the following decomposition of M:

$$M = U\Sigma V^*$$

where $rank(\Sigma) = 2$, and we know by a theorem well known that $Ker(M) = \langle v_3 \rangle$ where v_3 is the third column of V^*. Then, we compute the roots of $q(x_1, x_2)$=$\sum_{l=0}^{2} v_{3_l} x_1^l x_2^{2-l}$ which are $(\alpha_1, \beta_1) = (2, 1)$ and $(\alpha_2, \beta_2) = (0.25, 1)$. Lastly, we compute λ_1 and λ_2 by equating coefficients in the same monomials and we get the decomposition:

$$p(x_1, x_2) = (2x + y)^4 + 256(0.25x + y)^4$$

Chapter 4

Problem Formulations

In this chapter we present three different approaches to the problem. These approaches were given by the team of Bernard Mourrain in [1].

4.1 Polynomial Decomposition

We will explain how to get a decomposition of $f \in S_d$ as a sum of d-th powers of linear forms [1], i.e.:

$$f(\overline{x}) = \sum_{i=1}^{r} \lambda_i (k_{i0}x_0 + ... + k_{in}x_n)^d = \lambda_1 k_1(\overline{x})^d + ... + \lambda_r k_r(\overline{x})^d \tag{4.1}$$

where $k_i \neq 0$, and r is the smallest possible integer.

Remark 4.1. *In the case we work over $\mathbb{C}$ we may assume all $\lambda_i = 1$.*

Definition 4.2. *The minimal r is called the symmetric rank of $f \in S_d$, denoted $rank_S(f)$.*

Remark 4.3. *Note that the symmetric rank of $f \in S_d$ is the same as the symmetric rank of its corresponding tensor in $S^d(\mathbb{C}^{n+1})$.*

A first approach to solve the problem of decomposition consists ([1] page 86) : given $f \in S_d$, and we assume that r, the symmetric rank, is known. We consider the $r(n+1)$ coefficients $k_{i,j}$ of the linear forms of the equality 4.1, as unknowns. We expand the right hand side of this equation . The two polynomials on the left and right hand sides are equal. Thus by equating the coefficients of the same monomials we get a system with $r(n+1)$ unknowns and with $\binom{n+d}{d}$ equations. This approach describes the problem of decomposition in a non-optimal way, since:

- It introduces $r!$ redundant solutions, since every permutation of the linear form is a solution.
- We get an over-constrained polynomial system, where the polynomials involved are of high degree, that is, d.

The first approach motivates the definition of the following map, Φ ,which goes from the set of unknowns $(k_{i,j})$ to the set of $\binom{n+d}{d}$ equations. To be accurate: the expansion of the right hand side of the equation 4.1, in the basis of monomials $B(n;d) = \{\overline{x}^{\alpha}, |\alpha| = d\}$ defines a map Φ from the set $X = \mathbb{C}^{(n+1)r}$ of coefficients $k_{i,j}$ onto $\Upsilon = \mathbb{C}^{\binom{n+d}{d}}$:

$$\begin{array}{llll} \Phi : & X = \mathbb{C}^{(n+1)r} & \longrightarrow & \Upsilon = \mathbb{C}^{\binom{n+d}{d}} \\ & \overline{k} = ((k_{1,i}), ..., (k_{r,i})) & \longmapsto & (c_\alpha(\overline{k}))_{\alpha \in I} \end{array}$$

where $I = \{\alpha = (\alpha_0, ..., \alpha_n) : |\alpha_0 + ... + \alpha_n| = d\}$ is the set of index and $c_\alpha(\overline{k})$ is defined as the coefficient of the monomial $\overline{x}^\alpha$ of the expansion.

Definition 4.4. *A property is said to be true in the generic case, or for generic polynomials, if it is true in a dense algebraic open subset of Υ, in the Zariski topology.*

Definition 4.5. *The symmetric generic rank, denoted $g(n, d)$, is the minimal value to be given to r in the decomposition 4.1, in the generic case.*

Proposition 4.6. *The dimension of the image can not be greater than the numbers of parameters in function Φ (which is $(n + 1)r$).*

Proof. If $(n + 1)r < \binom{n+d}{d}$ then the image would lie in an hypersurface an would not be dense. Therefore, $\binom{n+d}{d} \leq (n + 1)r$. □

Example 4.7.

To show how careful we have to be, consider for instance a generic ternary quartic, one would expect that it could be decomposed in to 5 linear forms since $r \times (n + 1) = 5 \times 3 \geq \binom{6}{4} = \binom{n+d}{d}$, but the correct number of linear forms is 6 ([3] page 102).
We will see that the generic rank in S_d is known for any order and dimension due to the work of Alexander and Hirschowitz.

4.2 Geometric point of view

This section is written due to the information that you can find in [6] and [3].

4.2.1 Big Waring Problem

In 1770, E. Waring conjectured: "for all integers $d \geq 2$ there exists a positive integer $g(d)$ such that each $n \in \mathbb{N}$ can be written as $n = a_1^d + ... + a_{g(d)}^d$ with $a_i \geq 0$ and $i = 1, ..., g(d)$", [6].
The conjecture of Waring was showed to be true by Hilbert in 1909. An analogous problem can be formulated for homogeneous polynomials of given degree d in $S_d := K[x_o, ..., x_n]_d$: "Which is the minimum $r \in \mathbb{N}$ such that the generic form $F \in S_d$ is sum of at most r d-powers of linear forms?"

$$F = L_1^d + ... + L_r^d$$

This is the Big Waring Problem which was completely solved by J. Alexander and A. Hirchowitz in 1995.

4.2.2 Veronese and secant varieties

Definition 4.8. *The image of the following map is the d-th Veronese variety, $X_{n,d}$:*

$$\begin{aligned} \nu_d \colon \mathbb{P}^n &\longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \\ (u_o : \ldots : u_n) &\longmapsto (u_o^d : u_o^{d-1}u_1 : \ldots : u_n^d) \end{aligned}$$

This map can also be dually characterized as:

$$\begin{aligned} \nu_d \colon \mathbb{P}(S_1) = (\mathbb{P}^n)^* &\longrightarrow \mathbb{P}(S_d) = (\mathbb{P}^{\binom{n+d}{d}-1})^* \\ k(\overline{x}) &\longmapsto k(\overline{x})^d \end{aligned}$$

Therefore we can think to the Veronese variety as the variety that parameterizes d-th powers of linear forms. The polynomials of rank one are exactly those lying on $X_{n,d}$. If we want to study the variety that parameterizes sums of "r" d-powers of linear forms of $S := K[x_o, ..., x_n]$ we have to consider the r-th secant variety of $X_{n,d}$, which we will define below, [6].

Definition 4.9. *The set that parameterizes homogeneous polynomials $F \in S_d$ of rank at most "r" is:*

$$\sigma_s^0(X_{n,d}) := \bigcup_{[L_1^d],\ldots,[L_s^d] \in X_{n,d}} \langle [L_1^d], \ldots, [L_s^d] \rangle$$

but in general, $\sigma_s^0(X_{n,d})$ is not a variety.

Definition 4.10. *The r-th secant variety of $X_{n,d} \subset \mathbb{P}(S_d)$ is the Zariski clousure $\sigma_s^0(X_{n,d})$ denoted by $\sigma_s(X_{n,d})$*

From this point of view the smallest $r \in \mathbb{N}$ such that $\sigma_r(X_{n,d}) = \mathbb{P}(S_d)$ is the minimum integer "r" such that the generic form of degree d in $n+1$ variables is a linear combination of "r" powers of linear forms in the same number of variables. Then this minimun integer "r" answers the Big Waring Problem.

Definition 4.11. *Let $F \in S_d$ be a homogeneous polynomial, the minimum integer for which s, $[F] \in \sigma_s(X_{n,d})$ is the border rank of F, denoted $rank_B(F)$.*

Theorem 4.12. *(Alexander-Hirschowitz). If $X = \sigma_s(X_{n,d})$, for $d \geq 2$. Then:*

$$dimension(X) = min(sn + s - 1, \binom{n+d}{d} - 1)$$

except for:

- $d = 2,\ 2 \leq s \leq n$
- $n = 2, d = 4, s = 5$
- $n = 3, d = 4, s = 9$
- $n = 4, d = 4, s = 14$
- $n = 4, d = 3, s = 7$

This theorem is extremely complicated to prove, and the interested reader should refer to the two papers of Alexander and Hirschowitz :[11],[12]. The difficult of proving this theorem lies in establishing the fact that the four given exceptions to the expected formula are the only ones.

4.3 Decomposition using duality

In order to pass the problem to the dual problem, we need the following definition of the apolar inner product:

Definition 4.13. *Let f,$g \in S_d$ f=$\sum_{|\alpha|=d} f_\alpha x_0^{\alpha_0}...x_n^{\alpha_n}$ and g=$\sum_{|\alpha|=d} g_\alpha x_0^{\alpha_0}...x_n^{\alpha_n}$ the apolar inner product on S_d is:*

$$\langle f,g\rangle = \sum_{|\alpha|=d} f_\alpha g_\alpha \binom{d}{\alpha_0,...,\alpha_n}^{-1}$$

Note that $\langle\cdot,\cdot\rangle$ cannot be a inner product in the usual sense since $\langle f,f\rangle$ is in general complex valued (recall that for an inner product, we need $\langle f,f\rangle \geq 0$ for all f). However, we will show that it is a non-degenerate symmetric bilinear form.

Lemma 4.14. *The bilinear form $\langle\cdot,\cdot\rangle : S_d \times S_d \longrightarrow \mathbb{C}$ defined above is symmetric and non-degenerate. In other words, $\langle f,g\rangle = \langle g,f\rangle$ for every $f,g \in S_d$, and if $\langle f,g\rangle = 0$ for all $g \in S_d$, then $f = 0$.*

Proof. The bilinearity and symmetry is immediate from definition. Suppose $\langle f,g\rangle = 0$ for all $g \in S_d$. Choose g to be the monomials:

$$g_\alpha(\overline{x}) = \binom{d}{\alpha_1,...,\alpha_n}\overline{x}^\alpha$$

where $|\alpha| = d$ and we see immediately that:

$$f_\alpha = \langle f,g_\alpha\rangle = 0$$

Thus $f \equiv 0$. □

Using this non-degenerate inner product, we can associate an element of S_d with an element on S_d^*, and for any $f^* \in S_d^*$ we can associate an element on R_d^* through the following composition:

$$\begin{array}{ccccc} S_d & \xrightarrow{\tau} & S_d^* & \xrightarrow{\pi} & R_d^* \\ f & \longmapsto & f^* & \longmapsto & \Lambda_{f^*} \end{array} \tag{4.2}$$

such that: f^*:$g \longmapsto \langle f,g\rangle$ and Λ_{f^*}:$p \longmapsto$f*(p^h), where p^h is the homogenization in degree d of p.

Under, τ, the polynomial f=$\sum_{|\alpha|=d} c_\alpha \binom{d}{\alpha}\overline{x}^\alpha$ is mapped to f^*=$\sum_{|\alpha|=d} c_\alpha \overline{d}^\alpha \in S_d^*$.

Lemma 4.15. *Let $k(\overline{x})^d = (k_0x_0 + ... + k_nx_n)^d$. Then for any $f(\overline{x}) \in S_d$ we have:*

$$\langle f(\overline{x}), k(\overline{x})\rangle = f(k_0,...,k_n)$$

Proof. $\left\langle f(\overline{x}), k(\overline{x})^d\right\rangle = \left\langle \sum_{|\alpha|=d} f_\alpha x_0^{\alpha_0}...x_n^{\alpha_n}, (k_0x_0 + ... + k_nx_n)^d\right\rangle = \sum_{|\alpha|=d} f_\alpha k_\alpha \binom{d}{\alpha_0,...,\alpha_n}^{-1}$ where $k_\alpha = k_0^{\alpha_0}\cdots k_n^{\alpha_n}\binom{d}{\alpha_0,...,\alpha_n}$, thus $\left\langle f(\overline{x}), k(\overline{x})^d\right\rangle = \sum_{|\alpha|=d} f_\alpha k_0^{\alpha_0}\cdots k_n^{\alpha_n} = f(k_0,...,k_n)$. □

Notation 4.16.

We will denote: $k_i = (k_{i_0},...,k_{i_n}) \in \mathbb{K}^{n+1}$ the unknowns in the decomposition 4.1.

Corollary 4.17. *It holds that $\tau(k(\overline{x})^d)$=$ev(k) \in S_d^*$.*

Proof.

$$\begin{array}{rccc} \tau(k(\overline{x})): & S_d & \longrightarrow & \mathbb{K} \\ & f(\overline{x}) & \longmapsto & \langle f(\overline{x}), k(\overline{x})^d \rangle = f(k) \end{array}$$

□

Proposition 4.18. *Let $f \in S_d$ and $k_1, ..., k_r \in \mathbb{C}^{n+1}$. Then f can be written as:*

$$f = \sum_{i=1}^r \lambda_i (x_0 k_{i,0} + ... + x_n k_{i,n})^d$$

if and only if

$$f^* = \sum_{i=1}^r \lambda_i ev(k_i)$$

Proof. If f can be written as: $f = \sum_{i=1}^r \lambda_i (x_0 k_{i,0} + ... + x_n k_{i,n})^d$ then:

$$\tau(f) = f^* = \sum_{i=1}^r \lambda_i \tau(x_0 k_{i,0} + ... + x_n k_{i,n}) = \sum_i^r \lambda_i ev(k_i)$$

□

Corollary 4.19. *The problem of decomposition can then be restated as follows: Given $f^* \in S_d^*$, find the minimal number of non-zero vectors $k_1, ..., k_r \in \mathbb{C}^{n+1}$ and non-zero scalars $\lambda_1, ..., \lambda_r \in \mathbb{C} - \{0\}$ such that:*

$$f^* = \sum_{i=1}^r \lambda_i ev(k_i)$$

Definition 4.20. *We say that f^* has an affine decomposition if for every k_i in the decomposition $k_{i,0} \neq 0$*

By a generic change of coordinates, any decomposition of f^* can be transformed into an affine decomposition.

Proposition 4.21. *Let $f \in S_d$ and $k_1, ..., k_r \in \mathbb{C}^{n+1}$ such that $k_{i,0} = 1$ for all i. Then f can be written as:*

$$f = \sum_{i=1}^r \lambda_i (x_0 k_{i,0} + ... + x_n k_{i,n})^d$$

if and only if f^* can be written as:

$$\Lambda_f = \sum_{i=1}^r \lambda_i ev(\underline{k}_i)$$

where $\underline{k}_i = (k_{i,1}, ..., k_{i,n})$.

Proof. By the previous proposition f^* can be written as:

$$f^* = \sum_{i=1}^r \lambda_i ev(k_i)$$

with $k_{i,0} = 1$ for all i,then with the map π defined in 4.2 we get:

$$\pi(f^*) = \sum_{i=1}^r \lambda_i \pi(ev_{k_i}) = \sum_{i=1}^r \lambda_i \Lambda_{ev(k_i)}$$

such that:

$$\begin{array}{rlcl} \Lambda_{ev(k_i)}: & R_d & \longrightarrow & \mathbb{C} \\ & p & \longmapsto & ev(k_i)(p^h) = p^h(1, k_{i,1}, ..., k_{i,n}) = p(k_{i.1}, ..., k_{i,n}) \end{array}$$

Therefore, $\pi(f^*) = \sum_{i=1}^{r} \lambda_i \Lambda_{ev(\underline{k_i})}$

□

Corollary 4.22. *The problem of decomposition can be restated as follows: Let $\Lambda \in R_d^*$ find the minimal number of non-zero vectors k_1,...,$k_r \in \mathbb{K}^n$ and non zero scalars $\lambda_1, ..., \lambda_r \in \mathbb{K}$ such that $\Lambda = \sum_{i=1}^{r} \lambda_i ev(k_i)$*

We will see that the definition of symmetric rank is not vacuous because of the following lemma:

Lemma 4.23. *Let $f \in S_d$. Then there exist $k_1(\overline{x}), ..., k_s(\overline{x}) \in S_1$ linear forms such that:*

$$f = \sum_{i=1}^{s} k_i(\overline{x})^d:$$

with $s < \infty$.

