GNS construction

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- $\mathbb{R}[x] := \mathbb{R}[x_1, ..., x_n]$, $x = (x_1, ..., x_n)$
- $\mathbb{R}[x]_t := \{p \in \mathbb{R}[x] : deg(p) \leq t\}$
- $\mathbb{R}[x]_t^* := \{L : \mathbb{R}[x]_t \to \mathbb{R} \text{ linear form}\}$
- $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$
- $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{R}[x]_d \leftrightarrow \alpha \in \mathbb{N}_d^n$, (i.e. $|\alpha| := \alpha_1 + ... + \alpha_d \leq d$)

Definition

We say that a linear form $L \in \mathbb{R}[x]_{2d}^*$ *is integration with respect to a measure* μ, *if* $L(p) = \int p d\mu \ \forall p \in \mathbb{R}[x]_{2d}$.

Definition

Given a linear form $L \in \mathbb{R}[x]_{2d}^*$, *we can define its respective moment matrix, indexed by* $(x^\alpha)_{\alpha, |\alpha| \leq d}$, *in this way:*
$M_L = ((L(x^{\alpha+\beta}))_{\alpha,\beta,|\alpha|\leq d,|\beta|\leq d}$

Example

For $L = ev(0,1) \in \mathbb{R}[x_1, x_2]_2^*$, *its moment matrix is:*

$$M_L = \begin{pmatrix} L(1) & L(x) & L(y) \\ L(x) & L(x^2) & L(xy) \\ L(y) & L(xy) & L(y^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Bayer and Teichmann Theorem

A linear form $L \in \mathbb{R}[x]^*_{2d}$, *is a integration with respect to a measure if and only if there exists an integer* N, *nodes* $x_1, \ldots, x_N \in \mathbb{R}^n$ *and weights* $\lambda_1 > 0, \ldots, \lambda_N > 0$, *such that,* $L(p) = \sum_{i=1}^{N} \lambda_i p(x_i)$, $\forall p \in \mathbb{R}[x]_{2d}$.

Remark

If $L \in \mathbb{R}[x]^*_{2d}$, *and* L *is integration with respect to a measure, then* $L(\sum \mathbb{R}[x]^2_d) \subseteq \mathbb{R}_{\geq 0}$, *that is* $M_L \geq 0$

Given $L \in \mathbb{R}[x]_{2d}^*$, and $L(\sum \mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$

- Is L ,or at least $L_{|\mathbb{R}[x]_k}$ for some $k \leq 2d$, integration with respect to a measure?
- If L is integration with respect to a measure, How can I recover the nodes and weights of such a measure?

For this we will use a variation of the **G**elfand-**N**aimark-**S**egal construction.

Given $L \in \mathbb{R}[x]_{2d}^*$, and $L(\sum \mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$. We would like to find :

- $x_1, ..., x_N$ points in $\mathbb{R}^n$, $\lambda_1, ..., \lambda_N > 0$ weights such that $L = \sum_{i=1}^N \lambda_i ev(x_i)$

That is similar to find:

- A finite dimensional space V and commuting self-adjoint endomorphisms $M_1, ..., M_n$ of V and $a \in V$ such that $L(p) = < p(M_1, ..., M_n)a, a >$

Idea: if we had $L(p^2) > 0$ for all $p \neq 0$:

- $V := \mathbb{R}[x]$
- $< p, q > := L(pq)$
- $M_i : \mathbb{R}[x] \to \mathbb{R}[x] : p \mapsto x_i p$, $i \in \{1, ..., n\}$
- $a := 1 \in \mathbb{R}[x]$

Let $L \in \mathbb{R}[x]_{2d}^*$ with $L(\sum \mathbb{R}[x]^2) \subseteq \mathbb{R}_{\geq 0}$

- $U_L := \{p \in \mathbb{R}[x]_d : \forall q \in \mathbb{R}[x]_d : L(pq) = 0\}$ **GNS kernel**
- $V_L := \frac{\mathbb{R}[x]_d}{U_L}$ **GNS representation space of L**
- $< \overline{p}, \overline{q} >_L := L(pq)$ $(p, q \in \mathbb{R}[x]_d)$ **GNS scalar product**
- Then we have built $(V_L, <,>_L)$ a Euclidean Vector Space

We had a euclidean vector space $(V_L, <, >_L)$:

- Π_L:$V_L \to \{\overline{p} : p \in \mathbb{R}[x]_{d-1}\}$ **orthogonal projection**
- $M_{L,i}$:$\Pi_L V_L \to \Pi_L V_L : \overline{p} \to \Pi_L(\overline{X_i p})$ $(p \in \mathbb{R}[x]_{d-1})$ and $i \in \{1, ..., n\}$.

$M_{L,i}$, called **i-th truncated GNS multiplication operator**, is self-adjoint endomorphism of $\Pi_L V_L$

Simultaneaus Diagonalization

Let V be a finite euclidean vector space, and $M_1, ..., M_n$, commuting selfadjoint endomorphisms of V. Then there exists an orthonormal basis of V which contains all the common eigenvectors of M_i for $i = 1, ..., n$.

Proposition (Schweighofer)

Let $L \in \mathbb{R}[x]_{2d}^*$ with $L(\sum \mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$. Suppose that the truncated GNS multiplication operators of L commute. And we set:

$$W_L := \{\sum_{i=1}^m p_i q_i : m \in \mathbb{N}_0, p_i, q_i \in \mathbb{R}[x]_{d-1} + U_L\} \supseteq \mathbb{R}[x]_{2(d-1)}$$

Then $L_{|W_L}$ is integration with respect to a finitely supported measure.

Definition

Let $L \in \mathbb{R}[x]_{2d}^*$ *with* $L(\mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$ *. We say* L *is flat if* $\mathbb{R}[x]_{d-1} + U_L = \mathbb{R}[x]_d$.

