RIGIDITY RESULTS, INVERSE CURVATURE FLOWS AND ALEXANDROV-FENCHEL TYPE INEQUALITIES IN THE SPHERE

MATTHIAS MAKOWSKI AND JULIAN SCHEUER

ABSTRACT. We prove a rigidity result in the sphere which allows us to generalize a result about smooth convex hypersurfaces in the sphere by Do Carmo-Warner to convex C^2-hypersurfaces. We apply these results to prove $C^{1,\beta}$-convergence of inverse F-curvature flows in the sphere to an equator in $\mathbb{S}^{n+1}$ for embedded, closed, orientable, strictly convex initial hypersurfaces. The result holds for large classes of curvature functions including the mean curvature and arbitrary powers of the Gauss curvature. We use this result to prove Alexandrov-Fenchel type inequalities in the sphere.

CONTENTS

1. INTRODUCTION

This work deals with geometric problems on the $(n + 1)$-dimensional unit sphere $\mathbb{S}^{n+1} \subset \{x \in \mathbb{R}^{n+2} \colon |x| = 1\}$. We assume $n \geq 2$, unless stated otherwise. We are interested in the connection between (analytically) convex hypersurfaces and (geodesically) convex bodies. The notion of convexity of sets is significantly more subtle than in Euclidean space, due to the existence of focal points in the sphere.

A very well known result in this direction by do Carmo and Warner, [4, Theorem 1.1], is the following.

Theorem (Do Carmo, Warner)**.** *Let* $x \colon M^n \to \mathbb{S}^{n+1}$ *be an isometric immersion of a compact, connected, orientable n-dimensional C^∞-Riemannian*

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manifold into the $(n+1)$-sphere of sectional curvature equal to one, and assume that all sectional curvatures of M^n are greater than or equal to one. Then x is an embedding, M^n is diffeomorphic with $\mathbb{S}^n$ and $x(M^n)$ is either totally geodesic or contained in an open hemisphere. In the latter case $x(M^n)$ is the boundary of a convex body in $\mathbb{S}^{n+1}$.

Also compare [1], which deals with strictly convex hypersurfaces.
We will show that some parts of this result can be generalized to nonsmooth, geodesically convex bodies in the sphere. In particular we will prove the following result, for the exact definitions of weakly convex bodies see Section 3.

1.1. **Theorem.** *Let $n \geq 1$ and $\hat{M} \subset \mathbb{S}^{n+1}$ be a weakly convex body in a hemisphere. Let $x_0 \in \mathbb{S}^{n+1}$ be such that $\hat{M}$ is contained in the closed hemisphere $\mathcal{H}(x_0)$. Suppose that $\hat{M}$ satisfies an interior sphere condition at all points $p \in \hat{M} \cap \mathcal{S}(x_0)$. Then either $\hat{M}$ is equal to $\mathcal{H}(x_0)$ or $\hat{M}$ is contained in an open hemisphere.*

With the help of this result, we prove that the strong regularity assumption in [4] is not necessary.

1.2. **Corollary.** *Let $M \subset \mathbb{S}^{n+1}$ be a closed, orientable, connected, convex embedded C^2-hypersurface. Then M is either an equator or M is contained in an open hemisphere and bounds a convex body.*

We apply those rigidity results to treat an inverse curvature flow in the sphere $\mathbb{S}^{n+1}$ of the form

$$(1.1) \qquad \begin{aligned} \dot{x} &= -\Phi(F)\,\nu, \\ x(0) &= x_0, \end{aligned}$$

where $x_0 : \mathbb{S}^n \to \mathbb{S}^{n+1}$ is the embedding of an initial hypersurface $M_0 := x_0(\mathbb{S}^n)$ of class $C^{4,\alpha}$ for some $0 < \alpha < 1$, which is furthermore required to be strictly convex. ν is the corresponding outer normal, $\Phi \in C^\infty(\mathbb{R}_+, \mathbb{R})$, $\Phi(x) = -x^{-p}$, $p > 0$, F is a curvature function evaluated at the principal curvatures of the flow hypersurfaces M_t and $x(t)$ denotes the embedding of M_t. We will show that under certain assumptions, cf. 1.3, the flow exists up to a finite time and converges in $C^{1,\beta}$ to the embedding of an equator.
Curvature flows and their application to geometric inequalities have been treated for over thirty years. Following the ground breaking work by G. Huisken, [17], who considered the mean curvature flow, also inverse, or expanding flows have been considered. Here the works on the inverse curvature flow by C. Gerhardt, [9], as well as J. Urbas, [29], have to be mentioned, where also non-convex hypersurfaces were considered. Similar results have been shown in other ambient spaces and for general p-homogeneous curvature functions, e.g. [13], [14], or [26].
We consider a large class of curvature functions. We allow other homogeneities than 1, in particular our result holds for arbitrary powers of the Gaussian curvature without further pinching assumptions. The detailed assumptions on the curvature function are listed below, whereafter we state the convergence result.

1.3. **Assumption.** *Suppose $F \in C^{2,\alpha}(\Gamma)$, $0 < \alpha < 1$, is a symmetric function, where Γ is the positive cone $\Gamma_+ = \{\kappa = (\kappa_i) \in \mathbb{R}^n \colon \kappa_i > 0 \,\forall\, i \in \{1, \ldots, n\}\}$. We need the following assumptions for the curvature function F:*

- *F is positively homogeneous of degree 1, i.e. $\forall\, \kappa \in \Gamma_+$, $\forall\, \lambda \in \mathbb{R}_+$: $F(\lambda\kappa) = \lambda F(\kappa)$.*
- *F is strictly increasing in each argument: $\forall\, i \in \{1, \ldots, n\}$, $\forall\, \kappa \in \Gamma_+$ there holds $F_i(\kappa) = \frac{\partial F}{\partial \kappa_i}(\kappa) > 0$.*
- *F is positive, $F_{|\Gamma_+} > 0$, and F is normalized, $F(1, \ldots, 1) = n$.*
- *Either:*
    - *(i) F is concave and inverse concave, i.e. $F_{-1}(\kappa_i) := \frac{1}{F(\kappa_i^{-1})}$ is concave.*
    - *(ii) F is concave and F approaches zero on the boundary of Γ_+.*
- *If $p \neq 1$, we assume (ii) is valid.*

The most important examples of curvature functions F being concave and inverse concave are $\left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$, $n \geq k > l \geq 0$, or the power means $(\sum_{i=1}^n \kappa_i^r)^{\frac{1}{r}}$ for $|r| \leq 1$. For a proof of the inverse concavity of these functions see the proofs of [2, Theorem 2.6, Theorem 2.7]. Note that our definition of inverse concave curvature functions differs from Andrews' definition and since we do not understand the proof of [2, Theorem 2.3], we do not see that the definitions coincide in the case of curvature functions which are furthermore concave, homogeneous of degree one and monotonely increasing.

Our exact result concerning the curvature flows is:

1.4. **Theorem.** *Let $\mathbb{S}^n \hookrightarrow M_0 \subset \mathbb{S}^{n+1}$ be an embedded, strictly convex hypersurface of class $C^{4,\alpha}$. Let F be a curvature function satisfying 1.3. Then there exists some finite time $0 < T^* < \infty$ and a unique curvature flow*

$$x \in H^{2+\alpha, \frac{2+\alpha}{2}}([0, T^*] \times \mathbb{S}^n, \mathbb{S}^{n+1}),$$

which satisfies the flow equation

(1.2)
$$\dot{x} = F^{-p}\nu$$
$$x(0) = M_0,$$

where $0 < p < \infty$, $\nu(t, \xi)$ is the outward normal to $M_t = x(t, M)$ at $x(t, \xi)$ and there exists $0 < t_0 < T^*$ such that the leaves M_t, $t_0 \leq t < T^*$, are graphs over some suitable equator $\mathcal{S}(x_0)$, $x_0 \in \mathbb{S}^{n+1}$,

(1.3)
$$M_t = \operatorname{graph} u(t, \cdot).$$

For $t \to T^*$, the functions $u(t, \cdot)$ converge to $\frac{\pi}{2}$ in $C^{1,\beta}(\mathbb{S}^n)$ for arbitrary $0 < \beta < 1$ and we have for $1 \leq q < \infty$, that

(1.4)
$$\int_{M_t} H^q \to 0, \ t \to T^*.$$

In this theorem, $H^{2+\alpha, \frac{2+\alpha}{2}}([0, T^*] \times \mathbb{S}^n, \mathbb{S}^{n+1})$ denotes the parabolic Hoelder space as in [12, Def. 2.5.2].

Recently, Gerhardt also considered inverse curvature flows of strictly convex hypersurfaces in $\mathbb{S}^{n+1}$ by curvature functions satisfying the assumptions of

1.3(i), see [15]. He obtains the optimal convergence of the flow to an equator. However, his methods substantially differ from ours.

Theorem 1.4 allows us to prove Alexandrov-Fenchel type inequalities and to solve isoperimetric-type problems in the sphere, namely

1.5. **Theorem.** *Let $0 < \alpha < 1$. Let $M \subset \mathbb{S}^{n+1}$ be an embedded, closed, orientable, connected, convex C^2-hypersurface of the sphere. Then we have the inequality*

$$(1.5) \qquad (\tilde{V}_1(M))^2 \geq (\tilde{V}_0(M))^{2\left(\frac{n-1}{n}\right)} - (\tilde{V}_0(M))^2,$$

and equality holds if and only if M is a geodesic sphere.
Furthermore we have the inequality

$$(1.6) \qquad \tilde{V}_2(M) \geq \left(\tilde{V}_0(M)\right)^{\frac{n-2}{n}} - \tilde{V}_0(M),$$

and equality holds if and only if M is a geodesic sphere.
Let $k \in \mathbb{N}_+$ with $2k + 1 \leq n$ and let $\hat{M}$ be the convex body enclosed by M. Then we have the inequality

$$(1.7)$$
$$W_{2k+1}(\hat{M}) \geq \frac{\omega_n}{n+1} \sum_{i=0}^{k} (-1)^i \frac{n+1-2k}{n+1-2k+2i} \binom{k}{i} \left(\frac{n+1}{\omega_n} W_1(\hat{M})\right)^{\frac{n-2k+2i}{n}}.$$

and equality holds if and only if M is a geodesic sphere.

Here $\tilde{V}_k(M)$ denotes, up to a constant, the k-th mean curvature integral and is defined by

$$(1.8) \qquad \tilde{V}_k(M) := \omega_n^{-1} \int_M \tilde{H}_k \, d\mu,$$

where $\tilde{H}_k := \frac{H_k}{\binom{n}{k}}$ are the normalized elementary symmetric polynomials and $\omega_n := |\mathbb{S}^n|$. $W_k(\hat{M})$ denotes the k-th quermassintegral of $\hat{M}$, see Section 7 for a definition. For a more detailed account of the mean curvature integrals and their relation to the quermassintegrals in spaces of constant curvature, see for example [27]. Especially inequality (7.25) shows that the geometric inequalities for the quermassintegrals resemble the corresponding inequalities in hyperbolic space, see [8, Theorem 1.3].