Proof. What the lemma said is that the vector space generated by the d-th powers of linear forms : $\langle k(\overline{x})^d | k \in \mathbb{C}^{n+1} \rangle$ fills the ambient space $S_d := \mathbb{C}[x_0, ..., x_n]_d$, therefore what we actually have to prove, is that the vector space generated by the d-th powers of linear forms $k(\overline{x})$ (for all k $\in \mathbb{C}^{n+1}$) is not included in a hyperplane of S_d. This is indeed true, because otherwise there would exits a non-zero element of S_d, $f(\overline{x}) \neq 0$, which is orthogonal, under the bilinear form $\langle \cdot, \cdot \rangle$, to all $k(\overline{x})^d$ for $k \in \mathbb{C}^{n+1}$. Equivalently, by the lemma 2, there exists a non zero polynomial $f(\overline{x})$ of degree d such that $\langle f, k(\overline{x})^d \rangle = f(k) = 0$ for any $k \in \mathbb{C}^{n+1}$, but this is impossible, since a non-zero polynomial does not vanish identically on $\mathbb{C}^{n+1}$. □

Remark 4.24. *We can deduce $s \leq \binom{n+d}{d}$, but it was shown recently by Reznick [17] that*

$$s \leq \binom{n+d-2}{d-1} \tag{4.3}$$

which is a much tighter bound.

Proof. [Partial proof of Sylvestert́s Theorem]

For $r \leq d$:
We assume that $p(x_1, x_2) = \sum_{i=0}^{d} \binom{d}{i} c_i x_1^i x_2^{d-i}$ can be written as sum of r different forms:

$$p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j (\alpha_j x_1 + \beta_j x_2)^d$$

and we define $q(x_1, x_2) = \prod_{j=1}^{r} (\beta_j x_1 - \alpha_j x_2) = \sum_{l=0}^{r} g_l x_1^l x_2^{r-l}$. Then it is not hard to see that for any monomial $m(x_1, x_2)$ of degree $d - r$ in (x_1, x_2), we have $\langle m(x_1, x_2) q(x_1, x_2), p \rangle = 0$ since:

$$\left\langle m(x_1, x_2) q(x_1, x_2), \sum_{j=1}^{r} \lambda_j (\alpha_j x_1 + \beta_j x_2)^d \right\rangle =$$

$$\lambda_1 \left\langle m(x_1, x_2) q(x_1, x_2), (\alpha_1 x_1 + \beta_1 x_2)^d \right\rangle + \cdots + \lambda_r \left\langle (x_1, x_2) q(x_1, x_2), (\alpha_r x_1 + \beta_r x_2)^d \right\rangle =$$

$$\lambda_1 m(\alpha_1, \beta_1) q(\alpha_1, \beta_1) + ... + \lambda_r m(\alpha_r, \beta_r) q(\alpha_r, \beta_r) = 0$$

The last equality is due to for any $f \in S_d$ $\left\langle f(x_1,x_2),(\alpha_j x_1+\beta_j x_2)^d\right\rangle = f(\alpha_j,\beta_j)$ as the lemma 4.15 said.
Particularly if we take:

$$m_0(x_1,x_2)=x_2^{d-r},\ m_1(x_1,x_2)=x_2^{d-r-1}x_1,...,m_{d-r}(x_1,x_2)=x_1^{d-r}$$

we get respectively the equations:

- $g_0c_0+g_1c_1+...+g_rc_r=0$
- $g_0c_1+g_1c_2+...+g_rc_{r+1}=0$
- $\vdots$
- $g_0c_{d-r}+...........+g_rc_d=0$

Let us prove this for the case $m_0(x_1,x_2)=x_2^{d-r}$, (the other cases are analogous):

$$\left\langle x_2^{d-r}q(x_1,x_2),p(x_1,x_2)\right\rangle =$$

$$\left\langle g_0x_2^d+g_1x_2^{d-1}x_1+...+g_rx_1^rx_2^{d-r},\binom{d}{0}c_0x_2^d+...+\binom{d}{d}c_dx_1^d\right\rangle =$$

$$(g_0\binom{d}{0}c_0)\binom{d}{o}^{-1}+...+(g_r\binom{d}{r}c_r)\binom{d}{r}^{-1}=g_0c_0+...+g_rc_r$$

and this , it is the same as:

$$\begin{bmatrix} c_0 & c_1 & ... & c_r \\ \vdots & \dots & \dots & \vdots \\ c_{d-r} & ... & ... & c_d \end{bmatrix}[\overline{q}]=[\overline{0}]$$

Finally, note that $q(x_1,x_2)=\prod_{j=1}^r(\beta_jx_1-\alpha_jx_2)$ admits r distinct roots because the r linear forms are distinct. $\square$

Chapter 5

Inverse systems and duality

In this chapter we will see the necessary tools to understand and to prove the structure theorem 5.25, which will be used in the final decomposition algorithm. Most of these results can be found in the reference [4]. We recall that $\mathbb{K}$ is a field of characteristic 0.

5.1 Duality and formal series

Definition 5.1. *For all $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ we consider the linear form:*

$$\overline{\delta}^{\alpha} :\to \mathbb{K}$$

such that for all element $\overline{x}^{\beta}$ in the monomial basis $(\overline{x}^{\alpha})_{\alpha \in \mathbb{N}^n}$ is defined as follows:

$$\overline{\delta}^{\alpha}(\overline{x}^{\beta}) = \begin{cases} \alpha! = \alpha_1!...\alpha_n! & \textit{if } \alpha = \beta; \\ 0 & \textit{in other case.} \end{cases}$$

We write also $\overline{\delta}^{\alpha} = \delta_1^{\alpha_1} \cdots \delta_n^{\alpha_n}$ although we point out that this is just a notation.

Proposition 5.2. *Any $\Lambda \in R^*$ can be written in an unique way as:*

$$\Lambda = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\overline{x}^{\alpha}) \tfrac{1}{\alpha!} \overline{\delta}^{\alpha} \in \mathbb{K}[[\delta_1, ..., \delta_n]]$$

Reciprocally, any element of $\mathbb{K}[[\delta_1 \cdots \delta_n]]$ can be interpreted as an element of R^*.

Proof. We recall that $(\overline{d}^{\alpha}(f))_{\alpha \in \mathbb{N}^n}$ denote the coefficients of $f \in \mathbb{K}[x_1, ..., x_n]$ in the basis $(\overline{x}^{\alpha})_{\alpha \in \mathbb{N}^n}$. Then:

$$f(\overline{x}) = \sum_{\alpha \in \mathbb{N}^n} \overline{d}^{\alpha}(f) \overline{x}^{\alpha} .$$

As $char(\mathbb{K}) = 0$, clearly we have:

$$\overline{d}^{\alpha} = \frac{1}{\prod_{i=1}^{n} \alpha_i!} \overline{\delta}^{\alpha} = \frac{1}{(\alpha)!} \overline{\delta}^{\alpha} \tag{5.1}$$

And for all $\Lambda \in R^*$:

$$\Lambda(f) = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\overline{x}^{\alpha}) \overline{d}^{\alpha}(f)$$

Notice that this sum is finite for every $f \in R$. So that, we can write:

$$\Lambda = \sum_{\alpha\in\mathbb{N}^n} \Lambda(\overline{x}^{\alpha})\overline{d}^{\alpha} \in \mathbb{K}[[d_1, ..., d_n]]$$

and thanks to 5.1 we can write also:

$$\Lambda = \sum_{\alpha\in\mathbb{N}^n} \Lambda(\overline{x}^{\alpha})\frac{1}{\alpha!}\overline{\delta}^{\alpha} \in \mathbb{K}[[\delta_1, ..., \delta_n]] \tag{5.2}$$

□

Proposition 5.3. *For any $i \in \{1, ..., n\}$ and any $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$:*

$$x_i * \overline{\delta}^{\alpha} = \alpha_i \overline{\delta}^{\alpha'}$$

where $\alpha' = (\alpha_1, ..., \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, ..., \alpha_n)$

Proof. For any $p \in R$ such that $p = \sum_{\beta\in\mathbb{N}^n} c_\beta \overline{x}^{\beta}$: $x_i * \overline{\delta}^{\alpha}(p) = x_i * \overline{\delta}^{\alpha}(\sum_{\beta\in\mathbb{N}^n} c_\beta \overline{x}^{\beta}) = \overline{\delta}^{\alpha}(\sum_{\beta\in\mathbb{N}^n} c_\beta \overline{x}^{\beta} x_i) = \alpha! c_{\alpha_1,...,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},...,\alpha_n} = \alpha_i \delta^{(\alpha_1,...,\alpha_{i-1},\alpha_i-1,\alpha_{i+1},...,\alpha_n)}(p)$. □

Remark 5.4. *As $\overline{d}^{\alpha} = \frac{1}{\alpha!}\overline{\delta}^{\alpha}$ we have:*

$$x_i * \overline{d}^{\alpha} = d_1^{\alpha_1} ... d_{i-1}^{\alpha_{i-1}} d_i^{\alpha_i - 1} d_{i+1}^{\alpha_{i+1}} ... d_n^{\alpha_n}$$

Roughly speaking, "x_i " and "d_i^{-1}" are the "same",and the operation of R-module becomes on deriving the operator, such that $x_i * \delta^{\alpha} = \partial_i(\delta^{\alpha})$.

Definition 5.5. *For all $\alpha = (\alpha_1, ..., \alpha_n)$, and for all $\xi \in K^n$ we can define the linear form:*

$$\begin{array}{cccc} \overline{\delta}_{\xi}^{\alpha}: & R & \rightarrow & \mathbb{K} \\ & p & \longmapsto & \overline{\delta}_{\xi}^{\alpha}(p) = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}(p)(\xi) \end{array}$$

Remark 5.6. *Note that $\overline{\delta}_0^{\alpha} = \overline{\delta}^{\alpha}$*

Remark 5.7. *In the same way that 5.2 for all linear for $\Lambda \in R^*$, if $char(\mathbb{K}) = 0$:*

$$\Lambda = \sum_{\alpha\in\mathbb{N}} \Lambda((\overline{x} - \xi)^{\alpha})\tfrac{1}{\alpha!}\delta_{\xi}^{\alpha} \in \mathbb{K}[[\overline{\delta}_{\xi}]]$$

where $(\overline{x} - \xi)^{\alpha} = \prod_{i=1}^{n}(x_i - \xi_i)^{\alpha_i}$.

Theorem 5.8. *For all point $\xi \in \mathbb{K}^n$ there exists an isomorphism between $\mathbb{K}[[\delta]]$ and $\mathbb{K}[[\delta_{\xi}]]$.*

Proof. We realize that:

$$ev(\xi) = \sum_{\alpha\in\mathbb{N}^n} \xi^{\alpha}\overline{d}^{\alpha} = \sum_{\alpha\in\mathbb{N}^n} \xi^{\alpha}\tfrac{1}{\alpha!}\overline{\delta}^{\alpha}$$

And we define the homomorphism:

$$\begin{array}{rccc} \phi: & \mathbb{K}[[\overline{\delta}_\xi]] & \to & \mathbb{K}[[\overline{\delta}]] \\ & \overline{\delta}_\xi^\beta & \longmapsto & \sum_{\alpha\in\mathbb{N}} \frac{1}{\alpha!}\xi^\alpha\overline{\delta}^{\alpha+\beta} \end{array}$$

for all $p = \sum_{\alpha\in\mathbb{N}^n} p_\alpha \overline{x}^\alpha \in R$, and for all $\beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$, if we denote:

$$p^{(\beta)} = \partial_{x_1}^{\beta_1}...\partial_{x_n}^{\beta_n}(p) = \sum_{\alpha\in\mathbb{N}^n} p_\alpha^{(\beta)}\overline{x}^\alpha$$

Then we have:

$$\overline{\delta}_\xi^\beta(p) = \sum_{\alpha\in\mathbb{N}^n} p_\alpha^{(\beta)}\xi^\alpha = ev(\xi)(p^{(\beta)}) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\xi^\alpha\overline{\delta}^\alpha(\sum_{\alpha\in\mathbb{N}^n} p_\alpha^{(\beta)}\overline{x}^\alpha) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\xi^\alpha\overline{\delta}^{\alpha+\beta}(p).$$

Hence, ϕ is a bijection by the remark 5.7 and the proposition 5.2. □

5.2 Inverse systems

Definition 5.9. *Let $\mathbb{L}$ be the map defined as follows:*

$$\{I \subset R \ s.t\ I \text{ is an ideal }\} \xrightarrow{\mathbb{L}} \{D \subset R^* \ s.t\ D \text{ is } R\text{-submodule}\}$$

and for any I ideal of R, $\mathbb{L}(I) := \{\lambda \in R\text{: } \lambda(f) = 0\ \forall\ f \in I\}$.

Proposition 5.10. *The map $\mathbb{L}$ is well defined.*

Proof. Clearly $\mathbb{L}(I) \subset R^*$ thus, in order to see $\mathbb{L}$ is well defined, we have to see that the map:

$$\begin{array}{rl} * : R \times \mathbb{L}(I) \to \mathbb{L}(I) \\ (p, \Lambda) \longmapsto p * \Lambda \end{array}$$

is well defined in $\mathbb{L}(I)$ like R-submodule: for any $p \in R$ and any $\Lambda \in \mathbb{L}(I)$, $p * \Lambda \in \mathbb{L}(I)$ since: for all $f \in I$ $pf \in I$ and $p * \Lambda(f) = \Lambda(pf) = 0$ □

Definition 5.11. *Let $\mathbb{B}$ be the map defined as follows:*

$$\{D \subset R^* \ s.t\ D \text{ is } R\text{-submodule}\} \xrightarrow{\mathbb{B}} \{I \subset R \ s.t\ I \text{ is an ideal }\}$$

and for any $L \subset R^*$ and R-submodule, $\mathbb{B}(L) := \{f \in R : \lambda(f) = 0\ \forall\ \lambda \in L\}$

Proposition 5.12. *The map $\mathbb{B}$ is well defined.*

Proof. Let L be R^*-submodule, then $\mathbb{B}(L) \subset R$ is an ideal of R since: let $p_1, p_2 \in \mathbb{B}(L)$, and for all $\lambda \in L$ $\lambda(p_1 + p_2) = \lambda(p_1) + \lambda(p_2) = 0$, the first equality due to λ is linear and the second one due to $p_1, p_2 \in \mathbb{B}(L)$. If $g \in \mathbb{B}(L)$ and $p \in R$, then for all $\lambda \in L$, $p * \lambda \in L$ and $p * \lambda(g) = \lambda(pg) = 0$. □

Proposition 5.13. *Let I be an ideal of R and L a R-submodule:*

- *i)*$I = \mathbb{B}(\mathbb{L}(I))$
- *ii)*$\mathbb{L}(\mathbb{B}(L)) \supset L$

Proof. i)Let us see that $I \subset \mathbb{B}(\mathbb{L}(I))$, and let $f \in I$: $f \in \mathbb{B}(\mathbb{L}(I))$ iff $\lambda(f) = 0$ for all $\lambda \in \mathbb{L}(I)$ iff $\lambda(f) = 0\ \forall\ \lambda$ such that $\lambda(g) = 0$ for all $g \in I$. In particular $f \in I$ then $\lambda(f) = 0$ for all $\lambda \in \mathbb{L}(I)$. On the other hand $\mathbb{B}(\mathbb{L}(I)) \subset I$: let us see that if $f \notin \mathbb{B}(\mathbb{L}(I))$ then $f \notin I$. If $f \notin \mathbb{B}(\mathbb{L}(I))$ then there exists $\lambda \in \mathbb{L}(I)$ such that $\lambda(f) \neq 0$, but $\lambda(g) = 0$ for all $g \in I$, thus $f \notin I$.
ii) Let us see that $\mathbb{L}(\mathbb{B}(L)) \supsetneq L$ and let $\tau \in L$: $\tau \in L(\mathbb{B}(L))$ iff $\tau(f) = 0$ for all $f \in \mathbb{B}(L)$ iff $\tau(f) = 0$ for all $f \in$ such that $\Lambda(f) = 0$ for all $\Lambda \in L$. In particular $\tau \in L$, then $\tau(f) = 0$ for all $f \in \mathbb{B}(L)$. □

Example 5.14. *Let us see that $\mathbb{L}(\mathbb{B}(L)) = L$ is not true for all L R-module, i.e. $\mathbb{L}$ is not surjective:*

If we take:

$$L := \{\lambda \in R^* : \exists \eta \in \mathbb{N}^n : \lambda(\overline{x}^\alpha) = 0\ \forall \alpha \geq \eta\}$$

where $\alpha \geq \eta$ in the sense of some monomial order. Then, $\mathbb{B}(L) = \{0\}$, thus $\mathbb{L}(\mathbb{B}(L)) = R^*$.

