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Real Algebraic Geometry II – Exercise Sheet 1

Exercise 1 (4P). For $m \in \mathbb{N}_0$ and $g = (g_1, \dots, g_m) \in \mathbb{R}[\underline{X}]^m$, we define the *quadratic module*

$$M(g) := \sum \mathbb{R}[\underline{X}]^2 + \sum \mathbb{R}[\underline{X}]^2 g_1 + \ldots + \sum \mathbb{R}[\underline{X}]^2 g_m$$

generated by $g_1, \dots, g_m$ and the *basic closed semialgebraic set*

$$S(g) := \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\},$$

and we say that g is a *Putinar-tuple* if $S(g)$ is compact and every $f \in \mathbb{R}[\underline{X}]$ that is (pointwise) positive on $S(g)$ lies in $M(g)$.

Show that if $(g_1, \dots, g_m)$ is a Putinar-tuple and $h \in \mathbb{R}[\underline{X}]$, then $(g_1, \dots, g_m, h)$ is also a Putinar-tuple.

Hint: Let $f \in \mathbb{R}[\underline{X}]$ be positive on $S(g, h)$. Find $\lambda > 0$ and $k \in \mathbb{N}_0$ such that

$$f - (1 - \lambda h)^{2k} h > 0$$

on $S(g)$.

Exercise 2 (5P) Let $g \in \mathbb{R}[\underline{X}]^m$.

(a) Show that the following are equivalent:
  (i) For all $p \in \mathbb{R}[\underline{X}]$, there is $N \in \mathbb{N}$ such that $N + p \in M(g)$.
  (ii) There is $N \in \mathbb{N}$ such that $N - \sum_{i=1}^n X_i^2 \in M(g)$.
  (iii) There is $f \in M(g)$ such that $S(f)$ is compact.

  If one of the above conditions is fulfilled, we call $M(g)$ *Archimedean*.

(b) Prove *Putinar's Positivstellensatz*: Let $M(g)$ be Archimedean and $f \in \mathbb{R}[\underline{X}]$ be positive on $S(g)$. Then $f \in M(g)$.

Exercise 3 (6P) Suppose $g \in \mathbb{R}[X]^m$ is a tuple of univariate polynomials with compact $S(g)$. Show that $M(g)$ is Archimedean.

Exercise 4 (5P) Let $(A, \mathscr{O})$ be a topological space. If $\mathscr{O}$ comes from a metric on A we have the following well-known characterization of closed sets:

A set $B \subseteq A$ is closed if and only if every sequence $(x_n)_{n \in \mathbb{N}}$ in B with a limit $x \in A$ fulfills already $x \in B$.

However, we will see on a later exercise sheet that this characterization fails for arbitrary topological spaces. Generalize the above result in the language of (ultra)-filters (instead of sequences) so that it becomes true for every topological space $(A, \mathscr{O})$.

Exercise 5 (4P)

(a) Let I be a set, and for each $i \in I$ let X_i be a nonempty topological space. Let X be the product space $\prod_{i \in I} X_i$. Show that X is $\left\{\begin{matrix}\text{a Hausdorff space}\\ \text{quasicompact}\\ \text{compact}\end{matrix}\right\}$ if and only if each X_i is $\left\{\begin{matrix}\text{a Hausdorff space}\\ \text{quasicompact}\\ \text{compact}\end{matrix}\right\}$.

(b) Show that a topological space M is a Hausdorff space if and only if every ultrafilter on M converges in M to at most one point.

Please submit until Tuesday, May 2, 2017, 11:44 in the box named RAG II near to the room F411.