These theorems allow us to solve isoperimetric-type problems in the sphere: Since the isoperimetric problem is well understood in spherical space and the minimizers are known to be geodesic spheres, we can partially answer a question posed in [6], which can be regarded as a generalized isoperimetric problem:

1.6. **Corollary.** *Denote by $\mathcal{K}_s^n$ the set of convex bodies in $\mathbb{S}^{n+1}$ with C^2-boundary and with given volume $s \in \mathbb{R}_+$, such that there exists a geodesic ball B_s with volume s. Then the functionals V_1, V_2, W_{2k+1} for $k \in \mathbb{N}$ with $2k + 1 \leq n$ attain a minimum on $\mathcal{K}_s^n$ precisely at the geodesic ball B_s.*

Curvature flows have proven to be a useful method to obtain geometric inequalities. Probably the most known result in this direction is the proof of the Riemannian Penrose Inequality by Huisken and Ilmanen in [18] using an inverse mean curvature flow in asymptotically flat 3-manifolds.

But also Alexandrov-Fenchel type inequalities have been proven using curvature flows: In Euclidean space, McCoy showed in [23], that the Alexandrov-Fenchel inequalities for strictly convex hypersurfaces can be deduced from a mixed-volume preserving curvature flow. In 2009, Guan and Li (see [16]) used inverse F-curvature flows in Euclidean space to show these inequalities for k-convex, starshaped domains. Recently, one of the authors transferred the results about mixed-volume preserving curvature flows in Euclidean space from [23] to the hyperbolic space in [22]. Wang and Xia used these results in [30] to obtain the Alexandrov-Fenchel inequalities for horospherically convex hypersurfaces in hyperbolic space. Some of these inequalities have also been shown in hyperbolic space by using inverse F-curvature flows, see for example [7], [8], [21].

2. Setting and general facts

We now state some general facts about hypersurfaces, especially those that can be written as graphs. We basically follow the description of [13] and [26], but restrict to Riemannian manifolds. For a detailed discussion we refer to [12].

Let $N = N^{n+1}$ be Riemannian and $M = M^n \hookrightarrow N$ be a hypersurface. The geometric quantities of N will be denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$ etc., where greek indices range from 0 to n. Coordinate systems in N will be denoted by (x^α). Quantities for M will be denoted by (g_{ij}), (h_{ij}) etc., where latin indices range from 1 to n and coordinate systems will generally be denoted by (ξ^i), unless stated otherwise.

Covariant differentiation will usually be denoted by indices, e.g. u_{ij} for a function $u \colon M \to \mathbb{R}$, or, if ambiguities are possible, by a semicolon, e.g. $h_{ij;k}$. Usual partial derivatives will be denoted by a comma, e.g. $u_{i,j}$.

Let $x \colon M \hookrightarrow N$ be an embedding and (h_{ij}) be the second fundamental form with respect to a normal $-\nu$, i.e. we have the *Gaussian formula*

$$(2.1) \qquad x_{ij}^\alpha = -h_{ij}\nu^\alpha,$$

where ν is a differentiable normal, the *Weingarten equation*

$$(2.2) \qquad \nu_i^\alpha = h_i^k x_k^\alpha,$$

the *Codazzi equation*

$$(2.3) \qquad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the *Gauß equation*

$$(2.4) \qquad R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}) + \bar{R}_{\alpha\beta\gamma\delta}x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

Now assume that $N = (a, b) \times S_0$, where S_0 is compact Riemannian and that there is a Gaussian coordinate system (x^α) such that

$$(2.5) \qquad d\bar{s}^2 = e^{2\psi}((dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j),$$

where σ_{ij} is a Riemannian metric, $x = (x^i)$ are local coordinates for S_0 and $\psi \colon N \to \mathbb{R}$ is a function.

Let $M = \operatorname{graph} u_{|S_0}$ be a hypersurface

$$(2.6) \qquad M = \{(x^0, x) \colon x^0 = u(x), x \in S_0\},$$

then the induced metric has the form

$$(2.7) \qquad g_{ij} = e^{2\psi}(u_i u_j + \sigma_{ij})$$

with inverse

$$(2.8) \qquad g^{ij} = e^{-2\psi}(\sigma^{ij} - v^{-2}u^i u^j),$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$, $u^i = \sigma^{ij}u_j$ and

$$(2.9) \qquad v^2 = 1 + \sigma^{ij}u_i u_j \equiv 1 + |Du|^2.$$

We use, especially in the Gaussian formula, the normal

$$(2.10) \qquad (\nu^\alpha) = v^{-1}e^{-\psi}(1, -u^i).$$

Looking at $\alpha = 0$ in the Gaussian formula, we obtain

$$(2.11) \qquad e^{-\psi}v^{-1}h_{ij} = -u_{ij} - \bar\Gamma^0_{00}u_i u_j - \bar\Gamma^0_{0i}u_j - \bar\Gamma^0_{0j}u_i - \bar\Gamma^0_{ij}$$

and

$$(2.12) \qquad e^{-\psi}\bar h_{ij} = -\bar\Gamma^0_{ij},$$

where covariant derivatives are taken with respect to g_{ij}.

In our special situation $N = \mathbb{S}^{n+1}$ let $x_0 \in \mathbb{S}^{n+1}$, then by introducing geodesic polar coordinates we derive a representation of the metric in the form

$$(2.13) \qquad d\bar s^2 = dr^2 + \sin^2 r \, \sigma_{ij}dx^i dx^j,$$

where σ_{ij} is the canonical metric of $\mathbb{S}^n$ and $0 < r < \pi$. Then we obtain for a geodesic sphere given by a constant graph $u \equiv r$ with $0 < r < \pi$ that $\bar h_{ij} = \frac{\bar H}{n}\bar g_{ij}$ and

$$(2.14) \qquad \frac{\bar H}{n}(r) = \frac{\cos r}{\sin r}.$$

Using [4, Thm. 1.1], we conclude for an embedding of a smooth, strictly convex, closed hypersurface M, that it is contained in an open hemisphere and thus it can be written as a graph over $\mathbb{S}^n$ in the previously described coordinate system, i.e.

$$(2.15) \qquad M = \operatorname{graph} u_{|\mathcal{S}_0}.$$

Now we want to give some elementary facts about the curvature functions. Firstly, we provide the definition of these functions and mention some identifications, which will be used in the sequel without explicitly stating them again.

2.1. **Definition.** Let $\Gamma \subset \mathbb{R}^n$ be an open, convex, symmetric cone, i.e.

$$(2.16) \qquad (\kappa_i) \in \Gamma \implies (\kappa_{\pi i}) \in \Gamma \quad \forall\, \pi \in \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the set of all permutations of order n. Let $f \in C^{m,\alpha}(\Gamma)$, $m \in \mathbb{N}$, $0 \leq \alpha \leq 1$, be *symmetric*, i.e.,

$$(2.17) \qquad f(\kappa_i) = f(\kappa_{\pi i}) \quad \forall\, \pi \in \mathcal{P}_n.$$

Then f is said to be a *curvature function* of class $C^{m,\alpha}$. For simplicity we will also refer to the pair (f, Γ) as a curvature function.

Now denote by $\mathbf{S}$ the symmetric endomorphisms of $\mathbb{R}^n$ and by $\mathbf{S}_\Gamma$ the symmetric endomorphisms with eigenvalues belonging to Γ, an open subset of $\mathbf{S}$. If (f, Γ) is a smooth curvature function, we can define a mapping

$$(2.18) \qquad \begin{aligned} F : \mathbf{S}_\Gamma &\to \mathbb{R}, \\ A &\mapsto f(\kappa_i), \end{aligned}$$

where the κ_i denote the eigenvalues of A. For the relation between these different notions, especially the differentiability properties and the relation between their derivatives, see [12, Chapter 2.1]. Since the differentiability properties are the same for f as for F in our setting, see [12, Theorem 2.1.20], we do not distinguish between these notions and always write F for the curvature function. Hence at a point x of a hypersurface we can consider a curvature function F as a function defined on a cone $\Gamma \subset \mathbb{R}^n$, $F = F(\kappa_i)$ for $(\kappa_i) \in \Gamma$ (representing the principal curvatures at the point x of the hypersurface), as a function depending on (h_i^j), $F = F(h_i^j)$, or as a function depending on (h_{ij}) and (g_{ij}), $F = F(h_{ij}, g_{ij})$. However, we distinguish between the derivatives with respect to Γ or $\mathbf{S}$. We briefly summarize our notation and important properties:

For a smooth curvature function F we denote by $F^{ij} = \frac{\partial F}{\partial h_{ij}}$, a contravariant tensor of order 2, and $F_i^j = \frac{\partial F}{\partial h_j^i}$, a mixed tensor, contravariant with respect to the index j and covariant with respect to i. We also distinguish the partial derivative $F_{,i} = \frac{\partial F}{\partial \kappa_i}$ and the covariant derivative $F_{;i} = F^{kl} h_{kl;i}$. Furthermore F^{ij} is diagonal if h_{ij} is diagonal and in such a coordinate system there holds $F^{ii} = \frac{\partial F}{\partial \kappa_i}$. For a relation between the second derivatives see [12, Lemma 2.1.14]. Finally, if $F \in C^2(\Gamma)$ is concave (convex), then F is also concave (convex) as a curvature function depending on (h_{ij}).

3. RIGIDITY RESULTS

In this section we want to prove the rigidity result, Theorem 1.1 and Corollary 1.2. First we need some definitions, which also apply to the case $n \geq 1$.

3.1. **Definition.** (i) For a point $x \in \mathbb{S}^{n+1}$ we will denote the closed hemisphere with center in x by $\mathcal{H}(x)$,

$$(3.1) \qquad \mathcal{H}(x) := \{p \in \mathbb{S}^{n+1} : \langle p, x \rangle \geq 0\},$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^{n+2}$, and the corresponding equator by $\mathcal{S}(x) := \mathcal{H}(x) \setminus \operatorname{int} \mathcal{H}(x)$.

(ii) For points $p, q \in \mathbb{S}^{n+1}$, $\hat{\gamma}_{p,q} : [0, L] \to \mathbb{S}^{n+1}$ will denote a C^1 parametrization by arc length of the geodesic segment $\hat{\Gamma}_{p,q}$ with $\hat{\gamma}_{p,q}(0) = p$ and $\hat{\gamma}_{p,q}(L) = q$. The geodesic segment is not unique if $\operatorname{dist}(p, q) = \pi$.