Proposition 5.15. *If we restrict $\mathbb{L}$ from zero-dimensional ideals to L R-submodules such that $dim_{\mathbb{K}}(L) < \infty$ we get a bijection.*

Proof. If we denote $\mathbb{L}'$ the restriction of $\mathbb{L}$ to zero-dimensional ideals, and $\mathbb{B}'$ the restriction of $\mathbb{B}$ to R-submodules with $dim_{\mathbb{K}} L < \infty$:

$$\begin{array}{ccc} \{I \subset R, ideal\ s.t.\ Z(I) < \infty\} & \xrightarrow{\mathbb{L}'} & \{L \subset R^*\ R - submodule, s.t.\ dim_{\mathbb{K}} < \infty\} \\ \{L \subset R^*\ R - submodule, s.t.\ dim_{\mathbb{K}} < \infty\} & \xrightarrow{\mathbb{B}'} & \{I \subset R, ideal\ s.t. Z(I) < \infty\} \end{array}$$

For the previous proposition we know that $I = \mathbb{B}(\mathbb{L}(I))$ for every ideal $I \subset R$. In particular for all I zero-dimensional ideal we have $I = \mathbb{B}'(\mathbb{L}'(I))$. Then, we only have to prove that $\mathbb{L}'$ it is surjective:
Let $L \subset R^*$ such that $dim_{\mathbb{K}} L = \mu < \infty$, then we define:

$$I := \{f(\overline{x}) \in R : \lambda(f) = 0\ \forall \lambda \in L\}$$

I is zero dimensional if and only if for all $i \in \{1, ..., n\}$ $\mathbb{K}[x_i] \cap I \neq \{0\}$. If we fix $i \in \{1, ..., n\}$, then for all $\lambda_j \in L$ for $j = 1, ..., \mu$, $\{\lambda_j, x_i * \lambda_j, x_i^2 * \lambda_j, ..., x_i^\mu * \lambda_j\} \subset L$ because L is R-module and is a set linearly dependent.Then for all $j \in \{1, ..., \mu\}$ there exists η_j such that:

$$x_i^{\eta_j} * \lambda_j = a_0^j \lambda_j + a_1^j x_i \lambda_j + a_2^j x_i^2 ... + a_\mu^j x_i^\mu * \lambda_j$$

If we take,

$$f_j(x_i) = x_i^{\eta_j} - a_o^j - a_1^j x_i - a_2^j x_i^2 - ... - a_\mu^j x_i^\mu$$

Then we get $f_j(x_i) * \lambda_j = 0$ for all $j = 1, ..., \mu$, and if we define $g(x_i) := m.c.m(f_1(x_i), ..., f_\mu(x_i))$ then we obtain $\lambda_j(g(x_i)) = 0$ for all $j = 1, ..., \mu$. Therefore $\mathbb{K}[x_i] \cap I \neq \{0\}$. □

These results motivate the following definition:

Definition 5.16. *Let I be an ideal of R, then the orthogonal of I, is the following vector-subspace:*

$$I^\perp := \{\Lambda \in R^*; \forall p \in I, \Lambda(p) = 0\}$$

And for all vector-subspace D of R^*, then the orthogonal of D is the following vector-subspace:

$$D^{\perp} := \{p \in R, \forall \Lambda \in D, \Lambda(p) = 0\}$$

Proposition 5.17. *Let I be an ideal of R, then $I^{\perp}$ is isomorphic to $A^* = (R/I)^*$.*

Proof. Let π be the projection of R on $A = R/I$. The map:

$$\begin{aligned} \pi_* : A^* &\to I^{\perp} \\ \Lambda &\longmapsto \Lambda \circ \pi \end{aligned}$$

is an isomorphism of $\mathbb{K}$-vector spaces:
Clearly, it is well defined, moreover, if $\Gamma_1, \Gamma_2 \in A^*$ and $\alpha_1, \alpha_2 \in \mathbb{K}$: $\pi_*(\alpha_1\Gamma_1 + \alpha_2\Gamma_2) = (\alpha_1\Gamma_1 + \alpha_2\Gamma_2) \circ \pi = (\alpha_1\Gamma_1) \circ \pi + (\alpha_2\Gamma_2) \circ \pi = \alpha_1(\Gamma_1 \circ \pi) + \alpha_2(\Gamma_2 \circ \pi) = \alpha_1\pi_*(\Gamma_1) + \alpha_2\pi_*(\Gamma_2)$.Therefore, π_* is an isomorphism of $\mathbb{K}$-vector spaces. Clearly it is injective. Also, π_* is surjective, since: let $\Gamma \in I^{\perp}$ then $\Gamma(p) = 0$ for all $p \in I$ and $\Gamma \in R^*$, then if we restrict Γ to A^*, and we denote it Γ', then $\pi_*(\Gamma') = \Gamma$. □

Definition 5.18. *The vector-space L of R^* is stable if for all $\Lambda \in L$:*

$$x_i * \Lambda \in \langle L \rangle \text{ for } i = 1, ..., n$$

This definition allows us to obtain the following lemma:

Lemma 5.19. *$D = \langle \Lambda_1, ..., \Lambda_s \rangle$ is stable iff $D^{\perp}$ is an ideal.*

Proof. If we assume D stable then for all $p \in D^{\perp}$ and for all $i = 1, ..., n$,$j = 1, ..., s$

$$\Lambda_j(x_i p) = x_i * \Lambda_j(p) = \textstyle\sum_{k=1}^{s} \lambda_{ijk}\Lambda_k(p) = 0$$

($\lambda_{ijk} \in \mathbb{K}$) then $x_i p \in D^{\perp}$ for $i = 1, ..., n$ then $D^{\perp}$ is an ideal.
If we assume $D^{\perp}$ as an ideal then for all $p \in D^{\perp}$ and $i = 1, ..., n$ $x_i p \in D^{\perp}$ thus for all $j = 1, ..., s$ $\Lambda_j(x_i p) = x_j\Lambda_j(p) = 0$. Therefore, $x_i * \Lambda_j \in D^{\perp\perp} = D$. The last equality it holds true because D is a $\mathbb{K}$-vector space with finite dimension. □

5.3 Inverse system of a single point

we are in the case where the ideal $I \subset R$ defines a single point, $0 \in \mathbb{K}^n$. And we denote m_0 to the maximal ideal defining 0. We will compute the local structure of I at 0.

Proposition 5.20. *If I is m_0-primary then $I^{\perp} \subset \mathbb{K}[\overline{\delta}]$:*

Proof. There exists $N \in \mathbb{N}$ such that $m_0^N \subset I \subset m_0$, and then $\overline{x}^{\alpha} \in I$ with $|\alpha| = \alpha_1 + ... + \alpha_n \geq N$. Thanks to 5.2 for all $\Lambda \in I^{\perp}$ can be written:

$$\Lambda = \textstyle\sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}\Lambda(\overline{x}^{\alpha})\overline{\delta}^{\alpha}$$

but $\Lambda(\overline{x}^{\alpha}) = 0$ for $|\alpha| \geq N$, therefore:

$$\Lambda = \textstyle\sum_{\alpha \in \mathbb{N}^n; |\alpha| < N} \frac{1}{\alpha!}\Lambda(\overline{x}^{\alpha})\overline{\delta}^{\alpha} \in \mathbb{K}[\overline{\delta}]$$

□

Corollary 5.21. *If I is m_ξ-primary then $I^{\perp} \subset \mathbb{K}[\overline{\delta_\xi}]$*

Proof. It follows from the bijection between $\mathbb{K}[[\overline{\delta_\xi}]]$ and $\mathbb{K}[[\overline{\delta}]]$ □

Remark 5.22. *If I is a m_ξ-primary ideal and $dim_{\mathbb{K}}(R/I) = \mu$, where μ is the multiplicity of the root, thus $I^\perp$ is a vector space with dimension equal to μ.*

It is difficult to work directly with a m_0-primary ideal. The following result is for ideals having one m_0-primary component.

Theorem 5.23. *Let I be a zero-dimensional ideal of R and Q_0 its m_0-primary component then:*

$$(I^\perp \cap \mathbb{K}[\overline{\delta}])^\perp = Q_0$$

Proof. We denote $D_0 = I^\perp \cap \mathbb{K}[\overline{\delta}]$ and we will prove $D_0 = Q_0^\perp$.

As $I \subset Q_0$ then $Q_0^\perp \subset I^\perp$ since: for all $\Lambda \in Q_0^\perp$, $\Lambda(f) = 0$ for all $f \in Q_0$ and in particular for all $f \in I$, then $\Lambda \in I^\perp$. On the other hand $Q_0 \subset \mathbb{K}[\overline{\delta}]$ by the previous proposition, then $Q_0^\perp \subset I^\perp \cap \mathbb{K}[\overline{\delta}] = D_0$.

Now, let us see the other inclusion $D_0 \subset Q_0^\perp$. To prove this we have to take into account two properties:

1. $Q_0 = \{f \in R : \exists\, g \in R \text{ with } fg \in I \text{ and } g(0) \neq 0\}$
2. For all $\Lambda \in \mathbb{K}[\overline{\delta}]$ and for all $g \in R$, $(g * \Lambda)(f) = g(\partial_1, ..., \partial_n)(\Lambda)(f) = g(0)\Lambda(f) + (g - g(0))(\partial_1, ..., \partial_n)(\Lambda)(f)$.

The first property means that $Q_0 = I^{ec}$. And for the second property recall that proposition 5.3 states $x_i * \overline{\delta}^\alpha = \partial_i(\overline{\delta}^\alpha)$.
Let $\Lambda \in D_0$ we will argue by induction on the degree of Λ: If Λ has degree 0, then Λ is a scalar, exactly $\Lambda = \langle ev(0)\rangle$ .For all $f \in Q_0$, there exists $g \in R$ with $fg \in I$ and $g(0) \neq 0$ then $\Lambda(fg) = 0 = ev(0)(fg) = f(0)g(0)$ then $0 = f(0) = ev(0)(f) = \Lambda(f)$ and $\Lambda \in Q_0^\perp$. Now, we assume it is true for degree less than d. Let $\Lambda \in D_0$ of degree d and $f \in Q_0$ then there exists $g \in R$ such that $g(0) \neq 0$ and $fg \in I$: $\Lambda(fg) = 0 = g(0)\Lambda(f) + (g - g(0))(\partial_1, ..., \partial_n)(\Lambda)(f)$, but $\Lambda' := g - g(0)(\partial_1, ..., \partial_n)(\Lambda)$ is either zero if $g = g(0)$ or it has smaller degree than Λ then $\Lambda(f) = 0$ and $\lambda \in Q_0^\perp$. Then $D_0 = Q_0^\perp$ due to Q_0 is a zero-dimensional ideal, $D_0^\perp = Q_0^{\perp\perp} = Q_0 = (I^\perp \cap \mathbb{K}[\delta])^\perp$. ☐

Corollary 5.24. *Let I be a zero-dimensional ideal of R and Q_ξ its m_ξ-primary component then:*

$$(I^\perp \cap \mathbb{K}[\overline{\delta_\xi}])^\perp = Q_\xi$$

Proof. It follows from the bijection between $\mathbb{K}[[\overline{\delta_\xi}]]$ and $\mathbb{K}[[\overline{\delta}]]$ ☐

Theorem 5.25. *(Structure theorem).Let I be an ideal such that $Z(I) = \{\xi_1, ..., \xi_d\}$ then:*

$$I^\perp = Q_{\xi_1}^\perp \oplus ... \oplus Q_{\xi_d}^\perp$$

where Q_{ξ_i} is the m_{ξ_i}-primary component. Moreover, for all $\Lambda \in I^\perp$ there exists $p_i(\partial_1, ..., \partial_n)$ for $i = 1, ..., d$ such that Λ can be written as:

$$\Lambda = \sum_{i=1}^{s} ev(\xi_i) \circ p_i(\overline{\partial}) \tag{5.3}$$

Proof. As $I = Q_{\xi_1} \cap ... \cap Q_{\xi_d}$ then thanks to the properties of the operator $\perp$, $I^{\perp} = Q_{\xi_1}^{\perp} \cap ... \cap Q_{\xi_d}^{\perp} = Q_{\xi_1}^{\perp} + ... + Q_{\xi_d}^{\perp}$ .Moreover, for $i_1, ..., i_p \in \{1, ..., d\}$ and $i \neq i_1, ..., i_p$, $Q_i + (Q_{i_1} \cap ... \cap Q_{i_p}) = R$ then: $Q_i^{\perp} \cap (Q_{i_1}^{\perp} + ... + Q_{i_p}^{\perp}) = R^{\perp} = \{0\}$, therefore we have a direct sum:

$$I^{\perp} = Q_{\xi_1}^{\perp} \oplus ... \oplus Q_{\xi_d}^{\perp}$$

and by the corollary 5.24:

$$I^{\perp} = Q_{\xi_1}^{\perp} \oplus ... \oplus Q_{\xi_d}^{\perp} = (I^{\perp} \cap \mathbb{K}[\overline{\delta}_{\xi_1}]) \oplus ... \oplus (I^{\perp} \cap \mathbb{K}[\overline{\delta}_{\xi_d}])$$

then for all $\Lambda \in I^{\perp}$:

$$\Lambda = ev(\xi_1) \circ p_1(\overline{\partial}) \oplus ... \oplus ev(\xi_d) \circ p_d(\overline{\partial})$$

$\square$

Chapter 6

Gorenstein Algebras

This chapter is a brief look at some properties of the Gorenstein Algebras. All the results of this chapter are taken from [4].

Lemma 6.1. *If I_1,I_2,I are ideals of A, which is a commutative and unitary ring, then:*

- *i)* $(I : I_1) \cap (I : I_2) = (I : I_1 + I_2)$
- *ii) If* $I_1 + I_2 = A$, *then* $(I : I_1) + (I : I_2) = (I : I_1 \cap I_2)$

Proof. i) Let $x \in (I : I_1) \cap (I : I_2)$ then, $xI_1 \subseteq I$ and $xI_2 \subseteq I$, then: $x(I_1 + I_2) \subseteq I$, therefore $x \in (I : I_1 + I_2)$. Reciprocally, let $x \in (I : I_1 + I_2)$, then $xI_1 + xI_2 \subseteq I$, in particular $0 \in I_1$ and $0 \in I_2$, then $xI_1 \subseteq I$ and $xI_2 \subseteq I$, therefore $x \in (I : I_1) \cap (I : I_2)$.
ii) Let us prove that $(I : I_1 \cap I_2) \subset (I : I_1) + (I : I_2)$: as $I_1 + I_2 = A$, there exists $q_1 \in I_1$ and $q_2 \in I_2$ such that $1 = q_1 + q_2$. If $x \in (I : I_1 \cap I_2)$, then $x(I_1 \cap I_2) \subseteq I$ as $I_1 I_2 \subset I_1 \cap I_2$ then $x(I_1 I_2) \subseteq I$, then $xq_1 I_2 \subset I$, $xq_2 I_1 \subset I$ and as $x = xq_1 + xq_2$ then $x \in (I : I_1) + (I : I_2)$. The other inclusion is immediate. $\square$

Theorem 6.2. *If $A = R/I$ where I is a zero-dimensional ideal, with the following primary decomposition $I = Q_1 \cap ... \cap Q_d$. Then A is a direct sum of sub-algebras $A_1, ..., A_d$* [1]*:*

$$A = A_1 \oplus ... \oplus A_d$$

where $A_i := (\overline{0} : Q_i/I) = \{a \in A : qa \equiv 0 \text{ for all } q \in Q_i/I\}$

Proof. For all $i \in \{1, ..., d\}$ and $D \subset \{1, ..., d\} - \{i\}$, $Q_i + \cap_{j \in L} Q_j = \mathbb{K}[\overline{x}]$. Thus, due to lemma 6.1, we have:

$$A_1 + ... + A_d =$$

$$(\overline{0} : Q_1) + ... + (\overline{0} : Q_d) = (\overline{0} : Q_1 \cap ... \cap Q_d/I) = (\overline{0} : \overline{0}) = A$$

In order to prove, that the sum is direct, since: let $i \in \{1, ..., d-1\}$:

$$(A_1 + ... + A_d) \cap A_{i+1} = ((\overline{0} : Q_1/I) + ... + (\overline{0} : Q_d/I)) \cap (\overline{0} : Q_{i+1}) = (\overline{0} : ((Q_1 \cap ... \cap Q_i) + Q_{i+1})/I) = (\overline{0} : R/I) = 0.$$

Definition 6.3. *Let $A = R/I$ where I is a zero-dimensional ideal, then there exists a unique $(e_1, ..., e_d) \in A_1 \oplus ... \oplus A_d = A = R/I$ such that:*

[1] Strictly speaking A_i are ideals of the ring A whose identity element is $1 = e_1 + \cdots + e_d$. But, A_i can be seen as sub-algebras whose identity element is Ae_i.

$$1 = e_1 + ... + e_d$$

e_i for $i \in \{1, ..., d\}$ are the idempotents elements of the algebra A.

Remark 6.4. *$e_i^2 = e_i$ and $e_i e_j \equiv 0$ for $i \neq j$ since:*

$$1 = e_1 + ... + e_d = 1^2 = e_1^2 + ... e_d^2 + 2 \sum_{1 \leq i < j \leq d} e_i e_j$$

and $A_i \cap A_j = 0$ for $i \neq j$.