Proposition (Schweighofer)

Let $L \in \mathbb{R}[x]_{2d}^*$ *with* $L(\mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$ *and* $L' := L_{|\mathbb{R}[x]_{2(d-1)}}$*. Then the following are equivalent:*

- L *is flat.*
- $\Pi_L(V_L) = V_L$
- *For all* $\alpha \in^n :(|\alpha| = d \Rightarrow \exists p \in \mathbb{R}[x]_{d-1}$ *such that* $x^\alpha - p \in U_L)$
- $dim(V_{L'}) = dim(V_L)$
- $(L(X^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d-1}$ *and* $(L(X^{\alpha+\beta}))_{|\alpha|,|\beta| \leq d}$ *have the same rank.*

Proposition (Schweighofer)

Let $L \in \mathbb{R}[x]_{2d}^*$ *with* $L(\mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$. *If* L *is flat then the GNS multiplication operators commute.*

Corollary

Let $L \in \mathbb{R}[x]_{2d}^*$ *with* $L(\mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$ *in one variable. Then* $L_{|\mathbb{R}[x]_{2(d-1)}}$ *comes from a measure.*

Corollary (Curto and Fialkow,1996)

Let $L \in \mathbb{R}[x]_{2d}^*$ *with* $L(\mathbb{R}[x]_d^2) \subseteq \mathbb{R}_{\geq 0}$. *If* L *is flat then* $L_{|\mathbb{R}[x]_{2(d-1)}}$ *comes from a mesure.*

$M_{L,i}$ commute does not necessarily imply that L is flat:

Example

Let $L = \frac{1}{3}(ev(1,2) + ev(2,3) + ev(4,5)) \in \mathbb{R}[x_1, x_2]_4^*$, *(with points in the linie* $y = x + 1$*) here we have:*

$$M_{L,1} = \begin{pmatrix} 3 & 1{,}6330 \\ 1{,}6330 & 3 \end{pmatrix} \quad M_{L,2} = \begin{pmatrix} 4 & 1{,}6330 \\ 1{,}6330 & 4 \end{pmatrix}$$

$M_{L,1}M_{L,2} = M_{L,2}M_{L,1}$ *and L is not flat. And* $L' := \frac{1}{2}(ev(1.37, 2.37) + ev(4.63, 5.63))$ *with* $L' = L_{|\mathbb{R}[x]_2}$.

Example

Let $L = \frac{1}{6}(ev(1,1) + ev(-1,-1) + ev(-1,1) + ev(1,-1) + ev(2,-2) + ev(-2,2)) \in \mathbb{R}[x_1, x_2]_6^*$, *(points in* $x^2 = y^2$ *) here L is not flat and the operators commute.*

Interested

$L \in \mathbb{R}[x]^*_{2d}$ with $L(\sum \mathbb{R}[x]^2_d) \subseteq \mathbb{R}_{\geq 0}$ such that L is not necessarily flat and $M_{L,i}$ commute

Remark

Let $L = \sum_{i=1}^r \lambda_i ev_{x_i} \in \mathbb{R}[x,y]^*_{2d}$ (two variables) such that $x_i \in \{(x,y)|y = mx + n\} \subset \mathbb{R}^2$. Then the GNS multiplication operators commute.

Remark

Let $L = \sum_{i=1}^r \lambda_i ev_{x_i} \in \mathbb{R}[x]^*_{2d}$ with $r \leq dim(\mathbb{R}[x]_{d-1})$ then "almost always" L is flat. For example for $n = 2$ if we take points in the circle sometimes L is not flat.

Polynomial optimization problem: minimize $p[x]$ over $x \in \mathbb{R}^n$, such that $p \in \mathbb{R}[x]$. We denote $deg(p) := d$.

First SDP (Nie-Demmel-Sturmfels)

(NDS_k)
min $L(p)$ *with:*

- $L \in \mathbb{R}[x]^*_{2k}$
- $L(\sum \mathbb{R}[x]^2_k) \subseteq \mathbb{R}_{\geq 0}$
- $L(\frac{\partial p}{\partial x_i}(x^\alpha)) = 0$

for all α *with* $|\alpha| + d - 1 \leq 2k$

Second SDP (Heuristic-Program)

$(H_{k,\lambda})$
min $(1-\lambda)L(p) + \lambda E$

- $L \in \mathbb{R}[x]^*_{2k}$
- $L(\sum \mathbb{R}[x]^2_k) \subseteq \mathbb{R}_{\geq 0}$
- $L \approx$ *flat*

where $\lambda \in [0, 1]$ *is fixed.*

Some examples:

Example 1

With NDS_4 for $p = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$, the Moztkin polynomial. I get the minimizers : $(-16{,}8583, 0), (0, -16{,}8583), (\pm 1, \pm 1), (16{,}583, 0), (0, 16{,}583)$

Example 2

With NDS_4 for $p = x^6 + y^6 + 1 - x^4y^2 - x^2y^4 - x^4 - y^4 - x^2 - y^2 + 3x^2y^2$, the Robinson polynomial. I geth the minimizers: $(\pm 1, \pm 1), (1, 0), (0, 1), (-1, 0), (0, -1)$

Example 3

With NDS_4 fot the polynomial $x^2y^2(x^2 + y^2 - 1)$. I get the minimizers $(-14{,}89, 0), (0, -14{,}89), (0, 14{,}89), (14{,}89, 0), (\pm 0{,}5774, \pm 0{,}5774)$

Example 4

With $H_{3,1/60}$ for the Moztkin polynomial I get (aproximately) the minimizers $(\pm 1, \pm 1), (0, 0)$

Dankeschön!