(iii) For $x \in \mathbb{S}^{n+1}$ the stereographic projection mapping x to 0 will be denoted by $\mathcal{P}_x : \mathbb{S}^{n+1} \setminus \{-x\} \to \mathbb{R}^{n+1}$.

3.2. **Definition.** (i) Let $\hat{M} \subset \mathbb{S}^{n+1}$ be a set. We say that $\hat{M}$ is a weakly convex set in $\mathbb{S}^{n+1}$ [in a hemisphere], if [there exists $x \in \mathbb{S}^{n+1}$ such that $\hat{M} \subset \mathcal{H}(x)$ and] for arbitrary $p, q \in \hat{M}$ there exists a minimizing geodesic $\hat{\Gamma}_{p,q}$ connecting p and q, which is contained in $\hat{M}$.

(ii) Let $\hat{M} \subset \mathbb{S}^{n+1}$ be a set. We say that $\hat{M}$ is a convex set in $\mathbb{S}^{n+1}$ [in a hemisphere], if [there exists $x \in \mathbb{S}^{n+1}$ such that $\hat{M} \subset \mathcal{H}(x)$ and] for arbitrary $p, q \in \hat{M}$ all minimizing geodesics $\hat{\Gamma}_{p,q}$ connecting p and q [and contained in $\mathcal{H}(x)$] are contained in $\hat{M}$.

(iii) We say that $\hat{M} \subset \mathbb{S}^{n+1}$ is a (weakly) convex body [in a hemisphere], if it is a compact, (weakly) convex set [in a hemisphere] with nonempty interior.

(iv) A set $M \subset \mathbb{S}^{n+1}$ is a closed, geodesically convex hypersurface, if there exists a convex body $\hat{M} \subset \mathbb{S}^{n+1}$ in a hemisphere, such that $M = \partial\hat{M}$. The set $\hat{M}$ is called the convex body of M.

(v) Let $x \in \mathbb{S}^{n+1}$. Let $\hat{M} \subset \mathbb{S}^{n+1}$ be a set with $\overline{\hat{M}} \subset \mathbb{S}^{n+1} \setminus \{-x\}$. We say that $\hat{M}$ satisfies an interior sphere condition at a point $p \in \overline{\hat{M}}$ with respect to x, if the set $\mathcal{P}_x(\overline{\hat{M}}) \subset \mathbb{R}^{n+1}$ satisfies an interior sphere condition at $\mathcal{P}_x(p)$.

3.3. **Remark.** The following observations have to be made:

(i) Let $M \subset \mathbb{S}^{n+1}$ be a closed, geodesically convex hypersurface. Then the convex body of M is not unique, as can be seen by looking at $M = \mathcal{S}(x)$, where $x \in \mathbb{S}^{n+1}$ is arbitrary.

(ii) Note that there are different notions of convexity in the sphere: we do not demand that a geodesic connecting two points in the convex body has to be unique. It is well-known, see also Lemma 3.8, that a convex body $\hat{M}$ in the sphere, which does not contain a pair of antipodal points is contained in an open hemisphere. If on the other hand the convex body $\hat{M}$ in the sphere contains antipodal points, then it follows from the definition that $\hat{M} = \mathbb{S}^{n+1}$.

3.4. **Remark.** The following observations can be found in [12, p. 278, 279]. Let $x_0 \in \mathbb{S}^{n+1}$. Defining

$$(3.2) \qquad \rho = 2\tan\frac{r}{2},$$

where r is the geodesic distance to x_0, we obtain a representation of the spherical metric as

$$(3.3) \qquad d\bar{s}^2 = \frac{1}{\left(1 + \frac{1}{4}\rho^2\right)^2}(d\rho^2 + \rho^2\sigma_{ij}d\xi^i d\xi^j) \equiv e^{2\psi}\hat{g}_{\alpha\beta},$$

where $\hat{g}$ denotes the Euclidean metric in $\mathbb{R}^{n+1}$. A point $q \in \mathbb{S}^{n+1}$ is contained in $\mathcal{H}(x_0)$ if and only if $r \leq \frac{\pi}{2}$, which is equivalent to $\rho \leq 2$.

A C^2-hypersurface $M \subset \mathbb{S}^{n+1} \setminus \{-x_0\}$ can be seen as embedded in Euclidean space using the conformally flat parametrization of $\mathbb{S}^{n+1}$ via stereographic projection $\mathcal{P}_{x_0}$. We will denote the hypersurface $\mathcal{P}_{x_0}(M)$ by $\mathcal{M}$. The second fundamental form h^i_j of M and the corresponding Euclidean quantity $\hat{h}^i_j$ are related by

$$(3.4) \qquad e^{\psi}h^i_j = \hat{h}^i_j + \psi_\alpha\hat{\nu}^\alpha\delta^i_j,$$

where $\hat{\nu}$ denotes the Euclidean normal vector field of $\mathcal{M}$. Thus a simple calculation reveals that for a strictly convex and C^2-bounded M, the corresponding hypersurface $\mathcal{M}$ is strictly convex and bounded in C^2.

The closure of a weakly convex set is again a weakly convex set. However, this statement is not true for convex sets (neither in spheres nor in hemispheres). We want to prove a sufficient condition for a weakly convex body in a hemisphere to be a convex body in a hemisphere.

3.5. Theorem. *Let $n \geq 1$ and $\hat{M} \subset \mathbb{S}^{n+1}$ be a weakly convex body in a hemisphere $\mathcal{H}(x_0)$ for some $x_0 \in \mathbb{S}^{n+1}$. Suppose that $\hat{M}$ satisfies an interior sphere condition with respect to x_0 at all points $p \in \hat{M} \cap \mathcal{S}(x_0)$. Then $\hat{M}$ is a convex body in a hemisphere.*

Firstly, we need some lemmata.

3.6. Lemma. *Let $n \geq 1$ and $x_0 \in \mathbb{S}^{n+1}$. Let $p \in \mathcal{S}(x_0)$. Let $\gamma : [0, \pi] \to \mathbb{S}^{n+1}$ be a C^1-geodesic, parametrized by arc length, with $\gamma(0) = p$. Let $\tilde{p}$ denote the outward normal vector of $\mathcal{H}(x_0)$ at p. Then*

$$(3.5) \qquad \langle \dot{\gamma}(0), \tilde{p} \rangle < (>, =),$$

if and only if the geodesic satisfies $\gamma(t) \in \operatorname{int} \mathcal{H}(x_0) \left(\complement \mathcal{H}(x_0), \mathcal{S}(x_0) \right)$ for some (and hence every) $t \in (0, \pi)$.

Proof. First of all, we note the following fact: Since γ is a C^1-geodesic, γ is a segment of a great circle. Hence a third point lying on γ determines uniquely the great circle Γ such that $\gamma([0, \pi]) \subset \Gamma$. Thus the existence of $t \in (0, \pi)$, such that $\gamma(t) \in \operatorname{int} \mathcal{H}(x_0) (\complement \mathcal{H}(x_0), \mathcal{S}(x_0))$ implies $\gamma((0, \pi)) \subset \operatorname{int} \mathcal{H}(x_0) (\complement \mathcal{H}(x_0), \mathcal{S}(x_0))$.
Suppose firstly, that $\langle \dot{\gamma}(0), \tilde{p} \rangle < 0$. Then $\gamma(t) \in \operatorname{int} \mathcal{H}(x_0)$ for t close to 0. Hence by the observation made above, we obtain $\gamma((0, \pi)) \subset \operatorname{int} \mathcal{H}(x_0)$.
If on the other hand there exists $t \in (0, \pi)$, such that $\gamma(t) \in \operatorname{int} \mathcal{H}(x_0)$, then the geodesic $\tilde{\gamma} : [0, t] \to \mathbb{S}^{n+1}$ with $\tilde{\gamma}(0) = \gamma(t)$ and $\dot{\tilde{\gamma}}(0) = -\dot{\gamma}(t)$ satisfies $\langle \dot{\tilde{\gamma}}(t), \tilde{p} \rangle > 0$, hence we obtain $\langle \dot{\gamma}(0), \tilde{p} \rangle < 0$. $\qquad \square$

3.7. Lemma. *Let $n \geq 1$ and $\hat{M} \subset \mathbb{S}^{n+1}$ be a weakly convex body in the hemisphere $\mathcal{H}(x_0)$ for some fixed $x_0 \in \mathbb{S}^{n+1}$. Let $p \in \hat{M} \cap \mathcal{S}(x_0)$ and suppose that $\hat{M}$ satisfies an interior sphere condition at p with respect to x_0. Let $\gamma \in C^1([0, t_0], \mathbb{S}^{n+1})$, $0 < t_0$, be a geodesic with $\gamma(0) = p$ and $\gamma((0, t_0)) \in \operatorname{int} \mathcal{H}(x_0)$. Then there exists $0 < \delta \leq t_0$, such that $\gamma((0, \delta)) \subset \operatorname{int} \hat{M}$.*

Proof. We will assume without loss of generality that $x_0 = (0, \ldots, 0, 1)$ and $p = (1, 0, \ldots, 0)$. Let $\tilde{p}$ denote the outward normal vector of $\mathcal{H}(x_0)$ at p.
From Lemma 3.6 we obtain

$$(3.6) \qquad \langle \dot{\gamma}(0), \tilde{p} \rangle < 0.$$

Let us look at the situation in the coordinates given by the stereographic projection $\mathcal{P}_{x_0}$, where we assume that $\pi(p) = (2, 0, \ldots, 0)$.
Let $\rho > 0$ and $B_\rho(\bar{p})$ be an inball with respect to $\mathcal{P}_{x_0}(p)$ with center $\bar{p}$. Then we know that $\bar{p} = (2 - \rho, 0, \ldots, 0)$.
Let $\tilde{\gamma} := \mathcal{P}_{x_0} \circ \gamma$, then since the metric of the sphere is conformally equivalent to the Euclidean metric in stereographic coordinates, we obtain from (3.6)

$$(3.7) \qquad \dot{\tilde{\gamma}}^1(0) = \left\langle \dot{\tilde{\gamma}}(0), \frac{\mathcal{P}_{x_0}(p)}{|\mathcal{P}_{x_0}(p)|} \right\rangle = \langle \dot{\tilde{\gamma}}(0), \nu(\mathcal{P}_{x_0}(p)) \rangle < 0,$$

where $\nu(\mathcal{P}_{x_0}(p))$ denotes the outward normal of $B_2(0)$ at $\mathcal{P}_{x_0}(p)$ and is equal to $\frac{\mathcal{P}_{x_0}(p)}{|\mathcal{P}_{x_0}(p)|}$.