Proposition 6.5. *Let $A = R/I$ with I an ideal zero-dimensional such that $A = A_1 \oplus ... \oplus A_d$. Then $A_i = Ae_i$ for all $i \in \{1, ..., d\}$.*

Proof. Let $a \in A_i$, and $1 \equiv e_1 + ... + e_d$, then as $A_i \cap A_j = 0$ if $i \neq j$,

$$a \equiv ae_1 + ... + ae_d \equiv ae_i \in Ae_i$$

Reciprocally, if $ae_i \in Ae_i$, then $a \in A$ and $e_i \in A_i$, in particular A_i is and ideal of A, thus $ae_i \in A_i$ □

Definition 6.6. *Let A be an algebra such that $dim_{\mathbb{K}} A < \infty$, then A is a Gorenstein Algebra if A^* is a free module of rank 1.*

Proposition 6.7. *If $A = R/I$ is a Gorenstein algebra then the local subalgebras A_i, $i = 1, ..., d$ are Gorenstein algebras.*

Proof. If we assume A is a Gorenstein algebra then there exists Λ such that: $A^* = \Lambda * A$ and we can define:

$$\begin{array}{cccc} \Lambda_i : & A_i & \rightarrow & \mathbb{K} \\ & ye_i & \longmapsto & \Lambda(ye_i) \end{array}$$

Then we have $\Lambda_i * A_i = A_i^*$, since: for any $\phi_i \in A_i^*$, we define $\phi \in A^*$ as follows:

$$\begin{array}{cccc} \phi : & A \rightarrow & \mathbb{K} \\ & x & \longmapsto & \phi_i(xe_i) \end{array}$$

As A is a Gorenstein algebra, then there exists $a \in A$ with $\phi = a * \Lambda$. And then, we have $\phi_i = ae_i * \Lambda_i$, since: for any $z \in A_i$, there exists $y \in A$ such that $z = ye_i$, then:

$$(ae_i * \Lambda_i)(ye_i) = \Lambda_i(ye_i ae_i) = \Lambda_i(ye_i a) = \Lambda(ye_i a) = a * \Lambda(ye_i) = \phi(ye_i) = \phi_i(ye_i e_i) = \phi_i(ye_i).$$

□

Definition 6.8. *The linear form Λ such that $\Lambda * A = A^*$ is the residue of A.*

Remark 6.9. *If A is a Gorenstein algebra and Λ is a residue of A then $\Lambda_i = e_i * \Lambda$ is a residue of the sub-algebra A_i.*

Chapter 7

Hankel operators and quotient algebra

In this chapter, we recall the Hankel Operators, the quotient algebra and its necessary properties, to describe and analyze the final algorithm. We refer to [1] for the results in this chapter.

Definition 7.1. *For any $\Lambda \in R^*$ we define the bilinear form Q_Λ, such that:*

$$\begin{aligned} Q_\Lambda : R &\longrightarrow \mathbb{K} \\ (a,b) &\longmapsto \Lambda(a,b) \end{aligned}$$

The matrix of Q_Λ, in the monomial basis of R, is $\mathbb{Q}_\Lambda = (\Lambda(x^{\alpha+\beta}))_{\alpha,\beta}\alpha, \beta \in \mathbb{N}^n$.

Definition 7.2. *For any $\Lambda \in R^*$, we define the Hankel operator H_Λ from R to R^* as*

$$\begin{aligned} H_\Lambda : R &\longrightarrow R^* \\ p &\longmapsto p * \Lambda \end{aligned}$$

The matrix of H_Λ, in the monomial basis and in the dual basis, $\overline{d}^\alpha$, is $\mathbb{H}_\Lambda = (\Lambda(x^{\alpha+\beta}))_{\alpha,\beta}\alpha, \beta \in \mathbb{N}^n$.

In what follows we identify H_Λ and Q_Λ, since, for all a,$b \in R$, due to the R-module structure, it holds:

$$Q_\Lambda(a,b) = \Lambda(ab) = (a * \Lambda)(b) = (b * \Lambda)(a) = H_\Lambda(a)(b) = H_\Lambda(b)(a).$$

Definition 7.3. *Given $B = \{b_1, .., b_r\}$, $B' = \{b_1, .., b_r\} \subset R$ we define:*

$$H_\Lambda^{B,B'} : \langle B \rangle \longrightarrow \left\langle B' \right\rangle^*$$

This operator applies each element $b_i \in \langle B \rangle$ to the form $b_i * \Lambda \in R^*$ and then, thanks to $\left\langle B' \right\rangle^* \subset R^*$, we can restrict $b_i * \Lambda$ to $\left\langle B' \right\rangle$. Let $\mathbb{H}_\Lambda^{B,B'} = (\Lambda(b_i b_j'))1 \leq i \leq r, 1 \leq j \leq r'$. If $B' = B$, we use the notation H_Λ^B and $\mathbb{H}_\Lambda^{B',B}$.

Proposition 7.4. *Let I_Λ be the kernel of H_Λ. Then, I_Λ is an ideal of R*

Proof. From the definition of the Hankel operators, we can deduce that a polynomial $p \in R$ belongs to the kernel of $\mathbb{H}_\Lambda$ if and only if $p * \Lambda = 0$, which in turn holds if and only if for all $q \in R$, $\Lambda(p,q) = 0$.
Let p_1,$p_2 \in I_\Lambda$. Then for all $q \in R$, $\Lambda((p_1 + p_2)q) = \Lambda(p_1 q) + \Lambda(p_2 q) = 0$. Thus, $p_1 + p_2 \in I_\Lambda$. If $p \in I_\Lambda$ and $p' \in R$, then $\Lambda(pp'q) = 0$ holds for all $q \in R$. Thus, $pp' \in I_\Lambda$ and I_Λ is an ideal. $\square$

Let $A_\Lambda = R/I_\Lambda$ be the quotient algebra of polynomials modulo the ideal I_Λ, which, as Proposition 7.4 states, is the kernel of H_Λ. The rank of H_Λ is the dimension of A_Λ as a $\mathbb{K}$-vector space.

Proposition 7.5. *If $rank H_\Lambda = r < \infty$,$A_\Lambda = R/I_\Lambda$ is a Gorenstein algebra.*

Proof. In order to see this, let us see that the dual space A_Λ^*, can be identified with the set $D = \{ \ q * \Lambda \ s.t. \ q \in R \ \}$:

By definition $D^\perp = \{ \ p \in R \text{ s.t. } \forall \ q \in R, \ q * \Lambda(p) = \Lambda(pq) = 0 \ \}$. Therefore, $D^\perp = I_\Lambda$, which is the ideal of the kernel of H_Λ. Since $A_\Lambda^* \cong I_\Lambda^\perp$ by 5.17, A_Λ is the set of the linear forms in R^* which vanish on I_Λ, we deduce that $A^* = I_\Lambda^\perp = D^{\perp\perp} = D$. The last equality is true because D is a submodule of R, which has finite dimension equal to r like $\mathbb{K}$-vector space, since $rank H_\Lambda = r < \infty$.
Moreover if $p * \Lambda = 0$ then $p \equiv 0$ in A_Λ. Hence, A_Λ^* is a free rank 1 A_Λ-module (generated by Λ). Thus A_Λ is a Gorenstein algebra. □

Definition 7.6. *For any $B \subset R$ let $B^+ = B \cup x_1 B \cup \cdots \ x_n B$ and $\partial B = B^+ - B$.*

Proposition 7.7. *Assume that $rank(H_\Lambda) = r < \infty$ and let $B = \{b_1, ..., b_r\} \subset R$ such that $\mathbb{H}_\Lambda^B$ is invertible. Then $\{b_1, \ldots, b_r\}$ is a basis of A_Λ. If $1 \in \langle B \rangle$ then the ideal I_Λ is generated by $Ker H_\Lambda^{B^+}$.*

Proof. First we are going to prove that $\langle b_1, ..., b_r \rangle \cap I_\Lambda = \{0\}$. Let $p \in \langle \ b_1, ..., b_r \rangle \cap I_\Lambda$. Then $p = \sum_i p_i b_i$ with $p_i \in \mathbb{K}$ and $\Lambda(p b_j) = 0$. The second equation implies that $\mathbb{H}_\Lambda^B \cdot \overline{p} = \overline{0}$, where $\overline{p} = [p_1, ..., p_r]^t \in \mathbb{K}^r$. Since $\mathbb{H}_\Lambda^B$ is invertible, this implies that $\overline{p} = 0$ and $p = 0$.
Then we deduce that $b_1 * \Lambda, ..., b_r * \Lambda$ is a set linearly independent since otherwise there exists $[\mu_1, ..., \mu_r] \neq 0$ such that $\mu_i(b_1 * \Lambda_1) + ... + \mu_r(b_r * \Lambda_r) = (\mu_1 b_1 + ... + \mu_r(b_r)) * \Lambda = 0$ but this is not possible because $\langle b_1, ..., b_r \rangle \cap I_\Lambda = \{0\}$ and we have a contradiction. Hence, since $rank(H_\Lambda) = r$, $\{b_1 * \Lambda, ..., b_r * \Lambda\}$ span the image of H_Λ. For any, $p \in R$, it holds that $p * \Lambda = \sum_{i=1}^r \mu_i (b_i * \Lambda)$ for some $\mu_1, ..., \mu_r \in \mathbb{K}$. We deduce that $p - \sum_{i=1}^r \mu_i b_i \in I_\Lambda$. This yields the decomposition $R = B \oplus I_\Lambda$, and shows that $b_1, ..., b_r$ is a basis of A_Λ. □

Example 7.8.

Let $\tau = \delta^{\alpha_4} + \delta^{\alpha_5} + \delta^{\alpha_6} \in \mathbb{K}[x_1, x_2, x_3]^*$ where $\alpha_1 = (1,0,0)$,$\alpha_2 = (0,1,0)$,$\alpha_3 = (0,0,1)$,$\alpha_4 = (2,0,0)$,$\alpha_5 = (0,2,0)$,$\alpha_6 = (0,0,2)$, and $\alpha_0 = (0,0,0)$. We are going to compute the infinite matrix of $\mathbb{H}_\tau$, from the basis $(\overline{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ to the basis $(\delta^\alpha)_{\alpha \in \mathbb{N}^n}$. In order to do this, we realize that:

$$
\begin{aligned}
x_1 * \tau = 2\delta^{\alpha_1}, \ x_1^2 * \tau = 2\delta^0 \\
x_2 * \tau = 2\delta^{\alpha_2}, \ x_2^2 * \tau = 2\delta^0 \\
x_3 * \tau = 2\delta^{\alpha_3}, \ x_3^3 * \tau = 2\delta^0
\end{aligned}
$$

and for any monomial m, non-constant different from $\{x_1, x_2, x_3, x_1^2, x_2^2, x_3^2\}$, we have $m * \tau \equiv 0$, therefore the matrix $\mathbb{H}_\tau$ has a finite number of non-zero entries:

$$\left(\begin{array}{c|cccccccc} & 1 & x_1 & x_2 & x_3 & x_1^2 & x_2^2 & x_3^2 & \xrightarrow{\infty} \\ \hline 1 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 0 \\ \delta^{\alpha_1} & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta^{\alpha_2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \delta^{\alpha_3} & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \delta^{\alpha_4} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta^{\alpha_5} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \delta^{\alpha_6} & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right. \tag{7.1}$$

Clearly $rank(\mathbb{H}_\tau) = 5$, and the set $B = \langle 1, x_1, x_2, x_3, x_1^2 \rangle$ makes $\mathbb{H}_\tau^B$ invertible, then by the previous Proposition 7.7, B is a basis of A_τ, and by the Proposition 7.5, A_τ is a Gorenstein Algebra. Moreover, $1 \in B$, then the ideal I_τ is generated by $\left\langle Ker(H_\tau^{B^+}) \right\rangle$. By computing this kernel, we get; $f \in I_\tau$ if and only if f can be written as $f = a(x_1^2 - x_2^2) + b(x_1^2 - x_3^2) + c(x_1x_2) + d(x_1x_3) + e(x_2x_3) +$ terms of degree greater or equal to 3 where a, b, c, d, e are constant. Therefore $I_\tau = (x_1^2 - x_2^2, x_1^2 - x_3^2, x_1x_2, x_1x_3, x_2x_3)$.

The procedure followed by the example gives us a way to build Gorenstein Algebras: given a polynomial $p_i \in \mathbb{K}[\partial_1, ..., \partial_n]$, compute the ideal $I \in \mathbb{K}[x_1, ..., x_n]$ orthogonal to p_i and the quotient algebra $\mathbb{K}[x_1, ..., x_n]/I$ is a Gorenstein Algebra.

In order to compute the zeros of an ideal I_Λ when we know a basis of A_Λ, we exploit the properties of the operators of multiplication in A_Λ.

Definition 7.9. *Let $\Lambda \in R^*$ and $a \in A_\Lambda$, with $dim_{\mathbb{K}}(A_\Lambda) = r < \infty$, and let $(\overline{x}^\alpha)_{\alpha \in E}$, the monomial basis of A_Λ. The operator of multiplication in A_Λ is:*

$$\begin{aligned} M_a \colon A_\Lambda &\longrightarrow A_\Lambda \\ b &\longmapsto M_a(b) = ab \end{aligned}$$

The matrix of M_a, in the basis $(\overline{x}^\alpha)_{\alpha \in E}$ will be denoted $\mathbb{M}_a$.

Proposition 7.10. *The transposed endomorphism of M_a is:*

$$\begin{aligned} M_a^t \colon A_\Lambda^* &\longrightarrow A_\Lambda^* \\ \Lambda &\longmapsto M_a(\Lambda) = a * \Lambda = \Lambda \circ M_a \end{aligned}$$

The matrix of M_a^t in the basis $(\overline{d}^\alpha)_{\alpha \in E}$ is the transpose of $\mathbb{M}_a$. Therefore, the operators M_a^t and M_a have the same eigenvalues.

Proof. for any $\overline{x}^{\alpha_i} \in (\overline{x}^\alpha)_{\alpha \in E}$, then $a\overline{x}^{\alpha_i}$ can be written as:

$$a\overline{x}^{\alpha_i} = \sum_{\alpha \in E} \mu_{\alpha_i \alpha} \overline{x}^\alpha \tag{7.2}$$

the matrix $\mathbb{M}_a$ is:

$$\mathbb{M}_a = \begin{bmatrix} \mu_{\alpha_1\alpha_1} & \mu_{\alpha_2\alpha_1} & \cdots & \mu_{\alpha_r\alpha_1} \\ \mu_{\alpha_1\alpha_2} & \mu_{\alpha_2\alpha_2} & \cdots & \mu_{\alpha_r\alpha_2} \\ \vdots & \vdots & \vdots & \\ \mu_{\alpha_1\alpha_r} & \mu_{\alpha_2\alpha_r} & \cdots & \mu_{\alpha_r\alpha_r} \end{bmatrix}$$

Therefore, for any element $\overline{d}^{\alpha_i} \in (\overline{d}^{\alpha})_{\alpha\in E}$ (the dual basis of $(x^\alpha)_{\alpha\in E}$) $a * \overline{d}^{\alpha_i}$ can be written as:

$$a * \overline{d}^{\alpha_i} = \sum_{\alpha\in E} a * \overline{d}^{\alpha_i}(\overline{x}^\alpha)\overline{d}^{\alpha} = \sum_{\alpha\in E} \overline{d}^{\alpha_i}(a\overline{x}^\alpha)\overline{d}^{\alpha} = \sum_{\alpha\in E} \mu_{\alpha\alpha_i}\overline{d}^{\alpha}$$

The last equality is due to in 7.2 the component α_i-th of $a\overline{x}^\alpha$ is $\mu_{\alpha\alpha_i}$.Then the matrix of $\mathbb{M}_a^t$ in the basis $(\overline{d}^{\alpha})_{\alpha\in E}$ is:

$$\mathbb{M}_a^t = \begin{bmatrix} \mu_{\alpha_1\alpha_1} & \mu_{\alpha_1\alpha_2} & \cdots & \mu_{\alpha_1\alpha_r} \\ \mu_{\alpha_2\alpha_1} & \mu_{\alpha_2\alpha_2} & \cdots & \mu_{\alpha_2\alpha_r} \\ \vdots & \vdots & \vdots & \\ \mu_{\alpha_r\alpha_1} & \mu_{\alpha_r\alpha_2} & \cdots & \mu_{\alpha_r\alpha_r} \end{bmatrix}$$

Therefore, $\mathbb{M}_a^t$ is the transpose of $\mathbb{M}_a$. □

Theorem 7.11. *Let $Z(I_\Lambda)$=$\{\xi_1, ..., \xi_d\}$ the variety defined by the ideal I_Λ:*

- *i) If $a \in \mathbb{K}(\overline{x})$, then the eigenvalues of the operators M_a^t and M_a are $a(\xi_1)$,...,$a(\xi_d)$. In particular, the eigenvalues of M_{x_i}, $i = 1, ..., n$, are the ith-coordinates of the roots $\xi_1, ..., \xi_d$.*
- *ii) If $a \in \mathbb{K}(\overline{x})$, then the evaluations $ev(\xi_1)$,...,$ev(\xi_d)$ are the eigenvectors of the operators M_a^t respectively associated with the eigenvalues $a(\xi_1)$,...,$a(\xi_d)$. Moreover, these evaluations are the only eigenvectors common to all endomorphism M_a^t, $a \in \mathbb{K}(\overline{x})$.*

Proof. i) Let $i \in \{ 1, ..., d \}$. For any $b \in A_\Lambda$,

$$(M_a^t(ev(\xi_i)))(b) = ev(\xi_i)(ab) = (a(\xi_i)ev(\xi_i))(b)$$

this proves that $a(\xi_1)$,...,$a(\xi_d)$ are the eigenvalues of the operators M_a^t and M_a. Moreover, the $ev(\xi_i)$ are the eigenvectors of M_a^t and common to all endomorphism M_a^t.

