**Universität Konstanz**
**Department of Mathematics and Statistics**
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Tom-Lukas Kriel
María López Quijorna
Markus Schweighofer

# Real Algebraic Geometry II – Exercise Sheet 3

**Exercise 1** (4P) An subset of a topological space is called *clopen* if it is closed and open. Let $A$ be a commutative ring.

(a) Show that every constructible subset of $\operatorname{sper} A$ is quasicompact.

(b) Show that the constructible subsets of $\operatorname{sper} A$ are exactly the clopen subsets of $\operatorname{sper} A$ with respect to the constructible topology.

**Exercise 2** (4P) Let $A$ be a commutative ring. Show that the set of maximal prime cones of $A$ is a compact subset of $\operatorname{sper} A$.

**Exercise 3** (4P) Let $A$ be a commutative ring. Let $S \subseteq \operatorname{sper} A$ be closed with respect to the constructible topology. Prove that $\overline{S} = \{P \in \operatorname{sper}(A) \mid \exists Q \in S : Q \subseteq P\}$.

**Exercise 4** (6P) Let $A$ be a commutative ring. Show that the correspondence

$$\mathscr{F} \mapsto \bigcap \mathscr{F},$$
$$\{B \in \mathscr{C} \mid A \subseteq B\} \leftarrow\!\shortmid A$$

defines a bijection between the set of filters in the Boolean algebra $\mathscr{C}_A$ of constructible subsets of $\operatorname{sper} A$ and the set of nonempty closed subsets of $\operatorname{sper} A$ with respect to the constructible topology.

**Exercise 5** (4P) Show the following:

(a) If $P \in \operatorname{sper} \mathbb{R}[\underline{X}]$ and $\mathscr{U} := \mathscr{U}_P$, then $P$ is Archimedean if and only if the ultrafilter $\mathscr{U}$ contains a bounded semialgebraic set.

(b) $\mathscr{U} := \{S \in \mathscr{S}_2 \mid \exists N \in \mathbb{N} : \{(x^2, x) \mid x \in \mathbb{R}, x \geq N\} \subseteq S\}$ is an ultrafilter on the Boolean algebra of semialgebraic subsets of $\mathbb{R}^2$ such that $P := P_{\mathscr{U}}$ is a maximal element of $\operatorname{sper} \mathbb{R}[X_1, X_2]$ but $P$ is not Archimedean.

**Exercise 6** (2P) Let $R$ be a real closed field, $m \in \mathbb{N}$ and $P_1, \dots, P_m \in \operatorname{sper} R[\underline{X}]$. Show that there exists a real closed extension field $H$ of $R$ and $x_1, \dots, x_m \in H^n$ such that $P_i = \{f \in H[\underline{X}] \mid f(x_i) \geq 0\}$ for all $i \in \{1, \dots, m\}$.

**Please submit until Tuesday, May 16, 2017, 11:44 in the box named RAG II near to the room F411.**