Let us parametrize the boundary of the inball $B_\rho(\bar{p})$ locally around $\mathcal{P}_{x_0}(p)$ by a C^∞-curve $\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ for some small $\epsilon > 0$, such that $\alpha(0) = (2, 0, \ldots, 0)$ and, since the inball is tangent to $\partial B_2(0)$, we have $\dot{\alpha}^1(0) = 0$. Hence we obtain, that for some small $\delta > 0$, we have for $t \in (0, \delta)$:

$$(3.8) \qquad\qquad \tilde{\gamma}(t) \in B_\rho(\bar{p}).$$

$\square$

Proof of Theorem 3.5. Let $p, q \in \hat{M}$ be two arbitrary points, then we have to show that an arbitrary minimizing geodesic $\hat{\gamma}_{p,q}$ connecting p and q and contained in $\mathcal{H}(x_0)$ is contained in $\hat{M}$.

If no antipodal points exist in $\hat{M}$, we have nothing to prove. Hence let us assume that there exist points $p, q \in \hat{M}$ with $\text{dist}(p, q) = \pi$. We will show that then $\hat{M} = \mathcal{H}(x)$.

Since $\hat{M} \subset \mathcal{H}(x)$ we know that $p, q \in \mathcal{S}(x)$. Let $y \in \text{int}\,\mathcal{H}(x)$ be arbitrary. Then there exists a unique C^1-geodesic $\gamma : [0, \pi] \to \mathbb{S}^{n+1}$ starting at p and ending at q, such that $y \in \gamma((0, \pi)) \subset \text{int}\,\mathcal{H}(x)$. From Lemma 3.7 applied to p and q we obtain that $\gamma((0, \pi)) \subset \hat{M}$. Hence $\text{int}\,\mathcal{H}(x) \subset \hat{M}$ and we infer $\hat{M} = \overline{\hat{M}} = \mathcal{H}(x)$. $\square$

Hence we know that the weakly convex set $\hat{M}$ in Theorem 1.1 is a convex set in a hemisphere. Thus it remains to distinguish two cases: $\hat{M}$ does or does not contain a pair of antipodal points.

The proof of the following Lemma can be found in [5, Chapter 3, Corollary 1]. For the sake of completeness, we give a sketch of an elementary proof.

3.8. Lemma. *Let $n \geq 1$ and $\hat{M} \subset \mathbb{S}^{n+1}$ be a convex body in the sphere, which does not contain any antipodal points. Then $\hat{M}$ is contained in an open hemisphere.*

Proof. Firstly, since $\hat{M}$ is closed and does not contain pairs of antipodal points, we have $r := \max\{\text{dist}(p, q) : p, q \in \hat{M}\} < \pi$.

We will prove the Lemma by induction on $n \in \mathbb{N}$. For $n = 0$ the statement is obvious. Suppose we have proven the statement for $n - 1 \in \mathbb{N}$.

Let $p_1, q_1 \in \hat{M}$ be two points with $\text{dist}(p_1, q_1) = r$. Let $\epsilon := \frac{\pi - r}{2}$. We can assume without loss of generality, that $p_1 = (\sqrt{1 - \epsilon^2}, 0, \ldots, 0, \epsilon)$ and $q_1 = (-\sqrt{1 - \epsilon^2}, 0, \ldots, 0, \epsilon)$. Furthermore $\hat{M}^1 := \hat{M} \cap \{x^1 = 0\}$ is a closed subset of $\mathbb{S}^n \equiv \{0\} \times \mathbb{S}^n \subset \mathbb{S}^{n+1}$ and satisfies the requirements of the lemma for $m = n - 1$. Hence by the inductive assumption, $\hat{M}^1$ is contained in an open hemisphere and we can assume after a rotation about the x^1-axis, that $\hat{M}^1 \subset \{x \in \mathbb{S}^{n+1} : x^{n+2} \geq \epsilon\}$.

Suppose there exists a point $z \in \hat{M}$ with $z^{n+2} = 0$. In view of the observations made above there holds $0 < |z^1| < 1$, $\hat{z} = (z^2, \ldots, z^{n+1}) \neq 0$ and we can assume without loss of generality $-1 < z^1 < 0$. Then

$$(3.9) \qquad\qquad \hat{M} \cap \{x^1 \geq 0\} \subset \{x^{n+2} > 0\},$$

for otherwise there would exist a point $y \in \hat{M} \cap \{x^{n+2} = 0, x^1 > 0\}$ and either $\mathrm{dist}(y, z) = \pi$, which is excluded by the assumption of the lemma, or $\mathrm{dist}(y, z) < \pi$ and hence the geodesic segment $\hat{\Gamma}_{y,z}$ would be contained in $\hat{M} \cap \{y^{n+2} = 0\}$, which implies a contradiction to

$$(3.10) \qquad \hat{M}^1 \cap \{x^{n+2} = 0\} = \emptyset.$$

Now we rotate continuously in the positive x^1-direction such that $\hat{M} \cap \{x^1 > 0\} \subset \{x^{n+2} \geq 0\}$ and there exists $y \in \hat{M} \cap \{x^{n+2} = 0, x^1 > 0\}$. Note that we still have $\hat{M} \cap \{x^1 = 0\} = \emptyset$. By the same reasoning as above, in the new coordinate system $\hat{M} \cap \{x^{n+2} = 0, x^1 < 0\} = \emptyset$. This implies that $\hat{M}$ is contained in an open hemisphere. $\qquad \square$

Proof of Theorem 1.1. From Theorem 3.5 we obtain that $\hat{M}$ is a convex body in a hemisphere. Hence if $\hat{M}$ contains a pair of antipodal points, then $\hat{M} = \mathcal{H}(x_0)$. Otherwise it is contained in an open hemisphere in view of Lemma 3.8.

$$\square$$

For C^2-hypersurfaces we obtain a generalization of Theorem 1, see Corollary 1.2. For the proof of this result, we need one further Lemma.

3.9. Lemma. *Let $\hat{M}_n \subset \mathcal{H}(x_n) \subset \mathbb{S}^{n+1}$ be a sequence of sets, such that*

$$(3.11) \qquad \hat{M}_n \subset \mathrm{int}\, \hat{M}_{n+1},$$

then there exists $x_0 \in \mathbb{S}^{n+1}$, such that

$$(3.12) \qquad \hat{M}_n \subset \mathrm{int}\, \mathcal{H}(x_0) \quad \forall n \in \mathbb{N}.$$

Proof. A subsequence of points x_{n_k} converges to some $x_0 \in \mathbb{S}^{n+1}$. We claim that this x_0 is a point which satisfies (3.12). If this was not the case, then we use the monotonicity to derive the existence of $n_0 \in \mathbb{N}$ with the property

$$(3.13) \qquad \hat{M}_{n_0} \cap \mathcal{H}(x_0)^c \neq \emptyset.$$

Thus there exists a point

$$(3.14) \qquad y \in \hat{M}_n, \ n \geq n_0,$$

and $\epsilon > 0$ with the property

$$(3.15) \qquad \mathrm{dist}(y, \mathcal{H}(x_0)) \geq \epsilon.$$

This leads to a contradiction, since for large k we have

$$(3.16) \qquad y \in \hat{M}_{n_k - 1} \subset \mathrm{int}\, \mathcal{H}(x_{n_k}) \subset \mathrm{int}\, \mathcal{H}(x_{n_k}) \cup \mathrm{int}\, \mathcal{H}(x_0)$$

and the maximal distance of points in the latter set to x_0 converges to $\frac{\pi}{2}$. $\quad \square$

Proof of Corollary 1.2. Choose a differentiable normal vector field ν, such that the second fundamental form with respect to $-\nu$ is positive semi definite and let $\mathcal{U}$ be a tubular neighborhood around M with corresponding signed distance function $d \in C^2(\mathcal{U})$ and normal Gaussian coordinate system (x^α), compare [12, Theorem 1.3.13]. Note that $d = x^0$ and $\nabla d = \nu$. According to

[12, Lemma 2.4.3], the second fundamental form with respect to $-\nu$ of the coordinate slices

$$(3.17) \qquad \{x^0 = \text{const}\},$$

which can be seen as a solution to the 'curvature flow'

$$(3.18) \qquad \dot{x} = \nu, \ x(t,\xi) = (t,\xi),$$

evolves according to the evolution equation

$$(3.19) \qquad \dot{h}^i_j = -h^k_j h^i_k - \delta^i_j.$$

Thus the principle curvatures of the slices are strictly decreasing, which implies, that the hypersurfaces

$$(3.20) \qquad M_{-t} = \{x^0 = -t\}$$

are strictly convex with positive definite second fundamental form. Consider the image $\mathcal{M}_{-t}$ under a suitable stereographic projection $\mathcal{P}$, which is a strictly convex C^2 hypersurface in $\mathbb{R}^{n+1}$. For any $\delta > 0$ there exists $\epsilon(\delta) > 0$, such that for the convolution of the signed distance function, d_ϵ, there hold

$$(3.21) \qquad \mathcal{M}^\epsilon_{-t} \equiv \left\{ d_\epsilon = -\left(t + \frac{\delta}{2}\right) \right\} \subset \mathcal{P}(\{-t < d < -(t+\delta)\}),$$

$$(3.22) \qquad \langle \nabla d_\epsilon, \nabla d \rangle \geq c > 0$$

and

$$(3.23) \qquad \mathcal{M}^\epsilon_{-t} \text{ is strictly convex.}$$

Using (3.4) and the C^2-convergence of the convolution, those properties carry over to $M^\epsilon_{-t} \equiv \mathcal{P}^{-1}(\mathcal{M}^\epsilon_{-t})$, a hypersurface in $\mathbb{S}^{n+1}$, to which we may apply Theorem 1. Using the same construction, we obtain

$$(3.24) \qquad M^{\epsilon_1}_{-\frac{t}{2}}, \text{ where } \epsilon_1 = \epsilon\left(\frac{t}{2}\right).$$

Thus we derive a sequence of smooth and strictly convex hypersurfaces

$$(3.25) \qquad M_n \equiv M^{\epsilon_n}_{-\frac{t}{2^n}}, \text{ where } \epsilon_n = \epsilon\left(\frac{t}{2^n}\right),$$

with the property

$$(3.26) \qquad M_n \subset \text{int } \hat{M}_{n+1}.$$

Here we also used the generalized Jordan curve theorem, cf. [3, Chapter IV, 19]. Lemma 3.9 implies that there exists $x_0 \in \mathbb{S}^{n+1}$ such that $M \subset \mathcal{H}(x_0)$ and Theorem 3.5 shows that M bounds a convex body $\hat{M} = \overline{\bigcup_{n\in\mathbb{N}} \hat{M}_n}$ in a hemisphere, since $\hat{M}$ obviously satisfies the interior sphere condition with respect to x_0 at all points of $\hat{M} \cap \mathcal{S}(x_0)$. $\qquad \square$

4. The curvature flow and first estimates

Curvature functions. We now mention some elementary facts about curvature functions on a hypersurface.

The following lemma holds for inverse concave curvature functions, which are homogeneous of degree one.