Reciprocally, any eigenvalue of M_a is $a(\xi_i)$:
Let $p(\overline{x}) = \prod_{\xi\in Z(I_\Lambda)}(a(\overline{x}) - a(\xi)) \in \mathbb{K}(\overline{x})$ this polynomial vanishes over $Z(I_\Lambda)$. By the Hilberts Nullstellensatz, there exists $m \in \mathbb{N}$ such that $p^m \in I_\Lambda$. If $\mathbb{I}$ designates the identity on A_Λ , then the operator $p^m(M_a) = \prod_{\xi\in Z(I_\Lambda)}(M_a - a(\xi)\mathbb{I})$ is null, and the minimal polynomial of M_a divides to $\prod_{\xi\in Z(I_\Lambda)}(T - a(\xi))^m$. Therefore the eigenvalues of M_a are $a(\xi_i)$, with $\xi_i \in Z(I_\Lambda)$.

ii)Let $\Lambda \in A_\Lambda$ an eigenvector common to all endomorphism M_a^t, $a \in \mathbb{K}(\overline{x})$. If γ=$(\gamma_1, ..., \gamma_n) \in \mathbb{K}^n$ satisfies $M_{x_i}^t$=$\gamma_i\Lambda$, with i=$1, ..., n$, then any monomial $\overline{x}^\alpha$ satisfies:

$$(M_{x_i}^t(\Lambda))(\overline{x})^\alpha = \Lambda(x_i\overline{x}^\alpha) = \gamma_i\Lambda(\overline{x}^\alpha)$$

Then for any α=$(\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$,

$$\Lambda(\overline{x}^\alpha) = \gamma_1^{\alpha_1} \cdots \gamma_n^{\alpha_n}\Lambda(1) = \Lambda(1)ev(\gamma)(\overline{x}^\alpha).$$

Therefore Λ=$\Lambda(1)ev(\gamma)$, since $\Lambda \in A_\Lambda$=$I_\Lambda^\perp$, $\Lambda(p) = \Lambda(1)p(\gamma) = 0$ for any $p \in I_\Lambda$. Since $\Lambda(1) \neq 0$, $\gamma \in Z(I_\Lambda)$ and $ev(\gamma) \in A_\Lambda$. □

Theorem 7.12. *If $rankH_\Lambda = r < \infty$,then:*

- *i)A_Λ is of dimension r over $\mathbb{K}$ and the set of roots $Z(I_\Lambda) = \{\xi_1, .., \xi_d\}$ is finite with $d \leq r$.*

- *ii)There exists $p_i \in \mathbb{K}[\partial_1, .., \partial_n]$ such that $\Lambda = \sum_{i=1}^{d} ev(\xi_i) \circ p_i(\overline{\alpha})$.*
  Moreover, the multiplicity of ξ_i is the dimension of the vector space generated by $ev(\xi_i) \circ p_i(\overline{\alpha})$.

Proof. i)Since $rank(\mathbb{H}_\Lambda) < \infty$ the dimension of the vector space $A_\Lambda = R/I_\Lambda$ is also r. Thus, let us see that, the number of zeros of the ideal I_Λ, denoted $\{\xi_1, .., \xi_d\}$, is at most r, with $d \leq r$:
If r is the dimension of the $\mathbb{K}$-vector space A_Λ, then for any $i \in \{1, ..., n\}$, $\{1, x_i, x_i^2, ..., x_i^r\}$ is a set linearly dependent of A_Λ. Then, there exists, $c_0, ..., c_r \in \mathbb{K}$ such that, $q_i(x_i) = c_0 + c_1x_i + ... + c_rx_i^r \in I_\Lambda$. For any $i \in \{1, ..., n\}$ the ith-coordinates of the zeros of $Z(I_\Lambda)$, are roots of $q_i(x_i)$. Thus, if $\xi_j \in Z(I_\Lambda)$ then $q_i(\xi_{j_i}) = 0$ and like q_i has at most r roots, $|Z(I_\Lambda)| \leq r$.
ii) We can apply the structure theorem 5.3, in order to get the decomposition since obviously $\Lambda \in I_\Lambda^\perp$: $\Lambda \in I_\Lambda^\perp$ if $\Lambda(p) = 0$ for all $p \in I_\Lambda$ but $I_\Lambda = kerH_\Lambda$ then $p * \Lambda \equiv 0$ for all $p \in I_\Lambda$ in particular $p * \Lambda(1) = \Lambda(p) = 0$. On the other hand, we saw in the proof of the proposition 7.5 that Λ is the residue of A_Λ, then by the proof of 6.7 and due to the decomposition is unique, $p_i(\overline{\partial}) \circ ev(\xi_i)$ is the residue of the sub-algebra $Q_{\xi_i}^\perp$ that is, $(p_i \circ ev(\xi_i)) * Q_\xi^\perp = (Q_{\xi_i}^\perp)^*$, where Q_{ξ_i} is the component m_{ξ_i}-primary of I_Λ. Therefore, the dimension of the vector space generated by $p_i(\overline{\partial}) \circ ev(\xi_i)$ is the multiplicity of ξ_i. □

Remark 7.13. *If the field $\mathbb{K}$ is of characteristic 0, the inverse system $ev(\xi_i) \circ p_i(\overline{\alpha})$ is isomorphic to the vector space generated by p_i and its derivatives of any order with respect to the variables ∂_i*

Definition 7.14. *For $f \in S_d$, we call generalized decomposition of f^* a decomposition such that $f^* = \sum_{i=1}^{d} ev(\xi_i) \circ p_i(\overline{\alpha})$ where the sum for $i = 1, ..., d$ of the dimensions of the vector space spanned by the inverse system generated by $ev(\xi_i) \circ p_i(\overline{\alpha})$ is minimal. This minimal sum of the dimensions is called length of f.*

Remark 7.15. *The length of f^* is the rank of the corresponding Hankel operator H_Λ.*

Theorem 7.16. *Let $\Lambda \in R^*$. $\Lambda = \sum_{i=1}^{r} \lambda_i ev(\xi_i)$ with $\lambda_i \neq 0$ and ξ_i distinct points of $\mathbb{K}^n$, iff $rankH_\Lambda = r$ and I_Λ is a radical ideal.*

Proof. If $\Lambda = \sum_{i=i}^{r} \lambda_1 ev(\xi_1)$, with $\lambda_i \neq 0$ and ξ_i distinct points of $\mathbb{K}^n$. Let $\{e_1, ..., e_r\}$ be a family of interpolation polynomials at these points: $e_i(\xi_j) = 1$ if $i = j$ and 0 otherwise. Let I_ξ be the ideal of polynomials which vanish at $\xi_1, ..., \xi_r$, which is a radical ideal. Clearly we have $I_\xi \subset I_\Lambda$: let $p \in I_\xi$ then $p(\xi_i) = 0$ for any $i = 1, ..., r$ and $\Lambda(p) = \sum_{i=1}^{r} \lambda_i p(\xi_i) = 0$ thus $p \in I_\Lambda$. Let us see that $I_\Lambda \subset I_\xi$: for any $p \in I_\Lambda$, and $i = 1, ..., r$, we have $p * \Lambda(e_i) = \Lambda(pe_i) = \lambda_i p(\xi_i) = 0$, which proves that $I_\Lambda = I_\xi$, and I_Λ is a radical ideal. And the $rank(H_\Lambda) = r$ because the quotient A_Λ is generated by the interpolation polynomials $e_1, ..., e_r$.

Conversely if $rankH_\Lambda = r$ and I_Λ is radical, then by the previous theorem $\Lambda = \sum_{i=1}^{r} ev(\xi) \circ p_i(\overline{\partial})$, and due to the multiplicity of ξ is the dimension of the vector space spanned by the inverse system generated by $ev(\xi) \circ p_i(\overline{\alpha})$ the multiplicity of ξ_i is 1 and the polynomials p_i are of degree 0. □

Proposition 7.17. *For any linear form $\Lambda \in R^*$ such that rank $H_\Lambda < \infty$ and any $a \in A_\Lambda$, we have:*

$$H_{a*\Lambda}(p) = M_a^t \circ H_\Lambda(p)$$

Proof. $H_{a*\Lambda}(p) = a * (p * \Lambda) = M_a^t \circ H_\Lambda(p)$ □

Using the previous Proposition and Theorem 3, we can recover the points $\xi_i \in \mathbb{K}^n$ by eigenvector computation as follows:

Assume that $B = \langle b_1, ..., b_r \rangle \subset R$ with $|B| = rank(H_\Lambda)$ and H_Λ^B invertible, then by the previous proposition, $H_{a*\Lambda}(p) = M_a^t \circ H_\Lambda(p)$. Then by the theorem 7.11, the solutions of the generalized eigenvalue problem:

$$\mathbb{M}_a^t(\mathbb{H}_\Lambda^B v) = \lambda \mathbb{H}_\Lambda^B v \text{ if and only if } (\mathbb{H}_{a*\Lambda}^B - \Lambda \mathbb{H}_\lambda)v = \overline{0}$$

for any $a \in R$, yield the common eigenvectors $\mathbb{H}_\Lambda^B v$ of $\mathbb{M}_a^t$, that are the evaluation $ev(\xi)$ at the roots, $i = 1, ..., d$. Therefore these common eigenvectors $\mathbb{H}_\Lambda^B v$ are up to scalar, the vectors $[b_1(\xi_i), ..., b_r(\xi_i)]$ $(i = 1, ..., d)$, since:

If the dual basis to the basis $\langle b_1, ..., b_r \rangle$ is $\langle B \rangle^* = \left\langle \delta^1, ..., \delta^r \right\rangle$ then for any $\Lambda \in A^*$:

$$\Lambda = \Lambda(b_1)\delta^1 + ... + \Lambda(b_r)\delta^r \ ,$$

particularly :

$$ev(\xi_i) = b_1(\xi_i)\delta^1 + ... + b_r(\xi_i)\delta^r,$$

then the vectors $[b_1(\xi_i), ..., br(\xi_i)]$ for $i = 1, ..., d$ are the eigenvectors $ev(\xi_i)$ in the basis $\langle B \rangle^*$. Notice that it is enough to compute the common eigenvectors of $\mathbb{H}_{x_i*\Lambda}$ for $i = 1, ..., n$. Once the common eigenvectors $ev(\xi_i)$ for $i = 1, ..., d$ have been computed, in order to recover the points $\xi_i \in \mathbb{K}^n$ for $i = 1, ..., d$, it is necessary to compute the eigenvalue of $H_{x_j*\Lambda}$ for $j = 1, ..., n$ which is the j-th coordinate of the point ξ_i.

Particularly if $\Lambda = \sum_{i=1}^d \lambda_i ev(\xi_i)$ $(\lambda_i \neq 0)$, then the roots are simple, and the computation of the eigenvectors of one operator $\mathbb{M}_a$ for any $a \in R$ is sufficient, since: for any $a \in R$, $\mathbb{M}_a$ is diagonalizable and all the eigenvectors $\mathbb{H}_\Lambda^B v$ are, up to scalar factor, the evaluations $ev_(\xi_i)$ at the roots.

Chapter 8

Truncated Hankel Operators

As we saw in the section "Decomposition using duality", our problem of symmetric tensor decomposition can be restated as follows:

"Let $\Lambda_{f^*} \in R_d^*$ find the minimal number of non-zero vectors $k_1, ..., k_r \in \mathbb{K}^n$ and non-zero scalars $\lambda_1, ..., \lambda_r \in \mathbb{K}$ such that $\Lambda_{f^*} = \sum \lambda_i ev(k_i)$".

Then by virtue of the Theorem 7.16, $\Lambda = \sum_{i=1}^r \lambda_i ev(k_i)$ with $\lambda_i \neq 0$ and k_i distinct points of $\mathbb{K}^n$ if and only if $rank(\mathbb{H}_\Lambda) = r$ and I_Λ is a radical ideal.

In this section, we characterize the conditions under which $\Lambda_{f^*} \in R_d^*$ can be extended to $\Lambda \in R^*$ when the rank of $\mathbb{H}_\Lambda$ is r. To get this result, first we study how to parametrize the set of ideals I of R such that a given set B of monomials is a connected basis of the quotient R/I.

Lemma 8.1. *Let $B \subset R$ a finite set of monomials connected to 1. For $\overline{z} \in \mathbb{K}^{N:=|B|\times|\partial B|}$ we define the linear maps, for $i = 1, ..., n$:*

$$M_i^B(\overline{z}) : \langle B \rangle \to \langle B \rangle$$

such that:

$$M_i^B(\overline{z})(b) = \begin{cases} x_i b & \text{if } x_i \in B; \\ \sum_\beta z_{x_i b, \beta} \overline{x}^\beta & \text{if } x_i b \in \partial B. \end{cases}$$

And we define also the following subsets:

$$V^B := \{\overline{z} \in \mathbb{K}^N : M_j^B(\overline{z}) \circ M_i^B(\overline{z}) - M_i^B(\overline{z}) \circ M_j^B(\overline{z})\}$$

and

$$H^B := \{I \subset R \text{ ideal} : B \text{ is a basis of } R/I\}$$

Then H^B is in bijection with V^B.

Proof. We define the following application:

$$\begin{array}{rccc} \phi : & H^B & \to & V^B \\ & I & \longmapsto & \overline{z} \end{array}$$

where $\overline{z} = (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B}$ is defined as follows: for all $\alpha \in \partial B$ we get $z_{\alpha,\beta}$ due to the unique decomposition of x^α on B module I, that is:

$$\overline{x}^\alpha = \sum_{\beta\in B} z_{\alpha,\beta}\overline{x}^\beta$$

This application is well defined because R/I has structure of commutative algebra. We will show that ϕ is injective. In order to do this we only have to prove that $(\{h_\alpha(\overline{x})\}_{\alpha\in\partial B}) = I$, where for all $\alpha \in \partial B$:

$$h_\alpha(\overline{x}) = \overline{x}^\alpha - \sum_{\beta\in B} z_{\alpha,\beta}\overline{x}^\beta$$

It is easy to see that $(\{h_\alpha(\overline{x})\}_{\alpha\in\partial B}) \subset I$. Reciprocally, we will show $I \subset (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$. We define for all $P = \sum_\gamma a_\gamma \overline{x}^\gamma \in R$ the following application:

$$P(M): \ \langle B\rangle \ \rightarrow \ \langle B\rangle$$

where $P(M) = \sum_\gamma a_\gamma (M^B(\overline{z}))^\gamma$ and $(M^B(\overline{z}))^\gamma := M_1^B(\overline{z})^{\gamma_1}\circ\ldots\circ M_n^B(\overline{z})^{\gamma_n}$. As the multiplication operators commute the application is well defined. Note that $P(M)(1)$ is the decomposition of P in the basis B on R/I as $\mathbb{K}$-vector space of finite dimension. Then we will prove by induction on the degree of P, that:

$$P - P(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$$

We can assume P is a monomial, due to the linearity of the operators.

- If $P = k$ with k a constant, then it is clear that $P - P(M)(1) = k - k = 0 \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$
- If we assume it holds true for degree N. Let us see that for P of degree $N+1$ it holds true also. We can write $P = x_i P^{'}$ with $P^{'}$ of degree N. And we want to prove that $x_i P^{'} - P(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$. In order to prove this, we write:

  $$x_i P^{'} - P(M)(1) = x_i(P^{'} - P^{'}(M)(1)) + x_i P^{'}(M)(1) - P(M)(1)$$

  By induction hypothesis we have $P^{'} - P^{'}(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$, thus we only have to prove that:

  $$x_i P^{'}(M)(1) - P(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B}) \tag{8.1}$$

  where $P = x_i P^{'}$. We will prove 8.1 by induction with respect to the distance from $P^{'}$ to the border:

  - If $P^{'} \in B$ then either $x_i P^{'} \in \partial B$ or $x_i P^{'} \in B$:
    * If $x_i P^{'} \in B$ then:
      $$x_i P^{'}(M)(1) - P(M)(1) = x_i P^{'} - x_i P^{'} = 0 \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$$
    * If $x_i P^{'} \in \partial B$ then:
      $$x_i P^{'}(M)(1) - P(M)(1) = x_i P^{'} - \sum_{\beta\in B} \overline{z}_{x_i P^{'},\beta}\overline{x}^\beta \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$$
  - Assume 8.1 holds true for monomials P' such that the distance from P' to the ∂B is less than or equal to η, that is, for monomials $P^{'} = x_1^{\gamma_1}\ldots x_n^{\gamma_n} b$, where $b \in B$ and $|\gamma_1 + \ldots + \gamma_n| = \eta$.

    We are going to prove that it holds also true for monomials R' such that the distances to ∂B is less than or equal to $\eta + 1$. Namely, let $R' = x_j x_1^{\gamma_1}\ldots x_n^{\gamma_n} b$, then we want to prove that, $x_i R^{'}(M)(1) - R(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$, where $R = x_i R^{'}$. We have:

    $$x_i R^{'}(M)(1) - R(M)(1) =$$

$$x_i(M_j^B(\overline{z})\circ M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b)-(M_i^B(\overline{z})\circ M_j^B(\overline{z})\circ M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b) =$$

$$M_j^B(\overline{z})x_i(M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b)-M_j^B(\overline{z})(M_i^B(\overline{z})\circ M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b) =$$

$$M_j^B(\overline{z})[x_i(M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b)-(M_i^B(\overline{z})\circ M_1^B(\overline{z})^{\gamma_1}\circ...\circ M_n^B(\overline{z})^{\gamma_n})(b)] =$$

$$M_j^B(\overline{z})[x_iP'(M)(1)-P(M)(1)] = 0 \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$$

The last equality is due to by induction hypothesis: $x_iP'(M)(1)-P(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$, and moreover $(h_\alpha(\overline{x})\}_{\alpha\in\partial B}) \subset I$ and B is a basis of R/I.

Thus $P - P(M)(1) \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$.

Therefore, if $P \in I$, $P \in (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$. And, finally, $I = (\{h_\alpha(\overline{x})\}_{\alpha\in\partial B})$ and ϕ is injective. In order to prove, ϕ is surjective, we are going to build the application J such that $\phi(J(\overline{z})) = \overline{z}$ for all $\overline{z} \in V^B$. Let $\overline{z} = (z_{\alpha,\beta})_{\alpha\in\partial B,\beta\in B} \in V^B$, and we define the following application:

$$\begin{array}{rccc} \sigma_{\overline{z}}: & R & \to & \langle B\rangle \\ & P & \longmapsto & P(M)(1) \end{array}$$

It is well defined since the multiplication operators $(M_i^B(\overline{z}))_{1\le i\le n}$ commute. Then, we can define the following application:

$$\begin{array}{rccc} J: & V^B & \to & H^B \\ & \overline{z} & \longmapsto & ker(\sigma_{\overline{z}}) \end{array}$$

It is well defined since for all $\overline{z}$, $J(\overline{z}) = ker(\sigma_{\overline{z}})$ is an ideal due to $\sigma_{\overline{z}}$ is a ring homomorphism. Moreover, as for all $b \in B$, $b(M)(1) = b$, the application $\sigma_{\overline{z}}$ is surjective , then $R/J(\overline{z}) \cong \langle B\rangle$. Thus, $J(\overline{z}) \in H^B$, and for all $\alpha \in \partial B$, $\overline{x}^\alpha = \sum_{\beta\in B} \overline{z}_{\alpha,\beta}\overline{x}^\beta$ module $J(\overline{z})$, then $\phi(J(\overline{z})) = \overline{z}$. Therefore ϕ is a bijection.

□

Definition 8.2. *Let $B \subset R_d$ a set of monomials of degree at most d, and let $\Lambda \in R_d^*$, the Hankel matrix $\mathbb{H}_\Lambda^B(\overline{h})$ is the matrix defined as follows:*

$$\mathbb{H}_\Lambda^B(\overline{h})(\overline{x}^\gamma) = \begin{cases} \Lambda(\overline{x}^\gamma) & \textit{if } |\gamma| \le d; \\ h_\gamma & \textit{in other case.} \end{cases}$$

where h_γ is a variable, and $\overline{h}$ is the set of new variables. We will denote by $H_\Lambda^B(\overline{h}) : \langle B\rangle \to \langle B\rangle^*$ the linear form associated to the matrix $\mathbb{H}_\Lambda^B(\overline{h})$ in the basis B.

Definition 8.3. *Let $\Lambda \in R_d^*$ such that $\mathbb{H}_\Lambda^B(\overline{h})$ is invertible in $\mathbb{K}(\overline{h})$, that is the rational polynomial functions in $\overline{h}$ and $B \subset R_d$ a set of monomials. We define the multiplication operators:*

$$M_i^B(\overline{h}) := (H_\Lambda^B(\overline{h}))^{-1}H_{x_i*\Lambda}(\overline{h})$$

Remark 8.4. *With the previous definition of the multiplication operators we have: for all $i \in \{1, ..., n\}$ and for all $\overline{h} \in \mathbb{K}^N$ (for some $N \in \mathbb{N}$):*

- $M_i(\overline{h})(b) = x_ib$ *for all* $b \in B$ *if* $x_ib \in B$
- $M_i(\overline{h})(b) = \sum_{\beta\in B} h_{x_ib\overline{x}^\beta}\overline{x}^\beta$ *if* $x_ib \in \partial B$

Notation 8.5. *For any $\overline{h} \in \mathbb{N}$ (for some $N \in \mathbb{N}$) we write:*

$$h_{\overline{x}^{\alpha+\beta}} := h_{\alpha+\beta}$$

We are going to need the following property on the basis of A_Λ.

Definition 8.6. *Let $B \subset R$ a set of monomials. We say B is connected to 1 is for all $b \in B$ either $b = 1$ or there exists a variable x_i and $b \in B$ for $i = 1, ..., n$ such that $b = x_i b'$.*

Theorem 8.7. *Let $B = \{\overline{x}^{\beta_1}, ..., \overline{x}^{\beta_r}\}$ be a set of monomials of degree at most d, connected to 1 and let Λ be a linear form in $\langle BB^+\rangle_{\leq d}$. Let $\Lambda(\overline{h})$ be the linear form of $\langle BB^+\rangle^*$ defined as follows:*

$$\Lambda(\overline{h})(\overline{x}^\gamma) = \begin{cases} \Lambda(\overline{x}^\gamma) & \text{if } |\gamma| \leq d; \\ h_\gamma & \text{in other case.} \end{cases}$$

where $h_\gamma \in \mathbb{K}$ is a variable. Then Λ admits an extension $\widetilde{\Lambda} \in R^*$ such that $H_{\widetilde{\Lambda}}$ is of rank r with B a basis of $A_{\widetilde{\Lambda}}$ if and only if there exists a solution $\overline{h}$ for the following problem:

- $M_i^B(\overline{h}) M_j^B(\overline{h}) - M_j^B(\overline{h}) M_i^B(\overline{h}) = 0$, $(1 \leq i < j \leq n)$
- $\det(H_\Lambda^B(\overline{h}) \neq 0$.

Moreover, for every solution $\overline{h}_0 \in \mathbb{K}^N$ an extension such $\widetilde{\Lambda} = \Lambda(\overline{h_0})$ over $\langle BB^+\rangle$ is unique.

Proof. If there exists $\widetilde{\Lambda} \in R^*$ which extends Λ with $H_{\widetilde{\Lambda}}$ of rank r and B a basis of $A_{\widetilde{\Lambda}}$. We define $\overline{h}^0 \in \mathbb{K}^N$ (for some $N \in \mathbb{N}$) as follows:
for all $\overline{x}^\gamma \in \langle BB^+\rangle$ and $|\gamma| > d$:

$$h_\gamma^0 := \widetilde{\Lambda}(\overline{x}^\gamma)$$

then $\Lambda(\overline{h}^0) = \widetilde{\Lambda}$ over $\langle BB^+\rangle$ and $H^B_{\Lambda(\overline{h}^0)} = H^B_{\widetilde{\Lambda}}$ but $rank(H_{\widetilde{\Lambda}}) = r$ and B a basis of $H_{\widetilde{\Lambda}}$ then $H_{\widetilde{\Lambda}}$ is invertible and then $\Lambda(\overline{h}^0)$ is invertible. Therefore we can define the multiplication operators:

$$M_i^B(\overline{h^0}) := (H_\Lambda^B(\overline{h^0}))^{-1} H_{x_i * \Lambda}(\overline{h^0})$$

then:

$$\begin{array}{ccccccccccc} M_i^B(\overline{h^0}) M_j^B(\overline{h^0}): & \langle B\rangle & \overset{H^B_{x_j*\Lambda}(\overline{h^0})}{\rightarrow} & \langle B\rangle^* & \overset{(H^B_\Lambda(\overline{h^0}))^{-1}}{\rightarrow} & \langle B\rangle & \overset{H^B_{x_i*\Lambda}(\overline{h^0})}{\rightarrow} & \langle B\rangle^* & \overset{(H^B_\Lambda(\overline{h^0}))^{-1}}{\rightarrow} & \langle B\rangle \\ & b & \longmapsto & x_j b * \Lambda & \longmapsto & x_j b & \longmapsto & x_i x_j b * \Lambda & \longmapsto & x_i x_j b \end{array}$$

as A_Λ is a commutative algebra for all $b \in \langle B\rangle = A_{\widetilde{\Lambda}}$, $x_i x_j b = x_j x_i b$ and:

$$M_i^B(\overline{h^0}) M_j^B(\overline{h^0}) - M_i^B(\overline{h^0}) M_j^B(\overline{h^0}) = 0$$

Thus $\overline{h}^0$ is a solution of the problem.
Reciprocally, if there exists $\overline{h}^0 \in \mathbb{K}^N$ (for some $N \in \mathbb{N}$) such that the multiplication operators commute. By the theorem 8.1, there exists a bijection between the variety, $V^B := \{\overline{h} : M_i^B(\overline{h})M_j^B(\overline{h}) - M_i^B(\overline{h})M_j^B(\overline{h}), 1 \leq i < j \leq n\} = 0$ and the set $H^B := \{I \subset R : R/I$ is a free R-module of rank $\mu < \infty$ and B as basis $\}$. Therefore, there exists a unique ideal $I \subset R$ generated by the set border relations:

$$K := \{\overline{x}^\alpha - \textstyle\sum_{\beta \in B} h_{\alpha+\beta}\overline{x}^\beta \ \forall \alpha \in \partial B\} = \{x_i b - \textstyle\sum_{\beta \in B} \overline{x}^\beta \ \forall 1 \leq i \leq n \ and \ \forall b \in B\} = \{x_i b - M_i^B(\overline{h}^0)(b) \forall 1 \leq i \leq n \ and \ \forall b \in B\}$$ [1]

such that $R = \langle B \rangle \oplus I$, where $I = (K)$. We define $\widetilde{\Lambda} \in R^*$ as follows:

$$\forall p \in R \ \widetilde{\Lambda}[p] = \Lambda(\overline{h}^0)[p(M)(1)]$$

where $p(M)$ is the operator obtained by substitution of the variables x_i by the commuting operators M_i, then $p(M)$ is the operator of multiplication by p module I.
If $p \in I$, for any $q \in R$ then:

$$\widetilde{\Lambda}[pq] = \Lambda(\overline{h}^0)[0 \cdot q(M)(1)] = 0$$

then $I \subset KeH_{\widetilde{\Lambda}}$.
We will prove by induction on the degree of $b' \in B$:

$$\Lambda(\overline{h}^0)[b'b] = \Lambda(\overline{h}^0)[b'(M)(b)]$$

for all $b \in B$.

- for $b' = 1$ $\Lambda(\overline{h}^0)[b] = \Lambda(\overline{h}^0)[1(M)(b)] = \Lambda(\overline{h})^0[1b] = \Lambda(\overline{h}^0)[b]$
- if $b' \neq 1$ as B is connected to 1 then $b' = x_i b''$ for some variable x_i and some element $b'' \in B$. By construction of the operators $M_i^B(\overline{h}^0)$ and for all $b \in B$:

$$\Lambda(\overline{h}^0)[b'b] = \Lambda(\overline{h}^0)[b''x_i b] = \Lambda(\overline{h}^0)[b'' M_i^B(\overline{h}^0)(b)].$$

By induction hypothesis and as b'' has smaller degree than b' ,for all $b \in B$ we have:

$$\Lambda(\overline{h}^0)[b''b] = \Lambda(\overline{h}^0)[b''(M)(b)]$$

In particular, $M_i^B(\overline{h}^0)(b) \in B$ then:

$$\Lambda(\overline{h}^0)[b'' M_i^B(\overline{h}^0)(b)] = \Lambda(\overline{h}^0)[b''(M) \circ M_i^B(\overline{h}^0)(b)].$$

as $b' = x_i b''$, thus:

$$\Lambda(\overline{h}^0)[b''(M) \circ M_i^B(\overline{h}^0)(b)] = \Lambda(\overline{h}^0)[b'(M)(b)]$$

Therefore:

$$\Lambda(\overline{h}^0)[b'b] = \Lambda(\overline{h}^0)[b'(M)(b)]$$

On the other hand, let $b^+ \in B^+$, there exists $1 \leq i \leq n$ and $b \in B$ such that $x_i b = b^+$. By definition :

$$b(M)(1) = b \text{ for all } b \in B.$$

[1] Note that in the lemma 8.1 we write $z_{\alpha,\beta}$ and in this case it is convenient to write $z_{\alpha+\beta}$

Then for all $b' \in B$:
$\Lambda(\overline{h}^0)[b'b^+] = \Lambda(\overline{h}^0)[b'x_i b] = \Lambda(\overline{h}^0)[b'M_i^B(\overline{h}^0)(b)] = \Lambda(\overline{h}^0)[b'M_i^B(\overline{h}^0)\circ b(M)(1)] = \Lambda(\overline{h}^0)[b'b^+(M)(1)]$.
Then for all $b \in B$ and $b^+ \in B^+$

$$\Lambda(\overline{h}^0)[bb^+] = \widetilde{\Lambda}[bb^+].$$

Therefore, $\Lambda(\overline{h}^0) = \widetilde{\Lambda}$ over $\langle BB^+\rangle$ and $\widetilde{\Lambda}$ is an extension of Λ.And $\det(H^B_{\widetilde{\Lambda}}) = \det(H^B_{\Lambda(\overline{h}^0)}) \neq 0$.
Then we deduce that B is a basis of $A_{\widetilde{\Lambda}}$ and $H_{\widetilde{\Lambda}}$ has rank r.
Suppose there exists another $\Lambda' \in R^*$ which extends $\Lambda(\overline{h}) \in \langle BB^+\rangle^*$ such that $rankH_{\Lambda'} = r$ with B a basis of $H_{\Lambda'}$. By the Proposition 7.7:

$$I_{\Lambda'} = kerH_{\Lambda'} = (kerH^{BB^+}_{\Lambda'}) = (kerH^{BB^+}_{\widetilde{\Lambda}}) = I_{\widetilde{\Lambda}}$$

therefore $\Lambda' = \widetilde{\Lambda}$ because Λ' coincides with $\widetilde{\Lambda}$ on B. □

Example 8.8. *If we have the following $\Lambda(\overline{h})$ defined over $\langle B.B^+\rangle$ with $B = \langle 1, x_1, x_2, x_3, x_1^2\rangle$ and $B \subset R := \mathbb{K}[x_1, x_2, x_3]$ such that:*

$$\Lambda(\overline{h})(\overline{x}^\gamma) = \begin{cases} \Lambda(\overline{x}^\gamma) & \text{if } |\gamma| \leq 4; \\ h_\gamma & \text{in other case.} \end{cases}$$

where the matrix $\mathbb{H}^{BB^+}_\Lambda(\overline{h})$ is:

$$\mathbb{H}^{BB^+}_\Lambda(\overline{h}) = \left(\begin{array}{c|ccccc}
 & 1 & x_1 & x_2 & x_3 & x_1^2 \\ \hline
1 & 0 & 0 & 0 & 0 & 2 \\
x_1 & 0 & 2 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 2 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 2 & 0 \\
x_1^2 & 2 & 0 & 0 & 0 & 0 \\
x_1x_2 & 0 & 0 & 0 & 0 & 0 \\
x_1x_3 & 0 & 0 & 0 & 0 & 0 \\
x_2^2 & 2 & 0 & 0 & 0 & 0 \\
x_2x_3 & 0 & 0 & 0 & 0 & 0 \\
x_3^2 & 2 & 0 & 0 & 0 & 0 \\
x_1^3 & 0 & 0 & 0 & 0 & h_{500} \\
x_1^2x_2 & 0 & 0 & 0 & 0 & h_{410} \\
x_1^2x_3 & 0 & 0 & 0 & 0 & h_{401}
\end{array}\right)$$

We are going to compute $\overline{h} = (h_{500}, h_{410}, h_{401})$, in the case there exists solution, in the same way that the final symmetric tensor decomposition does it, in order to say that $\Lambda(\overline{h})$ admits an extension $\widetilde{\Lambda} \in R^*$:

The second condition of the previous theorem is satisfied by $H^B_\Lambda(\overline{h})$ since $\det(\mathbb{H}^B_\Lambda(\overline{h})) \neq 0$ and:

$$(\mathbb{H}^B_\Lambda(\overline{h}))^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Also we need that the multiplication operators commute, in order to do this we compute the matrix $\mathbb{H}^B_{x_1*\Lambda}$,$\mathbb{H}^B_{x_2*\Lambda}$,$\mathbb{H}^B_{x_3*\Lambda}$:

$$\mathbb{H}_{x_1*\Lambda}(\overline{h}) = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{500} \end{pmatrix}$$

$$\mathbb{H}_{x_2*\Lambda}(\overline{h}) = \begin{pmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{410} \end{pmatrix}$$

$$\mathbb{H}_{x_3*\Lambda}(\overline{h}) = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{401} \end{pmatrix}$$

We compute the multiplication operators:

$$M^B_i(\overline{h}) := (H^B_\Lambda(\overline{h}))^{-1} H_{x_i*\Lambda}(\overline{h})$$

and we form all the possible matrix equations:

$$M^B_i(\overline{h}) M^B_j(\overline{h}) - M^B_j(\overline{h}) M^B_i(\overline{h}) = 0,\ 1 \leq i < j \leq 3)$$

Then we get $\binom{3}{2}$ equations whose solutions are $h_{500} = h_{410} = h_{401} = 0$. Then by the theorem 8.7, $\Lambda(\overline{h})$ admits an extension $\widetilde{\Lambda} \in R^*$. Moreover, for this solution the extension is unique, and for $\overline{h} = (0,0,0)$, we have $H^{BB^+}_\Lambda(\overline{h})$=$H^{BB^+}_\tau$ where $\tau = \delta^{\alpha_1} + \delta^{\alpha_2} + \delta^{\alpha_3}$ defined in the example 7.1,then $\widetilde{\Lambda} = \tau$.