4.1. Lemma. *Let F be a curvature function as in 1.3 (i), then*

$$(4.1) \qquad \left\{ F^{rs,kl}\eta_{rs}\eta_{kl} + 2F^{rk}\tilde{h}^{sl}\eta_{rs}\eta_{kl} - \frac{2}{F}\left(F^{ij}\eta_{ij}\right)^2 \right\} \geq 0$$

for all symmetric matrices $(\eta_{ij}) \in \mathbf{S}$, where the curvature function F is evaluated at a positive definite, symmetric matrix $(h_{ij}) \in \mathbf{S}_{\Gamma_+}$ and $(\tilde{h}^{ij})$ denotes the inverse matrix.

To derive the geometric inequalities, we will need some properties of the elementary symmetric polynomials.

4.2. Lemma. *Let $1 \leq k \leq n$ be fixed.*

(i) *We define the convex cone*

$$(4.2) \qquad \Gamma_k = \{(\kappa_i) \in \mathbb{R}^n : H_1(\kappa_i) > 0, H_2(\kappa_i) > 0, \ldots, H_k(\kappa_i) > 0\}.$$

Then H_k is strictly monotone on Γ_k and Γ_k is exactly the connected component of

$$(4.3) \qquad \{(\kappa_i) \in \mathbb{R}^n : H_k(\kappa_i) > 0\}$$

containing the positive cone.

(ii) *The k-th roots $\sigma_k = H_k^{\frac{1}{k}}$ are concave on Γ_k.*

(iii) *For $1 < s < t < n$ and $\tilde{\sigma}_k = \left(\frac{H_k}{\binom{n}{k}}\right)^{\frac{1}{k}}$ there holds*

$$(4.4) \qquad \tilde{\sigma}_n \leq \tilde{\sigma}_t \leq \tilde{\sigma}_s \leq \tilde{\sigma}_1,$$

where the principal curvatures have to lie in $\Gamma_n \equiv \Gamma_+$ for the first, in Γ_t for the second and in Γ_s for the third inequality.

(iv) *For fixed i, no summation over i, there holds*

$$(4.5) \qquad H_k = \frac{\partial H_{k+1}}{\partial \kappa_i} + \kappa_i \frac{\partial H_k}{\partial \kappa_i}.$$

Proof. The convexity of the cone Γ_k and (i) follows from [19, Section 2], (ii) and (iii) from [20, Lemma 15.12, Theorem 15.16] and (iv) follows directly from the definition of the H_k. $\qquad\square$

A consequence of the preceding lemma is the following

4.3. Lemma. *Let N be a semi-Riemannian space of constant curvature, then the symmetric polynomials $F = H_k$, $1 \leq k \leq n$, are divergence free for every admissible hypersurface M of N. In case $k = 2$ it suffices to assume that N is an Einstein manifold.*

Proof. The proof of the lemma can be found in [11, Lemma 5.8]. The proof consists of induction on k and (iv) of Lemma 4.2. $\qquad\square$

Now we state a well-known inequality for general curvature functions:

4.4. **Lemma.** *Let $F \in C^2(\Gamma_+)$ be a strictly monotone, concave (respectively convex) curvature function, positively homogeneous of degree 1 with $F(1, \ldots, 1) > 0$, then*

$$(4.6) \qquad F(\kappa) \leq (\text{respectively } \geq) \frac{F(1, \ldots, 1)}{n} H(\kappa).$$

and

$$(4.7) \qquad \sum_{i=1}^{n} F_i(\kappa) \geq (\text{respectively } \leq) F(1, \ldots, 1),$$

where $\kappa = (\kappa_k) \in \Gamma_+$.

Proof. See [12, Lemma 2.2.19, Lemma 2.2.20]. $\qquad \square$

Evolution equations for the curvature flow. The following evolution equations are valid for curvature flows in the sphere, for a derivation see [12, Chapter 2].

4.5. **Lemma.** *(Evolution equations)*

$$(4.8) \quad \frac{d}{dt}\tilde{h}^i_i - \Phi' F^{kl} \tilde{h}^i_{i;kl} = -\Phi' F^{kl} h_{kr} h^r_l \tilde{h}^i_i + \{\Phi' F - \Phi\}\delta^i_i$$
$$- \{\Phi' F + \Phi\}\tilde{h}_{ki}\tilde{h}^{ki} + \Phi' F^{kl} g_{kl}\tilde{h}^i_i$$
$$- \left\{\Phi' F^{rs,kl} h_{rs;p} h_{kl;q} + 2\Phi' F^{rk} \tilde{h}^{sl} h_{rs;p} h_{kl;q} + \Phi'' F_p F_q\right\} \tilde{h}^{pi}\tilde{h}^q_i$$

$$(4.9) \qquad \frac{d}{dt} h^i_j - \Phi' F^{kl} h^i_{j;kl} = \Phi' F^{kl} h^r_k h_{rl} h^i_j + (\Phi - \Phi' F) h^i_k h^k_j$$
$$+ \Phi' F^{kl,rs} h_{kl;r} h_{rs;j} g^{ri} + \Phi'' F^i F_j$$
$$+ ((\Phi + \Phi')\delta^i_j - \Phi' F^{kl} g_{kl} h^i_j).$$

$$(4.10) \qquad \frac{d}{dt}\Phi - \Phi' F^{kl}\Phi_{;kl} = \Phi' F^{kl} h_{kr} h^r_l \Phi + \Phi' F^{kl} g_{kl}\Phi$$

$$(4.11) \qquad \frac{d}{dt}\left(\frac{\tilde{H}}{-\Phi}\right) - \Phi' F^{kl}\left(\frac{\tilde{H}}{-\Phi}\right)_{;kl} = -\Phi' F^{kl}\left(\frac{\tilde{H}}{-\Phi}\right)_k \frac{(-\Phi)_l}{-\Phi}$$
$$- \left\{\Phi' F^{rs,kl} h_{rs;p} h_{kl;q} + 2\Phi' F^{rk}\tilde{h}^{sl} h_{rs;p} h_{kl;q}\right.$$
$$\left. + \Phi'' F_p F_q\right\} \frac{\tilde{h}^{pi}\tilde{h}^q_i}{-\Phi} + (p+1)n - 2p F^{kl} h_{rk} h^r_l \frac{\tilde{H}}{F}$$
$$- \left(\Phi' F + \Phi\right)\frac{|\tilde{A}|^2}{-\Phi}$$

Curvature estimates. Since the flow reduces to a scalar parabolic equation, see for example [12, Chapter 2.5], short-time existence is guaranteed and we know that the flow exists on a maximal time interval $[0, T^*)$ for some $0 < T^* \leq \infty$.

Firstly, we show that the flow exists only up to a finite time T^*.

4.6. **Lemma.** *There holds*

$$(4.12) \qquad\qquad T^* < \infty.$$

Proof. Using Lemma 4.4 we can compare the evolution of $-\Phi$, see (4.10), with the solution of the ordinary differential equation

$$(4.13) \qquad\qquad \dot{\varphi} = n\varphi^3.$$

This shows that $\sup_{x \in M_t} (-\Phi(x))$ becomes unbounded in finite time. $\qquad\square$

We show that the hypersurfaces remain strictly convex and the principal curvatures are uniformly bounded from above. For curvature functions with $F_{|\partial\Gamma} = 0$ the strict convexity of the flow hypersurfaces follows immediately. For inverse concave curvature functions this follows from the following

4.7. Lemma. *For this lemma suppose that F is a curvature function as in 1.3 (i) with $p = 1$. Let $c_2 > 0$ be such that $\sup_{M_0} \frac{\tilde{H}}{-\Phi} \leq c_2$. Then there holds*

$$(4.14) \qquad\qquad \sup_{t \in [0,T^*),\, x \in M_t} \frac{\tilde{H}}{-\Phi} < c_2 + 2nT^*.$$

Proof. Let $0 < T < T^*$ be arbitrary. Suppose there exists $t_0 \in (0, T]$ and $x_0 \in M_{t_0}$ such that

$$(4.15) \qquad\qquad \sup_{t \in [0,t_0],\, x \in M_t} \left(\frac{\tilde{H}}{-\Phi} - 2nt \right) \leq \frac{\tilde{H}}{-\Phi}(x_0) - 2nt_0.$$

Then we obtain from the maximum principle, that at x_0 there holds in view of (4.11)

$$(4.16) \qquad\qquad 0 \leq -2F^{kl}h_{rk}h_l^r \frac{\tilde{H}}{F} < 0,$$

since at x_0 we have $\nabla \left(\frac{\tilde{H}}{-\Phi} \right) = 0$ and the term in the brackets in the evolution equation is nonnegative in view of Lemma 4.1. $\qquad\square$

Next, we derive upper bounds for the principal curvatures.

4.8. Lemma. *Let $c_1 := \sup_{M_0} H$. Then there holds*

$$(4.17) \qquad\qquad \sup_{t \in [0,T^*),\, x \in M_t} H(x) \leq c_1.$$

Proof. Let $0 < T < T^*$ be arbitrary. Suppose there exists $t_0 \in (0, T]$ and $x_0 \in M_{t_0}$ such that

$$(4.18) \qquad\qquad \sup_{t \in [0,t_0],\, x \in M_t} H \leq H(x_0).$$

Then we obtain from the maximum principle and the concavity of F, that at x_0 there holds in view of (4.9)

(4.19)

$$0 \leq pF^{kl}h_k^r h_{rl} \frac{H}{F^{p+1}} - (p+1)\frac{|A|^2}{F^p} - pF^{kl}g_{kl}\frac{H}{F^{p+1}} - (1-p)\frac{n}{F^p}$$

$$= \frac{p}{F^{p+1}} \left(F^{kl}h_k^r h_{rl}H - F|A|^2 \right) - \frac{n+|A|^2}{F^p} - \frac{p}{F^{p+1}} \left(F^{kl}g_{kl}H - nF \right).$$

However, we note that

$$F^{kl}h_k^r h_{rl}H - F\,|A|^2 = \sum_{i,j}\left(f_i\kappa_i^2\kappa_j - f_i\kappa_i\kappa_j^2\right)$$

(4.20)

$$= \sum_{i<j}\kappa_i\kappa_j(\kappa_i - \kappa_j)\,(f_i - f_j) \leq 0,$$

since for concave curvature functions there holds $f_i \geq f_j$ for $i < j$. Furthermore, in view of Lemma 4.4 we obtain

(4.21)
$$F^{kl}g_{kl}H - nF \geq 0.$$

Hence we obtain a contradiction. $\qquad\square$

5. Elementary flow properties and further curvature estimates

5.1. Lemma. *Let M_t, $0 \leq t < T^*$, be a flow hypersurface and $\hat{M}_t$ be the enclosed convex body, cf. Corollary 1.2. Then those convex bodies are strictly monotonically ordered, i.e.*