Theorem 8.9. *Let $B = \{\overline{x}^{\beta_1}, ..., \overline{x}^{\beta_r}\}$ be a set of monomials of degree at most d, connected to 1, and let $\Lambda \in \langle B^+B^+\rangle^*_{\leq d}$ and $\Lambda(\overline{h}) \in \langle B^+B^+\rangle^*$ defined as follows:*

$$\Lambda(\overline{h})(\overline{x}^\gamma) = \begin{cases} \Lambda(\overline{x}^\gamma) & \textit{if } |\gamma| \leq d; \\ h_\gamma & \textit{in other case.} \end{cases}$$

Then, Λ admits an extension $\widetilde{\Lambda} \in R^*$ such that $H_{\widetilde{\Lambda}}$ is of rank r, with B a basis of $A_{\widetilde{\Lambda}}$ if and only if there exists a solution $\overline{h}$ to the problem:

- *i) All $(r+1) \times (r+1)$ minors of $H_{\Lambda}^{B^+}(\overline{h})$ vanish.*
- *ii) $\det(H_{\Lambda}^{B})(\overline{h}) \neq 0$*

Moreover, for every solution $\overline{h}_0 \in \mathbb{K}^N$ an extension such $\widetilde{\Lambda} = \Lambda(\overline{h_0})$ over $\langle B^+B^+\rangle$ is unique.

Proof. If there exists $\widetilde{\Lambda} \in R^*$ extension. We define $\overline{h}^0 \in \mathbb{K}^M$ (for some $M \in \mathbb{N}$) as follows: for all $\overline{x}^\gamma \in \langle B^+B^+\rangle$ such that $|\gamma| > d$:

$$h_\gamma^0 := \widetilde{\Lambda}(\overline{x}^\gamma)$$

As $H_{\widetilde{\Lambda}}$ is of rank r and $A_{\widetilde{\Lambda}}$ has B as basis then all $(r+1) \times (r+1)$ minors of $H_{\widetilde{\Lambda}}^{B^+} = H_{\Lambda(\overline{h}^0)}^{B^+}$ vanish and $H_{\widetilde{\Lambda}}^{B} = H_{\Lambda(\overline{h}^0)}^{B}$ is invertible. Thus $\overline{h}^0$ is solution for the problem i) and ii).
Reciprocally, if there exists $\overline{h}^0 \in \mathbb{K}^N$ solution for the problem i) and ii).We define: $\overline{h}^1 \in \mathbb{K}^N$ $(N \leq M)$: for all $\overline{x}^\gamma \in \langle BB^+\rangle$ and $|\gamma| > d$:

$$h_\gamma^1 := h_\gamma^0$$

We are going to prove that the multiplication operators $(M_i^B(\overline{h}^1))_i$ commute and then we apply the previous theorem. In order to do this, we realize that for all $b, b^{'}$ and for all $1 \leq n$:

$$\Lambda(\overline{h}^1)[M_i^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^0)[M_i^B(\overline{h}^1(b)b^{'}] = \Lambda(\overline{h}^0)[x_ibb^{'}]$$

then:

$$\Lambda(\overline{h}^0)[(x_ib - M_i^B(\overline{h}^1)(b))b^{'}] = 0$$

Moreover, as all $(r+1) \times (r+1)$ of $H_{\Lambda(\overline{h}^0)}^{B^+B^+}$ vanish and $H_{\Lambda(\overline{h}^0)}^{B}$ is invertible, then:

$$\Lambda(h^0)[(x_ib - M_i^B(\overline{h}^1(b))b^{''}] = 0$$

for all $b^{''} \in B^+$:

$$\Lambda(\overline{h}^0)[M_i^B(\overline{h}^1)(b)b^{''}] = \Lambda(\overline{h}^0)[x_ibb^{''}] \tag{8.2}$$

for all $b^{''} \in B^+$.
If we fix $b \in B$ and $1 \leq i < j \leq n$. We have:
$\Lambda(\overline{h}^1)[M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^0)[M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^0)[M_j^B(\overline{h}^1)(b)x_ib^{'}]$. For all $b^{'} \in B$. By 8.2 we have:

$$\Lambda(\overline{h}^1)[M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^0)[M_j^B(\overline{h}^1)(b)x_ib^{'}] = \Lambda(\overline{h}^0)[x_jbx_ib^{'}]$$

Then we get:
$\Lambda(\overline{h}^1)[M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^0)[x_jbx_ib^{'}] = \Lambda(\overline{h}^0)[x_ibx_jb^{'}] = \Lambda(\overline{h}^1)[M_j^B(\overline{h}^1) \circ M_i^B(\overline{h}^1)(b)b^{'}]$
thus:

$$\Lambda(\overline{h}^1)[M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b)b^{'}] = \Lambda(\overline{h}^1)[M_j^B(\overline{h}^1) \circ M_i^B(\overline{h}^1)(b)b^{'}]$$

for all $b' \in B$. As $H^B_{\Lambda(\overline{h}^1)} = H^B_{\Lambda(\overline{h}^0)}$ is invertible, we obtain:

$$M_i^B(\overline{h}^1) \circ M_j^B(\overline{h}^1)(b) = M_j^B(\overline{h}^1) \circ M_i^B(\overline{h}^1)(b)$$

for all $b \in B$ and $1 \leq i < j \leq n$. $\square$

Example 8.10. *Let $B = \langle 1 \rangle$ and $\Lambda \in \langle B^+ \rangle_{\leq 0}$ defined as follows:*

$$\begin{array}{cccc} \Lambda : & \langle 1 \rangle & \longmapsto & \mathbb{K} \\ & 1 & \longmapsto & 1 \end{array}$$

Does $\Lambda \in \langle B \rangle^*$ admit an extension $\widetilde{\Lambda} \in R^*$ with $H_{\widetilde{\Lambda}}$ of rank r and B a basis of $A_{\widetilde{\Lambda}}$?. And in the affirmative case, is there unique?

First, we have:

$$H^B_\Lambda = (1)$$

then $\det(H_\Lambda)^B = 1 \neq 0$.
On the other hand, taking $\overline{h} = (h_1, ..., h_n) \in \mathbb{K}^n$, with:

$$H^{B^+}_\Lambda = \left(\begin{array}{c|ccccc} & 1 & x_1 & x_2 & \cdots & x_n \\ \hline 1 & 1 & h_1 & h_2 & \cdots & h_n \\ x_1 & h_1 & h_1^2 & h_1 h_2 & \cdots & h_1 h_n \\ x_2 & h_2 & h_2 h_1 & h_2^2 & & h_2 n_n \\ \vdots & \vdots & & \ddots & & \vdots \\ x_n & h_n & h_1 h_n & \cdots & \cdots & h_n^2 \end{array} \right)$$

All the $(2) \times (2)$ minors of $H^{B^+(\overline{h})}_\Lambda$ vanish for all $\overline{h} \in \mathbb{K}^n$. Then by the previous theorem Λ admits an extension $\widetilde{\Lambda} \in (\mathbb{K}[x_1, ..., x_n])^*$. Note, that in this case the extensions are $\widetilde{\Lambda} = ev(h_1, ...h_n)$. Moreover, if we take for example $h_1 = ... = h_n = 0$ the extension is unique , and in this case is $ev(0, ..., 0)$ such that $\Lambda(0, ...0) = ev(0, ..., 0)$ over $\langle B^+ B^+ \rangle$.

Proposition 8.11. *Let $B = \{\overline{x}_1^\beta, ..., \overline{x}_r^\beta\}$ be a set of monomials of degree at most d, connected to 1. Then, the linear form $\Lambda \in \langle B^+ B^+ \rangle^*_{\leq d}$ admits an extension $\widetilde{\Lambda} \in R^*$ such that $H_{\widetilde{\Lambda}}$ is of rank r with B a basis of $A_{\widetilde{\Lambda}}$ if and only if there exists $\overline{h}$:*

- *i)*

$$\mathbb{H}^{B^+}_{\Lambda(\overline{h})} = \begin{pmatrix} \mathbb{H} & \mathbb{G} \\ \mathbb{G}^t & \mathbb{J} \end{pmatrix} \tag{8.3}$$

  where $\mathbb{H} = \mathbb{H}^B_{\Lambda(\overline{h})}$ and $\mathbb{G} = \mathbb{H}\mathbb{W}$ and $\mathbb{J} = \mathbb{W}^t \mathbb{H} \mathbb{W}$ for some matrix $\mathbb{W} \in \mathbb{K}^{|B| \times |\partial B|}$

- *ii) $H^B_{\Lambda(\overline{h})}$ is invertible*

where $\Lambda(\overline{h}) \in \langle B^+ B^+ \rangle^*$ is definided as follows:

$$\Lambda(\overline{h})(\overline{x}^{\gamma}) = \begin{cases} \Lambda(\overline{x}^{\gamma}) & \textit{if } |\gamma| \leq d; \\ h_{\gamma} & \textit{in other case.} \end{cases}$$

Proof. If there exists $\overline{h}$, such that $\mathbb{G} = \mathbb{H}\mathbb{W}$, and $\mathbb{J} = \mathbb{W}^t\mathbb{H}\mathbb{W}$ for some matrix $\mathbb{W} \in K^{|B|\times|\partial B|}$, and $\det(\mathbb{H}^{B}_{\Lambda(\overline{h})}) \neq 0$, then:

$$\mathbb{H}^{B+}_{\Lambda(\overline{h})} = \begin{pmatrix} \mathbb{H} & \mathbb{H}\mathbb{W} \\ \mathbb{W}^t\mathbb{H} & \mathbb{W}^t\mathbb{H}\mathbb{W} \end{pmatrix} \tag{8.4}$$

$H^{B+}_{\Lambda(\overline{h})}$ is of rank r and then $\overline{h}$ is a solution for the previous theorem, then there exists an extension $\widetilde{\Lambda} \in R^*$ such that $H_{\widetilde{\Lambda}}$ is of rank r and B a basis of $A_{\widetilde{\Lambda}}$.
Conversely, if there exists an extension $\Lambda \in R^*$ such that $H_{\widetilde{\Lambda}}$ is of rank r and B a basis of $A_{\widetilde{\Lambda}}$. We define $\overline{h}^0 \in \mathbb{K}^M$ (for some $M \in \mathbb{N}$) as follows: for all $\overline{x}^{\alpha} \in \langle B^+B^+\rangle$ such that $\alpha > d$ we have:

$$h^0_{\alpha} := \Lambda(\overline{x}^{\alpha})$$

then $\overline{h}^0$ is solution for the previous theorem, then $rank(\mathbb{H}^{B+}_{\Lambda(\overline{h}^0)}) = rank(\mathbb{H}^{B}_{\Lambda(\overline{h}^0)}) = r$. Let us decompose $\mathbb{H}^{B+}_{\Lambda(\overline{h}^0)}$ as 8.3: we know that $\mathbb{H}^{B+}_{\Lambda(\overline{h})}$ is of the form:

$$\mathbb{H}^{B+}_{\Lambda(\overline{h})} = \overbrace{\begin{pmatrix} \mathbb{H} & \mathbb{G} \\ \mathbb{G}^t & \mathbb{J} \end{pmatrix}}^{B \qquad \partial B} \tag{8.5}$$

but, as $rank(\mathbb{H}^{B+}_{\Lambda(\overline{h}^0)}) = rank(\mathbb{H}) = r$, then the image of $\mathbb{G}$ is in the image of $\mathbb{H}$, then there exists $\mathbb{W} \in \mathbb{K}^{|B|\times|\partial B|}$ such that $\mathbb{G} = \mathbb{H}\mathbb{W}$. We realize that $\mathbb{W} \in \mathbb{K}^{|B|\times|\partial B|}$ is the matrix of the following map:

$$\Omega_{\partial B} : \ \langle \partial B \rangle \ \rightarrow \ \langle B \rangle = A_{\widetilde{\Lambda}}//$$

which is the projection of the border in B, then we have , for all $b, b^{'} \in \partial B$:

$$\widetilde{\Lambda}[bb^{'}] = \widetilde{\Lambda}[\Omega_{\partial B}(b)\Omega_{\partial B}(b^{'})] = \Lambda(\overline{h}^0)[\Omega_{\partial B}(b)\Omega_{\partial B}(b^{'})].$$

Therefore:

$$\mathbb{J} = \mathbb{W}^t\mathbb{H}\mathbb{W}$$

□

Example 8.12. *Let $\tau = \delta_1^2 + \delta_2^2 + \delta_3^2$ defined as 7.1, let us see that can be decomposed as 8.3. We have the following matrix:*

$$\mathbb{H}_{\tau}^{B^{+}} = \left(\begin{array}{c|ccccccccccccc}
 & 1 & x_1 & x_2 & x_3 & x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 & x_1^3 & x_1^2x_2 & x_1^2x_3 \\
\hline
1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\
x_1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2^2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_3^2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^2x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^2x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

with $B = \left\langle 1, x_1, x_2, x_3, x_1^2 \right\rangle$ *basis of* $A_\tau = R[x_1, x_2, x_3]/I_\tau$ *and* $I_\tau = (x_1^2 - x_2^2, x_1^2 - x_3^2, x_1x_2, x_1x_3, x_2x_3)$

In order to compute $\mathbb{W} \in \mathbb{K}^{|B| \times |\partial B}$, we know that $\mathbb{W}$ is the matrix of the projection:

$$\Omega_{\partial B} : \quad \langle \partial B \rangle \quad \to \quad \langle B \rangle \ \textit{module}\ \mathrm{I}_\tau$$

$$\mathbb{W} = \left(\begin{array}{c|cccccccc}
 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 & x_1^3 & x_1^2x_2 & x_1^2x_3 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1^2 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{array}\right)$$

and indeed:

$$\mathbb{G} = \mathbb{H}\mathbb{W} \text{ and } \mathbb{J} = \mathbb{W}^t\mathbb{H}W$$

Chapter 9

Symmetric tensor decomposition algorithm

This algorithm for decomposing a symmetric tensor as sum of rank one symmetric tensors generalizes the algorithm of Sylvester, and was devised by Bernard Mourrain and his team. First of all, we will introduce two easy examples for decomposing of homogeneous polynomials, and then we will describe this algorithm.