(5.1)
$$s < t \Rightarrow \hat{M}_s \subset \mathrm{int}\ \hat{M}_t.$$

Proof. Let $0 \leq s < T^*$. Then the flow hypersurface M_s is strictly convex, as was shown in the previous section. Thus, using Corollary 1.2, we first conclude that M_s does indeed enclose a convex body $\hat{M}_s$ and that this body has to lie compactly in an open hemisphere $\mathrm{int}\,\mathcal{H}(\tilde{x}_s)$. Choose $x_s \in \mathrm{int}\ \hat{M}_s$. Thus M_s, which in particular is starshaped with respect to x_s, can be written as a graph over $\mathcal{S}(x_s)$,

(5.2)
$$M_s = \mathrm{graph}\,u(s,\mathcal{S}(x_s)),$$

and thus there is an $\epsilon > 0$, such that for all $s \leq t < s + \epsilon$ the hypersurfaces M_t may be written as a graph over $S(x_s)$, compare [12, Thm. 2.5.19]. In these coordinates u locally satisfies the scalar flow equation

(5.3)
$$\frac{\partial u}{\partial t} = \frac{v}{F^p},$$

cf. [12, p. 98-99], and thus the function u is strictly increasing for fixed $x \in \mathcal{S}(x_s)$. Since

(5.4)
$$\hat{M}_s = \{(r,(x^i)) \in \mathbb{R} \times \mathcal{S}(x_s)\colon 0 \leq r \leq u(s,(x^i))\},$$

where $(r,(x^i))$ describe the corresponding geodesic polar coordinates around x_s, the claim follows. $\qquad\square$

5.2. Proposition. *There is a uniquely determined limit surface M_{T^*}, which can be written as a graph in geodesic polar coordinates,*

(5.5)
$$M_{T^*} = \mathrm{graph}\,u(T^*,\mathcal{S}(y_0)),$$

where $y_0 \in \mathrm{int}\ \hat{M}_{T^*}$. Furthermore there holds

(5.6)
$$u(t,\cdot) \to u(T^*,\cdot)\ \ \mathrm{in}\ C^{1,\beta}(\mathcal{S}(y_0))\ \ \forall 0 \leq \beta < 1.$$

Proof. In those geodesic polar coordinates the metric of $\mathbb{S}^{n+1}$ is given by

$$(5.7) \qquad d\bar{s}^2 = dr^2 + \sin^2 r \sigma_{ij} dx^i dx^j, \ 0 < r < \pi.$$

Let $y_0 \in \operatorname{int} \hat{M}_0$ and $\hat{y}_0$ denote the antipodal point of y_0, then by Lemma 3.9 we know that

$$(5.8) \qquad \hat{M}_t \subset K \Subset \mathbb{S}^{n+1}\backslash\{\hat{y}_0\} \ \ \forall 0 \le t < T^*.$$

Thus we have a uniform parametrization of the flow hypersurfaces as graphs over $\mathcal{S}(y_0)$

$$(5.9) \qquad M_t = \operatorname{graph} u(t, \cdot).$$

The quantity $v^2 = 1 + \sin^{-2} u \sigma^{ij} u_i u_j$ is bounded by convexity, see [12, Theorem 2.7.10]. The second fundamental form of a graph hypersurface satisfies

$$(5.10) \qquad h_j^i = \frac{\dot{\vartheta}}{v\vartheta}\delta_j^i + \frac{\dot{\vartheta}}{v^3\vartheta^3}u^i u_j - \frac{\tilde{\sigma}^{ik}}{v\vartheta^2}u_{kj},$$

where $\vartheta = \sin u$ and $\tilde{\sigma}^{ik}$ is the inverse of

$$(5.11) \qquad \tilde{\sigma}_{ik} = \varphi_i \varphi_k + \sigma_{ik}, \quad \varphi = \int_{r_0}^u \sin^{-1}(s)ds,$$

cf. [26, (3.112)]. Here covariant differentiation and index raising is performed with respect to the metric σ_{ij} and by Lemma 4.8 we obtain uniform C^2 estimates for the $u(t, \cdot)$. This gives the existence of a convergent subsequence with uniquely determined C^1 limit $u(T^*, \cdot)$, using monotonicity. $\square$

5.3. Proposition. *Suppose that on some time interval $[s, t] \subset (0, T^*)$ there is a point $x_0 \in \operatorname{int} \hat{M}_s$, such that for a common parametrization of the surfaces M_τ,*

$$(5.12) \qquad M_\tau = \operatorname{graph} u(\tau, \mathcal{S}(x_0)),$$

there is a constant $\epsilon > 0$ satisfying $u \le \frac{\pi}{2} - \epsilon$, then the curvature function of those hypersurfaces is uniformly positive,

$$(5.13) \qquad F \ge \tilde{c} > 0,$$

where $\tilde{c}$ depends on ϵ.

Proof. Using [12, Lemma 3.3.2] and (4.10) we deduce the evolution equations for $-\Phi$ and u to be

$$(5.14) \qquad \frac{d}{dt}(-\Phi) - \Phi'F^{ij}(-\Phi)_{ij} = \Phi'F^{ij}h_{ik}h_j^k(-\Phi) + \Phi'F^{ij}g_{ij}(-\Phi)$$

and

$$(5.15) \qquad \frac{d}{dt}u - \Phi'F^{ij}u_{ij} = (p^{-1} + 1)\Phi'Fv^{-1} - \Phi'F^{ij}\bar{h}_{ij}.$$

Set

$$(5.16) \qquad w = \log(-\Phi) + f(u),$$

where f will be specified later. Then

(5.17)
$$\frac{d}{dt}w - \Phi' F^{ij} w_{ij} = \Phi' F^{ij} h_{ik} h_j^k + \Phi' F^{ij} g_{ij} + \Phi' F^{ij} \frac{\Phi_i}{\Phi} \frac{\Phi_j}{\Phi}$$
$$+ (p^{-1}+1) f' \Phi' F v^{-1} - f' \Phi' F^{ij} \bar{h}_{ij} - f'' \Phi' F^{ij} u_i u_j.$$

We want to bound the function w. Thus suppose without loss of generality, that

(5.18)
$$\sup_{(\tau,\xi)\in(s,t]\times M} w(\tau,\xi) = w(t_0,\xi_0).$$

Then at this point we have

(5.19)
$$\frac{\Phi_i}{\Phi} = -f'(u)u_i$$

and thus, also using $\bar{h}_{ij} = \frac{\bar{H}}{n}\bar{g}_{ij} = \frac{\bar{H}}{n}g_{ij} - \frac{\bar{H}}{n}u_i u_j$, we find at (t_0,ξ_0)

(5.20)
$$0 \le \Phi' F^{ij} h_{ik} h_j^k + \left(1 - f'\frac{\bar{H}}{n}\right) \Phi' F^{ij} g_{ij} + (p^{-1}+1) f' \Phi' F v^{-1}$$
$$+ \left((f')^2 + f'\frac{\bar{H}}{n} - f''\right) \Phi' F^{ij} u_i u_j.$$

Now define the function

(5.21)
$$f(u) = -\log\left(\cos u - \cos\left(\frac{\pi}{2} - \frac{\epsilon}{2}\right)\right) = -\log(\cos u - c),$$

where $c := \cos\left(\frac{\pi}{2} - \frac{\epsilon}{2}\right)$, such that by assumption f is strictly positive and uniformly bounded for $\tau \in [s,t]$. We have

(5.22)
$$f' = \frac{\sin u}{\cos u - c}$$

and

(5.23)
$$f'' = \frac{1}{(\cos u - c)^2} - \frac{c \cdot \cos u}{(\cos u - c)^2}$$

and thus in view of (2.14)

(5.24)
$$f'\frac{\bar{H}}{n} = \frac{\cos u}{\cos u - c} > 1 + \delta, \ \delta = \delta(\epsilon)$$

and

(5.25)
$$(f')^2 + f'\frac{\bar{H}}{n} - f'' = 0.$$

Since $F^{ij} h_{ik} h_j^k \le FH$ due to the convexity of the hypersurfaces, we conclude at (t_0,ξ_0)

(5.26)
$$0 \le \Phi' F^{ij} h_{ik} h_j^k + (p^{-1}+1) f' \Phi' F v^{-1} - n\delta\Phi'$$
$$\le \frac{p}{F^p}H + (p+1)\frac{f'}{vF^p} - n\delta\frac{p}{F^{p+1}}.$$

Supposing that $w(t_0,\xi_0)$ is very large, $-\Phi$ must also be very large, which leads to a contradiction, since H is bounded by Lemma 4.8. Hence w, and thus also $-\Phi$, must be bounded. $\square$

Now we characterize T^*, where in particular we show, that the flow exists as long as the hypersurfaces remain strictly convex.

5.4. **Proposition.** *Suppose that on some interval* $[s, t) \subset [0, T^*)$ *we have*

$$(5.27) \qquad F(\tau, \xi) \geq \epsilon > 0 \quad \forall (\tau, \xi) \in [s, t) \times \mathbb{S}^n,$$

then there holds

$$(5.28) \qquad\qquad\qquad T^* > t.$$

Proof. Without loss of generality suppose that all $M_\tau, s \leq \tau < t$, are uniformly parametrized over $\mathcal{S}(y_0)$. The second fundamental form has, with respect to the corresponding spherical metric, the form

$$(5.29) \qquad\qquad h_j^i = \frac{\dot\vartheta}{v\vartheta} \delta_j^i + \frac{\dot\vartheta}{v^3 \vartheta^3} u^i u_j - \frac{\tilde\sigma^{ik}}{v\vartheta^2} u_{kj}.$$

The scalar flow equation is

$$(5.30) \qquad\qquad \frac{\partial}{\partial t} u = \frac{v}{F^p(h_j^i)} \equiv G(x, u, Du, D^2 u).$$

Thus

$$(5.31) \qquad \frac{\partial G}{\partial u_{ij}} = \frac{pv}{F^{p+1}} F_l^k \frac{\partial h_k^l}{\partial u_{ij}} = \frac{pv}{F^{p+1}} F_l^k \frac{\tilde\sigma^{lm}}{v\vartheta^2} \delta_m^i \delta_k^j = \frac{1}{\vartheta^2 F^{p+1}} F_l^j \tilde\sigma^{li},$$

where the latter is uniformly positive definite by assumption, as well as by Lemma 4.7 or $F|_{\partial\Gamma_+} = 0$. Then, using Krylov-Safonov and Schauder, we obtain uniform $C^{4,\alpha}$ estimates on $[s, t)$ and thus the flow extends beyond t. $\qquad\square$

6. Convergence to an equator

In view of Lemma 5.1 we know that there exists $x_0 \in \mathbb{S}^{n+1}$, such that $\hat M_t \subset \operatorname{int} \mathcal{H}(x_0)$. Now we want to show, that the limit hypersurface M_{T^*} is equal to $\mathcal{S}(x_0)$. We first view the hypersurfaces as embedded in Euclidean space using the conformally flat parametrization of $\mathbb{S}^{n+1}$ via stereographic projection $\mathcal{P}_{x_0}$.

6.1. **Lemma.** *The enclosed, weakly convex body of M_{T^*}, $\hat M_{T^*}$, satisfies a uniform interior sphere condition.*

Proof. We will denote $\mathcal{P}_{x_0}(M_t)$ by $\mathcal{M}_t$ for $t \in [0, T^*]$. Since all M_t range within distance less than $\frac{\pi}{2}$ around x_0, the metrics $\bar g_{\alpha\beta}$ and $\hat g_{\alpha\beta}$ are uniformly equivalent on the set of consideration. Thus we also obtain the C^1-convergence of $\mathcal{M}_t \to \mathcal{M}_{T^*}$. Let $x \in \mathcal{M}_{T^*}$ be arbitrary and $t_n, x_n \in \mathcal{M}_{t_n}$ be sequences such that

$$(6.1) \qquad\qquad t_n \to T^*, \; x_n \to x.$$

By Remark 3.4 we obtain a sequence of inballs with center $y_n \in \operatorname{int} \hat{\mathcal{M}}_{t_n}$ and uniform radius R, such that

$$(6.2) \qquad B_R(y_n) \subset \operatorname{int} \hat{\mathcal{M}}_{t_n} \; \wedge \; \partial B_R(y_n) \cap \mathcal{M}_{t_n} = \{x_n\}.$$

Without loss of generality we have

$$(6.3) \qquad\qquad y_n \to y \in \operatorname{int} \hat{\mathcal{M}}_{T^*}.$$

First of all, let $z \in B_R(y)$. By the triangle inequality for large n there holds

$$z \in B_R(y_n) \subset \operatorname{int} \hat{\mathcal{M}}_{t_n} \subset \operatorname{int} \hat{\mathcal{M}}_{T^*}. \tag{6.4}$$

Thus

$$B_R(y) \subset \operatorname{int} \hat{\mathcal{M}}_{T^*}. \tag{6.5}$$

There holds

$$\operatorname{dist}(x, y) \leq \operatorname{dist}(x, x_n) + \operatorname{dist}(x_n, y_n) + \operatorname{dist}(y_n, y) \to R \tag{6.6}$$

and thus

$$\operatorname{dist}(x, y) = R, \tag{6.7}$$

since $x \in \mathcal{M}_{T^*}$. We summarize:

$$B_R(y) \subset \operatorname{int} \hat{\mathcal{M}}_{T^*} \ \wedge \ \{x\} \subset \partial B_R(y) \cap \mathcal{M}_{T^*}. \tag{6.8}$$

If we now choose an inball

$$B_{\frac{R}{2}}(\tilde{y}) \subset B_R(y) \tag{6.9}$$

with the property

$$\partial B_{\frac{R}{2}}(\tilde{y}) \cap \partial B_R(y) = \{x\}, \tag{6.10}$$

we obtain the desired uniform interior sphere condition with radius $\frac{R}{2}$. $\qquad \square$

6.2. **Lemma.** $\hat{M}_{T^*} \subset \mathcal{H}(x_0)$ *is a convex body in a hemisphere.*

Proof. Let $y_0 \in \operatorname{int} \hat{M}_0$ be arbitrary but fixed. In view of Lemma 5.1 we know that the weakly convex body of M_{T^*} with respect to y_0 can be described as

$$\hat{M}_{T^*} = \overline{\bigcup_{t \in [0, T^*)} \hat{M}_t}. \tag{6.11}$$

In view of the monotonicity, see Lemma 5.1 and Lemma 3.9, we obtain that $\bigcup_{t \in [0, T^*)} \hat{M}_t$ is a convex set in a hemisphere. Lemma 6.1 and Theorem 3.5 imply that $\hat{M}_{T^*} \subset \mathbb{S}^{n+1}$ is a convex body in a hemisphere. $\qquad \square$

The lemmata of this section show, that $\hat{M}_{T^*}$ satisfies the requirements of Theorem 1.1. Finally we will show that $\hat{M}_{T^*}$ is not contained in an open hemisphere:

6.3. **Lemma.** *There is no hemisphere $\mathcal{H}(y_0)$, such that*

$$\hat{M}_{T^*} \equiv \overline{\bigcup_{t \in [0, T^*)} \hat{M}_t} \subset \operatorname{int} \mathcal{H}(y_0). \tag{6.12}$$

Proof. Suppose contrary that this was the case. First we show, that we may assume without loss of generality, that $y_0 \in \operatorname{int} \hat{M}_{T^*}$. Considering the stereographic projection with y_0 corresponding to the origin, $\mathcal{P}_{y_0}$, we see that

$$\mathcal{P}_{y_0}(\hat{M}_{T^*}) \equiv \hat{\mathcal{M}}_{T^*} \subset B_{2-\delta}(0), \tag{6.13}$$

for some $\delta > 0$ and thus also

$$\hat{\mathcal{M}}_t \subset B_{2-\delta}(0) \ \ \forall t < T^*. \tag{6.14}$$

Since the $\hat{\mathcal{M}}_t$ are closed and convex subsets of $\mathbb{R}^{n+1}$, we may use the distance minimizing projection of 0 to $\partial\hat{\mathcal{M}}_t$, $z_t \equiv \mathrm{pr}(0)$, and conclude for arbitrary $x \in \hat{\mathcal{M}}_t$, that

$$\|x - z_t\|^2 = \|x\|^2 - 2\langle x, z_t \rangle + \|z_t\|^2 \le \|x\|^2. \tag{6.15}$$

Thus

$$\hat{\mathcal{M}}_t \subset B_{2-\delta}(z_t) \tag{6.16}$$

and using approximation of z_t from $\mathrm{int}\,\hat{\mathcal{M}}_t$ we obtain that

$$\hat{\mathcal{M}}_t \subset B_{2-\frac{\delta}{2}}(\bar{y}_t), \tag{6.17}$$

for suitable $\bar{y}_t \in \mathrm{int}\,\hat{\mathcal{M}}_t$. A subsequence of $\bar{y}_t$ converges to some $\bar{y} \in \hat{\mathcal{M}}_{T^*}$ and we have

$$\hat{\mathcal{M}}_{T^*} \subset B_{2-\frac{\delta}{3}}(\bar{y}), \tag{6.18}$$

where we may again, by perturbation of $\bar{y}$, assume that $\bar{y} \in \mathrm{int}\,\hat{\mathcal{M}}_{T^*}$. There holds

$$\mathcal{P}_{y_0}^{-1}\left(B_{2-\frac{\delta}{3}}(\bar{y})\right) \subset \mathrm{int}\,\mathcal{H}(\mathcal{P}_{y_0}^{-1}(\bar{y})) \tag{6.19}$$

and thus we are now in the situation, that we may parametrize the surfaces

$$M_t, \ T < t < T^*, \tag{6.20}$$

uniformly as a graphs over the same equator and may apply Propositions 5.3 and 5.4 to conclude that the flow would exist longer than T^*. $\qquad\square$

Hence we obtain:

6.4. Theorem. *There exists $x_0 \in \mathbb{S}^{n+1}$ such that $M_{T^*} = \mathcal{S}(x_0)$.*

7. Geometric inequalities

In this section we want to deduce Alexandrov-Fenchel type inequalities from the convergence of the flow to an equator. We use the inverse mean curvature flow. The mixed volumes of a hypersurface M in the sphere $\mathbb{S}^{n+1}$ are defined for $k \in \{0, \ldots, n\}$ by

$$V_k(M) := \int_M \tilde{H}_k d\mu, \tag{7.1}$$

where $\tilde{H}_k := \frac{H_k}{\binom{n}{k}}$ are the normalized elementary symmetric polynomials. Firstly, we note that a geodesic ball B_ρ of radius $0 < \rho \le \frac{\pi}{2}$ satisfies for $k \in \{0, \ldots, n\}$

$$V_k(B_\rho) = \omega_n \cos^k \rho \sin^{n-k} \rho, \tag{7.2}$$

where ω_n is the volume of $\mathbb{S}^n$. For the sake of brevity, we define $\tilde{V}_k := \frac{V_k}{\omega_n}$. For geodesic spheres, it is easy to obtain a relation between different mixed volumes. For example there holds

$$(\tilde{V}_1)^2 = (\tilde{V}_0)^{2\left(\frac{n-1}{n}\right)} - (\tilde{V}_0)^2, \tag{7.3}$$

and

$$\tilde{V}_2 = \left(\tilde{V}_0\right)^{\frac{n-2}{n}} - \tilde{V}_0. \tag{7.4}$$

Furthermore, we will establish geometric inequalities between certain quermassintegrals.

In $\mathbb{S}^{n+1}$ we have the following definition of the quermassintegrals, compare [27] (also for a more detailed definition of the measure dL_k):

7.1. Definition. Let $\hat{M} \subset \mathbb{S}^{n+1}$ be a compact domain. For $k \in \{1, \ldots, n\}$ set

$$(7.5) \qquad W_k(\hat{M}) = \frac{(n+1-k)\omega_{k-1}\cdots\omega_0}{(n+1)\omega_{n-1}\cdots\omega_{n-k}} \int_{\mathcal{L}_k} \chi(L_k \cap \hat{M})dL_k,$$

where $\mathcal{L}_k$ is the space of k-dimensional totally geodesic subspaces L in $\mathbb{S}^{n+1}$, dL_k is the natural measure on $\mathcal{L}_k$ and χ is the Euler characteristic. Furthermore set

$$(7.6) \qquad\qquad\qquad W_0(\hat{M}) = \mathrm{Vol}(\hat{M})$$

and

$$(7.7) \qquad\qquad\qquad W_{n+1}(\hat{M}) = \frac{\omega_n}{n+1}\chi(\hat{M}).$$

In Euclidean space the quermassintegrals differ only by constants with respect to the corresponding curvature integrals. This relation is more complicated in curved spaces, however, we still have the following relation between the curvature integrals and the quermassintegrals in the space $\mathbb{S}^{n+1}$, see for example [27, Proposition 7, Corollary 8] for a proof of this relation.