Notation 9.1. *For all $f \in S_d$ we denote $\underline{f} := f(1, x_1, ..., x_n)$.*

Example 9.2. *Consider a tensor of dimension* 3 *and order* 3 *,which corresponds to the following homogeneous polynomial:*

$$f(x_0, x_1, x_2) = x_0^3 + 3x_0^2x_1 + 3x_0^2x_2 + 3x_0x_1^2 + 6x_0x_1x_2 + 3x_0x_2^2 + x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3$$

We may assume without loss of generality, that at least one variable, say x_0 , all its coefficients in the decomposition are non-zero, then we deshomogenize f with respect to this variable:

$$\underline{f} := f(1, x_1, x_2)$$

And under τ defined in 4.2 $\underline{f}$ is mapped to:

$$\underline{f}^* = \overline{d}^{(0,0)} + \overline{d}^{(1,0)} + \overline{d}^{(0,1)} + \overline{d}^{(2,0)} + \overline{d}^{(1,1)} + \overline{d}^{(0,2)} + \overline{d}^{(3,0)} + \overline{d}^{(2,1)} + \overline{d}^{(1,2)} + \overline{d}^{(0,3)}$$

defined in $K[x_1, x_2]_{\leq 3}$. First, we prove with $B = \langle 1 \rangle$ as a basis, and we obtain:

$$\mathbb{H}_{\underline{f}^*}^{B^+} = \left(\begin{array}{c|ccc} & 1 & x_1 & x_2 \\ \hline 1 & 1 & 1 & 1 \\ x_1 & 1 & 1 & 1 \\ x_2 & 1 & 1 & 1 \end{array} \right) \tag{9.1}$$

In this case $\mathbb{H}_{\underline{f}^*}^{B} = (1)$, $\mathbb{H}_{x_1 * \underline{f}^*}^{B} = (1)$ and $\mathbb{H}_{x_2 * \underline{f}^*}^{B} = (1)$. Then:

$$M_{x_1}^B = (H_{\underline{f}^*}^B)^{-1} H_{x_1 * \underline{f}^*}^B = (1)$$
$$M_{x_2}^B = (H_{\underline{f}^*}^B)^{-1} H_{x_2 * \underline{f}^*}^B = (1)$$

The multiplication operators commute and by the theorem 8.7, $\underline{f}^*$ admits an extension $\Lambda \in R^*$, with $rank(\mathbb{H}_\Lambda) = r$. Moreover, this extension is of the form $\Lambda = \sum_{i=1}^r \lambda_i ev(\xi_i)$ with $\lambda_i \neq 0$ and ξ_i distinct points of K^2 if and only if $rank(H_\Lambda) = r$ and I_Λ is a radical ideal. I_Λ is a radical ideal since $I_\Lambda = kernel(H_\Lambda^B) = kernel(H_{\underline{f}^*}^B) = (x_1 - 1, x_2 - 1)$. In this case $r = 1$, and in order to

recover the point ξ we recall that the eigenvalues of the operators M_{x_i} are the $i-th$ coordinates of the root ξ, and the common eigenvector are the $ev(\xi)$. The eigenvalue of M_{x_1} is 1, then $\xi_1 = 1$ and the eigenvalue of M_{x_2} is 1, then $\xi_2 = 1$ .Then $\Lambda = \lambda_i ev(1,1)$.
Recall that the coefficient of x_0 are considered to be one. Thus the polynomial admits a decomposition:

$$f(x_0, x_1, x_0) = \lambda_1(x_0 + x_1 + x_2)^3$$

We can compute λ_1 easily equating coefficients in the same monomials. Doing that we deduce:

$$f(x_0, x_1, x_2) = (x_0 + x_1 + x_2)^3$$

that is the corresponding tensor is of rank 1.

Example 9.3. *Consider a tensor of dimension 3 and order 3, which corresponds to the following homogeneous polynomial:*

$$f(x_0, x_1, x_2) = 3x_0^2x_1 + 3x_0^2x_2 + 3x_0x_1^2 + 6x_0x_1x_2 + 3x_0x_2^2 + x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3$$

We deshomogenize f with respect to the variable x_0, and we denote:

$$\underline{f} = f(1, x_1, x_2)$$

Under τ defined in 4.2 $\underline{f}$ is mapped to:

$$\underline{f}^* = \overline{d}^{(1,0)} + \overline{d}^{(0,1)} + \overline{d}^{(2,0)} + \overline{d}^{(1,1)} + \overline{d}^{(0,2)} + \overline{d}^{(3,0)} + \overline{d}^{(2,1)} + \overline{d}^{(1,2)} + \overline{d}^{(0,3)}$$

$\underline{f}^* \in (\mathbb{K}[x_1, x_2]_{\leq 3})^*$. If we take $B = \langle 1, y \rangle$ then:

$$\mathbb{H}^{B+}_{\underline{f}^*(\underline{h})} = \left(\begin{array}{c|ccccc} & 1 & x_1 & x_2 & x_1x_2 & x_2^2 \\ \hline 1 & 0 & 1 & 1 & 1 & 1 \\ x_1 & 1 & 1 & 1 & 1 & 1 \\ z_2 & 1 & 1 & 1 & 1 & 1 \\ x_1x_2 & 1 & 1 & 1 & h_{22} & h_{31} \\ x_1^2 & 1 & 1 & 1 & h_{31} & h_{20} \end{array} \right)$$

In this case,

$$\mathbb{H}^{B}_{\underline{f}^*(\overline{h})} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $\mathbb{H}^{B}_{\underline{f}^*(\overline{h})}$ is invertible. Moreover, we have:

$$\mathbb{H}_{x_1 * \underline{f}^*} = \left(\begin{array}{c|cc} & x_1 & x_1^2 \\ \hline 1 & 1 & 1 \\ x_1 & 1 & 1 \end{array} \right)$$

$$\mathbb{H}_{x_1 * \underline{f}^*} = \left(\begin{array}{c|cc} & x_2 & x_1x_2 \\ \hline 1 & 1 & 1 \\ x_1 & 1 & 1 \end{array} \right)$$

. Therefore:

$$\mathbb{M}^B_{x_1} = (\mathbb{H}^B_{\underline{f^*}})^{-1}\mathbb{H}^B_{x_1*\underline{f^*}} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{M}^B_{x_2} = (\mathbb{H}^B_{\underline{f^*}})^{-1}\mathbb{H}^B_{x_2*\underline{f^*}} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Obviously, the multiplication operators commute and by the theorem 8.7, $\underline{f^*}$ admits an extension $\Lambda \in R^*$ with H_Λ of rank r. This extension can be written as $\Lambda = \sum_{i=1}^r ev(\xi_i)$ by the theorem 7.16 if and only if H_Λ is of rank r and I_Λ is a radical ideal. Then we only have to see that $I_\Lambda = kernel(\mathbb{H}^{B+}_{\underline{f^*}})$ is a radical ideal.Then $\underline{v} \in kernel(\mathbb{H}^{B+}_{\underline{f^*}})$ if and only if:

$$\mathbb{H}^{B+}_{\underline{f^*}}\, v = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The solutions are:

$$v_1 = 0 \; v_2 = -a - b - c \; v_3 = a \; v_4 = b \; v_5 = c$$

for $a, b, c \in \mathbb{K}$. Then $p \in kernel(\mathbb{H}^{B+}_{f^*})$ if and only if, $p = a(x_2 - x_1) + b(x_1x_2 - x_1) + c(x_2^2 - x_2) +$ terms of degree greater than 3. Thus we obtain $kernel(\mathbb{H}^{B+}_{f^*}) = kernel(\mathbb{H}^{B+}_{\Lambda}) = I_\Lambda = (x_2 - x_1, x_2x_1 - x_1, x_1^2 - x_1)$ which is an radical ideal.

Therefore $\Lambda = \sum_{i=1}^r \lambda_i ev(\xi)$, where $r = 2$ we can recover the points ξ_1, ξ_2 by two different ways:

- 1) The eigenvalues of $M^B_{x_1}$ are $\alpha_1 = 0$ and $\alpha_2 = 1$, and the eigenvector of $(M^B_i)^t$, associated with $\alpha_1 = 0$ is:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

  and the eigenvector associated with $\alpha_2 = 1$ is only:

$$\xi_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- 2) We know due to the theorem 7.12 that ξ_1 and ξ_2 are the roots of I_Λ: $Z(I_\Lambda) = \{(0,0), (1,1)\}$

Recall that the coefficient of x_0 are considered to be one. Thus the polynomial admits a decomposition:

$$f(x_0, x_1, x_2) = \lambda_1(x_0 + x_1 + x_2)^3 + \lambda_2(x_0)^3$$

We can compute λ_1 and λ_2 easily . Doing that:

$$f(x_0, x_1, x_2) = (x_0 + x_1 + x_2)^3 - (x_0)^3$$

which is a tensor of rank 2.

9.1 Symmetric tensor decomposition algorithm

The algorithm for decomposition a symmetric tensor as a sum or rank one symmetric tensors generalizes the algorithm of Sylvester, devised for dimension two tensors.

In this algorithm we may assume without loss of generality, that for at least one variable, say x_0, all its coefficients in the decomposition are non-zeros, i.e. $k_{i,0} \neq 0$ for $1 \leq i \leq r$.

Symmetric tensor decomposition algorithm
Input: A homogeneous polynomial $f(x_0, .., x_n)$ of degree d
Output: A decomposition of f as $f = \sum_{i=1}^{r} \lambda_i k_i(\overline{x})^d$ with r minimal

1. Compute the coefficients of $\underline{f}^*$: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$.
2. Initialize $r := 0$
3. Increment $r := r + 1$
4. Specialization:
   - Take any basis B connected to 1 with $|B| = r$
   - Build the matrix $H^{B+}_{\underline{f^*}(\overline{h})}$ with the coefficients c_α.
   - If there exists any minor of order $r+1$ in $H^{B+}_{\underline{f^*}(\overline{h})}$, without coefficients depending on $\overline{h}$, different to zero, try another specialization. If cannot be obtained go to step 3.
   - Else if all minors of order $r+1$ in $H^{B+}_{\underline{f^*}(\overline{h})}$, without coefficients depending on $\overline{h}$, vanish, compute $\overline{h}$ s.t:
     - $\det(H^{B}_{\underline{f^*}(\overline{h})}) \neq 0$
     - the operators $M_i^B(\overline{h}) := (H^{B}_{\underline{f^*}(\overline{h})})^{-1}(H_{x_i * \underline{f^*}(\overline{h})})$ commute
     - the eigenvalues of $M_i^B(\overline{h})$ are simple

     If there not exist such $\overline{h}$ try another specialization. If cannot be obtained go to step 3.
   - Else if there exists such $\overline{h}$ compute the eigenvalues $\xi_{i,j}$ and the eigenvectors v_j s.t $M_i^B v_j = \xi_{i,j} v_j$ for $i = 1, ..., n$ and $j = 1, ..., r$.
5. Solve the linear system in (λ_j) s.t $f(\overline{x}) = \sum_{i=1}^{r} \lambda_j k_i(\overline{x})^d$ where $k_i(\overline{x}) = (x_0 + v_{i,1}x_1 + ... + v_{i,n}x_n)$.

Remark 9.4. *This algorithm stops as we saw in Lemma 4.23, let $f \in S_d$ there exists $k_1(\overline{x}), ..., k_s(\overline{x})$ with $s < \infty$ such that: $f = \sum_{i=1}^{s} k_i(\overline{x})^d$. Once, the algorithm has computed the parameters $\overline{h}$ such that $\det(H^{B}_{\underline{f^*}(\overline{h})}) \neq 0$ and the operators $M_i = (H^{B}_{\underline{f^*}(\overline{h})})^{-1} H^{B}_{x_i * \underline{f^*}(\overline{h})}$ commute, we need to ensure that I_Λ is a radical ideal, and this holds true when the eigenvalues are simple.*

Remark 9.5. *It can be pointed out that ith-coordinate of several distinct points could be the same, i.e. $\xi_{j,i} = \xi_{k,i}$ with $\xi_j \neq \xi_k$, and then the eigenvalues of M_i are not simple. For this reason, sometimes it is convenient to check that the eigenvalues are simple in the matrix M_p instead of M_i, with a random polynomial p, for example $p = \sum_{i=1}^{n} a_i x_i$. In this case, it would be improbable that if the points are distinct not to obtain simple eigenvalues.*

Example 9.6. *Let us apply the algorithm in order to obtain the decomposition of the homogeneous polynomial of dimension 3 and order 4:*
$f(x,y,z) = 3x^4 + 4x^3y - 4x^3z + 6x^2y^2 - 12x^2yz + 18x^2z^2 + 4xy^3 - 12xy^2z + 12xyz^2 - 4xz^3 + y^4 - 4y^3z + 6y^2z^2 - 4yz^3 + 3z^4$

We deshomogenize with $x = 1$ and compute the coefficients $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$. And we get the following element of R_4^*:

$\underline{f^*} = 3\overline{d}^{(0,0)} + \overline{d}^{(1,0)} - \overline{d}^{(0,1)} + \overline{d}^{(2,0)} - \overline{d}^{(1,1)} + 3\overline{d}^{(0,2)} + \overline{d}^{(3,0)} - \overline{d}^{(2,1)} + \overline{d}^{(1,2)} - \overline{d}^{(0,3)} + \overline{d}^{(4,0)} - \overline{d}^{(3,1)} + \overline{d}^{(2,2)} - \overline{d}^{(1,3)} + 3\overline{d}^{(0,4)}$.

Taking a connected basis with $r = 1$ and $r = 2$ elements, we find minors of order 2 and 3 respectively, in $H_{f^*}^{B^+}$ different from zero hence, f has not rank equal to 1 or 2.

We follow to $r = 3$ and we take the connected basis $B = \{1, y, z\}$, then $B^+ = \{1, y, z, yz, y^2, z^2\}$, we obtain the following matrix:

$$H_{\underline{f^*}}^{B^+} = \begin{pmatrix} 3 & 1 & -1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 3 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & -1 \\ 3 & 1 & -1 & 1 & 3 & -1 \\ -1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix} \tag{9.2}$$

All the minors of order 4 vanish, then we can continue with the algorithm,and we realize that:

$$\det(H_{\underline{f^*}}^B) = \det \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{pmatrix} \neq 0 \tag{9.3}$$

We need that the multiplication operators commute that is $M_y^B M_z^B = M_z^B M_y^B$, and we have:

$$M_y^B = (H_{\underline{f^*}}^B)^{-1} H_{y*\underline{f^*}} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$M_y^B = (H_{\underline{f^*}}^B)^{-1} H_{y*\underline{f^*}} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & 0 \\ \frac{-1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -1 & -1 & 3 \\ -1 & -1 & 1 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

And it holds true that the multiplication operators commute, that is $M_y^B M_z^B = M_z^B M_y^B$. It should be noted, that in this step the algorithm has to compute $\overline{h}$ such that the multiplication operators commute but in this case all our entries are known. The following step is to ensure the eigenvalues of $(M_z^B)^t$ and $(M_y^B)^t$ are simple, but in this case the eigenvalues of $(M_z^B)^t$ are $x_1 = -1$, $x_2 = -1$ and $x_3 = 1$, and the eigenvalues of $(M_y^B)^t$ are $x_1 = 0$, $x_2 = 0$ and $x_3 = 1$, we are in the case of the 9.5 because if we take for example $p = y + z$ then the eigenvalues of M_p^B are $x_1 = 2$, $x_2 = -2$ and $x_3 = 0$ and these are simple. Then we can continue with the algorithm and compute the eigenvectors of M_z^t which are:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \xi_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xi_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The coordinates of the eigenvectors correspond to the elements $\{1, y, z\}$. Thus we can recover the coefficients of y and z in the decomposition from coordinates of the eigenvectors. Recall that the coefficients of x are considered to be one. Thus, the polynomial admits a decomposition:

$$f = \lambda_1(x + z)^4 + \lambda_2(x + y - z)^4 + \lambda_3(x - z)^4$$

It remains to compute $\lambda's$. We can do this easily by solving an over-determined linear system, which we know has always a solution, since the decomposition exists. Doing this last step, we deduce:

$$f(x, y, z) = (x + z)^4 + (x + y - z)^4 + (x - z)^4 \tag{9.4}$$

9.2 Future work

There are some questions that remain open: the complexity of the algorithm, the computing of the decomposition when some entries of the tensor are not known (case of missing data) and to extend the algorithm to non-symmetric tensors.

The theorem of Alexander and Hirschowitz states [12], that the generic rank is always the expected one, with a finite list of exceptions. However, it has not received any answer yet, either for non symmetric tensors, or for decompositions in the real field. Nevertheless, we know there is always an open subset where the general rank is the same as the complex one. In other words, for given order and dimension the smallest typical rank in the real field coincides with the generic rank in the complex field, (see [14],[15],[16],[10]). We can see in [14], in order to exhibit more than two typical ranks, that it seems necessary to consider tensors of order higher than 3. An elementary example would be:

$$2x^3 - 6xy^2 = (x + \sqrt{-1}y)^3 + (x - \sqrt{-1})^3 = (2x^3) - (x + y)^3 - (x - y)^3 \tag{9.5}$$

In this case the complex rank is 2 and the real rank is 3.

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