7.2. Lemma. *If $\hat{M} \subset \mathbb{S}^{n+1}$ is a compact domain with C^2-boundary. Then there holds $W_1(\hat{M}) = \frac{1}{n+1}V_0(\partial M)$ and for $k \in \{1, \ldots, n\}$ there holds*

$$(7.8) \qquad V_k(\partial\hat{M}) = (n+1)\left(W_{k+1}(\hat{M}) - \frac{k}{n+2-k}W_{k-1}(\hat{M})\right).$$

Furthermore for $k \in \mathbb{N}$ with $2k+1 \leq n$ we have

$$(7.9) \qquad W_{2k+1}(\hat{M}) = \frac{1}{n+1}\sum_{i=0}^{k} \frac{(2k)!!(n-2k)!!}{(2k-2i)!!(n-2k+2i)!!}V_{2k-2i}(\partial\hat{M}).$$

We also need the following evolution equations. These evolution equations can be computed by an induction argument using Lemma 4.5 and Lemma 7.2, see also the proof of [30, Proposition 3.1].

$$(7.10) \qquad \frac{d}{dt}\int_{M_t} H_k d\mu_t = \int_{M_t} \left((k+1)\frac{H_{k+1}}{F} - (n+1-k)\frac{H_{k-1}}{F}\right) d\mu_t.$$

$$(7.11) \qquad \frac{d}{dt}W_k(M_t) = \frac{n+1-k}{n+1}\int_{M_t} \frac{\tilde{H}_k}{F}d\mu_t.$$

Finally, let us state a decay lemma for the inverse curvature flow.

7.3. Lemma. *For all $1 \leq q < \infty$ there holds*

$$(7.12) \qquad\qquad\qquad \lim_{t\to T^*}\int_{M_t} H^q = 0.$$

Proof. By the previous results we know

(7.13)
$$u \to c$$

in the C^1 norm, where we use a graph representation as in Prop. 5.2. The second fundamental form is

(7.14)
$$h_{ij}v^{-1} = -u_{ij} + \bar{h}_{ij},$$

where covariant differentiation is performed with respect to the induced metric. Thus

(7.15)
$$\int_{M_t} Hv^{-1} = \int_{M_t} \bar{h}_{ij}g^{ij}$$

and

(7.16)
$$\int_{M_t} H = \int_{M_t} H(1 - v^{-1}) + \int_{M_t} \left(\bar{H} - \frac{\bar{H}}{n}\|Du\|^2 \right).$$

Since

(7.17)
$$g_{ij} = u_i u_j + \sinh^2 u\,\sigma_{ij},$$

the volume element is uniformly bounded and thus the right hand side converges to 0. The other L^q norms converge to 0 by interpolation.

$\square$

This leads to our first geometric inequality:

7.4. Theorem. *Let $M \subset \mathbb{S}^{n+1}$ be a closed, orientable, convex C^2-hypersurface. Then we have the inequality*

(7.18)
$$(\tilde{V}_1(M))^2 \geq (\tilde{V}_0(M))^{2\left(\frac{n-1}{n}\right)} - (\tilde{V}_0(M))^2,$$

and equality holds if and only if M is a geodesic sphere.

Proof. First of all, we can assume that the hypersurface is smooth and strictly convex, since otherwise we can use convolutions as in the proof of Corollary 1.2 to obtain a sequence of approximating smooth, strictly convex hypersurfaces converging in C^2 to M.

We consider the flow of the initial hypersurface M by the inverse mean curvature. Let M_t, $t \in [0, T^*]$, be the level hypersurfaces of the flow, where we know that M_{T^*} is a geodesic sphere with radius $\frac{\pi}{2}$ and the graphs over a geodesic sphere $M_t = \text{graph}|_{\mathbb{S}^n} u(t, \cdot)$ converge in C^1 to $u(T^*, \cdot) \equiv \frac{\pi}{2}$. Furthermore we know that the mean curvature of the level hypersurfaces converges almost everywhere to 0 by Lemma 7.3.

For $t \in [0, T^*]$ we define

(7.19)
$$\phi(t) := \frac{(V_1(M_t))^2}{(V_0(M_t))^{2\left(\frac{n-1}{n}\right)}} + (V_0(M_t))^{\frac{2}{n}}.$$

Then $\phi(t) \to (V_0(B_{\frac{\pi}{2}}))^{\frac{2}{n}}$ for $t \to T^*$, since $V_0(M_t) \to V_0(M_{T^*})$ for $t \to T^*$ in view of the C^1-convergence and $V_1(M_t) \to 0$ for $t \to T^*$ in view of the convergence of the mean curvature to zero almost everywhere.

Hence if we can show that ϕ is monotonically non-increasing, we obtain

(7.20)
$$\frac{(V_1(M))^2}{(V_0(M))^{2\left(\frac{n-1}{n}\right)}} + (V_0(M))^{\frac{2}{n}} = \phi(0) \geq \phi(T^*) = \omega_n^{\frac{2}{n}}.$$

This implies the geometric inequality.

We have in view of (7.10) and (iii) from Lemma 4.2 for $F = H$

$$(V_0(M_t))^{2\left(\frac{n-1}{n}\right)+1}\frac{d\phi}{dt} = 2V_0(M_t)\int_{M_t}\frac{H}{n}d\mu_t\int_{M_t}\left(\frac{2H_2}{nF} - \frac{1}{F}\right)d\mu_t$$

$$(7.21) \qquad - 2\frac{n-1}{n^3}\left(\int_{M_t}Hd\mu_t\right)^2\int_{M_t}\frac{H}{F}d\mu_t + \frac{2}{n}(V_0(M_t))^2\int_{M_t}\frac{H}{F}d\mu_t$$

$$\leq \frac{2}{n}V_0(M_t)\left((V_0(M_t))^2 - \int_{M_t}Hd\mu_t\int_{M_t}\frac{1}{H}d\mu_t\right) \leq 0.$$

In the last inequality we have used the Cauchy-Schwarz inequality and hence the inequality is strict unless H is constant on M_t. Hence we obtain that ϕ is monotonically decreasing unless M_t is a geodesic sphere. $\qquad\square$

We can also prove a geometric inequality relating $\tilde{V}_2$ and $\tilde{V}_0$.

7.5. Theorem. *Let $M \subset \mathbb{S}^{n+1}$ be a closed, orientable, convex C^2-hypersurface. Then we have the inequality*

$$(7.22) \qquad \tilde{V}_2(M) \geq \left(\tilde{V}_0(M)\right)^{\frac{n-2}{n}} - \tilde{V}_0(M),$$

and equality holds if and only if M is a geodesic sphere.

Proof. Again we assume M to be smooth and strictly convex and we use the same curvature flow as in the proof of Theorem 7.4. For $t \in [0, T^*]$ we define

$$(7.23) \qquad \phi(t) := \frac{(V_2(M_t)) + (V_0(M_t))}{(V_0(M_t))^{\frac{n-2}{n}}}.$$

Again we have to show that ϕ is monotonically non-increasing to obtain the geometric inequality (7.22). We have in view of (7.10) for $F = H$

(7.24)

$$(V_0(M_t))^{\frac{n-2}{n}+1}\frac{d\phi}{dt} = V_0(M_t)\int_{M_t}\frac{3H_3}{\binom{n}{2}F} - V_0(M_t)\int_{M_t}\left(1 - \frac{(n-1)}{\binom{n}{2}}\right)\frac{H}{F}d\mu_t$$

$$- \frac{n-2}{n}\int_{M_t}\frac{H}{F}d\mu_t\left(\int_{M_t}\tilde{H}_2d\mu_t + V_0(M_t)\right)$$

$$= \frac{V_0(M_t)}{\binom{n}{2}}\int_{M_t}\left(\frac{3H_3}{H} - \frac{n-2}{n}H_2\right)d\mu_t \leq 0,$$

where the last inequality follows from (iii) from Lemma 4.2. Again, this inequality is strict unless the hypersurface is totally umbilic, which implies that ϕ is only stationary for geodesic spheres. $\qquad\square$

7.6. Theorem. *Let $M \subset \mathbb{S}^{n+1}$ be a closed, orientable, convex C^2-hypersurface. Let $k \in \mathbb{N}_+$ with $2k + 1 \leq n$ and let $\hat{M}$ be the convex body enclosed by M. Then we have the inequality*

(7.25)

$$W_{2k+1}(\hat{M}) \geq \frac{\omega_n}{n+1}\sum_{i=0}^{k}(-1)^i\frac{n+1-2k}{n+1-2k+2i}\binom{k}{i}\left(\frac{n+1}{\omega_n}W_1(\hat{M})\right)^{\frac{n-2k+2i}{n}}.$$

and equality holds if and only if M is a geodesic sphere.

Proof. For the sake of brevity, let us denote the right hand side of the equation (7.25) by $A_k(\hat{M})$.

We prove the result by induction on k. For $k = 1$ the result follows from Theorem 7.5 and Lemma 7.2. Let us assume we have proved the result for $k - 1 \in \mathbb{N}$ with $2k + 1 \le n$.

We assume the hypersurface to be smooth and strictly convex and use the same curvature flow as in the proof of Theorem 7.4. For $t \in [0, T^*]$ we define

$$(7.26) \qquad \phi(t) := \frac{W_{2k+1}(\hat{M}_t) - A_k(\hat{M}_t)}{W_1(\hat{M}_t)^{\frac{n-2k}{n}}}.$$

From now on we will drop the arguments and write simply W_k instead of $W_k(\hat{M})$. We will show that ϕ is monotonically non-increasing. We obtain from (7.11) with $F = H$ and Lemma 7.2

$$(7.27)
\begin{aligned}
\frac{d}{dt}\phi &\le \frac{(n-2k)}{(n+1)} \frac{\int_{M_t} \tilde{H}_{2k} d\mu_t - (n+1)W_{2k+1}}{W_1^{\frac{n-2k}{n}}} - \frac{d}{dt}\left(\frac{A_k}{W_1^{\frac{n-2k}{n}}}\right) \\
&= \frac{(n-2k)2k}{(n+2-2k)} \frac{W_{2k-1}}{W_1^{\frac{n-2k}{n}}} - \frac{d}{dt}\left(\frac{A_k}{W_1^{\frac{n-2k}{n}}}\right).
\end{aligned}$$

Note that the inequality at time t is an equality if and only M_t is totally umbilic. Using the induction hypothesis we obtain after a simple calculation

$$(7.28) \qquad \frac{d}{dt}\phi \le 0,$$

again with equality if and only if M_t is totally umbilic. Employing (7.9) and the decay Lemma, we obtain

$$(7.29)
\begin{aligned}
W_{2k+1}(\hat{M}_t) - A_k(\hat{M}_t) \to & \frac{\omega_n}{n+1} \frac{(2k)!!(n-2k)!!}{n!!} \\
& - \frac{\omega_n}{n+1} \sum_{i=0}^{k} (-1)^i \frac{n-2k}{n-2k+2i} \binom{k}{i}
\end{aligned}$$

for $t \to T^*$. However, the right hand side equals zero, as a proof by induction shows. Hence we obtain the inequality (7.25) and equality holds if and only if M is a geodesic sphere. $\qquad\square$

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Dr. Matthias Makowski, Universität Konstanz, Fachbereich Mathematik und Statistik, 78457 Konstanz, Germany
E-mail address: matthias.makowski@uni-konstanz.de

Dr. Julian Scheuer, Ruprecht-Karls-Universität, Institut für Angewandte Mathematik, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
E-mail address: scheuer@math.uni-heidelberg.de