Linear Spaces and Grassmannians

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Let V be a vector space of dimension n over a field K. In Lecture 2 we constructed the projective space $\mathbb{P}(V)$. Its points are the 1-dimensional subspaces of V, and this is the principal example of a compact algebraic variety when $K = \mathbb{C}$. Our aim is to generalize this construction from lines to subspaces of arbitrary dimension k. We will construct a projective variety $G(k, V)$ whose points correspond bijectively to k-dimensional subspaces of V. This variety is called the Grassmannian, after the 19th century mathematician Hermann Grassmann, and it is also denoted $G(k, n)$ when $V = K^n$.

We start with an explicit construction in coordinates, by fixing a basis $e_1, \ldots, e_n$ of V. Let us consider any k linearly independent vectors $v_1, \ldots, v_k \in V$. We represent them in a form of a $k \times n$ matrix M_v of rank k. To these vectors, or equivalently to a full rank matrix, we associate the linear subspace $\langle v_1, \ldots, v_k \rangle$ in V. This association is clearly surjective, but not injective, as we may replace the $v_i' s$ by general linear combinations. In other words, the group $GL(k)$ acts on the set of $k \times n$ matrices by left multiplication, and this does not change the linear span of the rows.

Fortunately we know polynomial functions that do not change (up to scaling) under taking linear combinations of the rows: these are the $k \times k$ minors of the $k \times n$ matrix. Suppose that W is a k-dimensional subspace of V. Pick any basis and express W as the row space of a $k \times n$-matrix. We then write $\mathfrak{i}(W)$ for the vector of all $k \times k$-minor of that matrix, up to scale. This construction defines a map

$$\mathfrak{i} : \{k\text{-dimensional subspaces of } V\} \to \mathbb{P}(K^{\binom{n}{k}}).$$

The map $\mathfrak{i}$ is well-defined since $\mathfrak{i}(W)$ does not depend on the choice of the basis of W.

Lemma 1. *The map* $\mathfrak{i}$ *is injective.*

Proof. Consider two k-dimensional subspaces $W_1, W_2 \subset V$. Assume $\mathfrak{i}(W_1) = \mathfrak{i}(W_2)$. As the matrices M_{W_1} and M_{W_2} representing respectively W_1 and W_2 are of full rank, without loss of generality we may assume that the minor given by first k columns is nonzero. By performing linear operations on the rows of both matrices, we transform M_{W_i} to a matrix $\tilde{M}_{W_i}$ whose left-most $k \times k$ submatrix is the identity. We observe that any entry of $\tilde{M}_{W_i}$ not in the first k columns, is equal to some maximal minor or its negation. Thus, if $\mathfrak{i}(W_1) = \mathfrak{i}(W_2)$ the two matrices $\tilde{M}_{W_1}, \tilde{M}_{W_2}$ must be equal. This implies $W_1 = W_2$. $\square$

The image of $\mathfrak{i}$ is known as the *Grassmannian* $G(k,n)$ and the inclusion in $\mathbb{P}(K^{\binom{n}{k}})$ as the *Plücker embedding*. Our aim is now to prove that the Grassmannian is a projective variety. Equivalently, we need to express the fact that $\binom{n}{k}$ numbers are minors of a matrix, by vanishing of (homogeneous) polynomials.

Theorem 2. *The Grassmannian $G(k,n) \subset \mathbb{P}(K^{\binom{n}{k}})$ is Zariski closed and irreducible.*

Proof. Lemma 1 gives us an idea how to proceed. Namely, first let us assume that the matrix M_W representing W is of the form:

$$\left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & \\ \vdots & & \ddots & \vdots & \text{A} \\ 0 & \dots & 0 & 1 & \end{array}\right], \tag{1}$$

where A is a $k \times (n-k)$ matrix. Each maximal minor of M_W is now, up to sign, a minor of A of some size. Further, by Laplace expansion, a $q \times q$ minor of A for $q > 1$ may be expressed, as a quadratic polynomial, in terms of smaller minors. This exactly provides us a collection of $\sum_{q=2}^{\min(k,n-k)} \binom{k}{q}\binom{n-k}{q}$ inhomogeneous quadratic equations in the entries of the $k \times (n-k)$-matrix A. These quadratic equations define the part of image of our map $\mathfrak{i}$ that lies in the affine open set $K^{\binom{n}{k}-1} \subset \mathbb{P}(K^{\binom{n}{k}})$ given by the nonvanishing of the first coordinate.

If $\mathfrak{i}(W)$ has its first coordinate zero then some other coordinate will be nonzero. In other words, the matrix M_W must have some invertible $k \times k$ submatrix. If we multiply M_W on the left by the inverse of that matrix then we obtain a matrix that looks like (1) but with its columns permuted. The same construction as before gives us a system of $\sum_{q=2}^{\min(k,n-k)} \binom{k}{q}\binom{n-k}{q}$ inhomogeneous quadratic equations in the $k(n-k)$ entries of the new matrix A.

Each of the quadratic equations in $k(n-k)$ variables obtained above can be written as a homogeneous quadratic equation in the $\binom{n}{k}$ coordinates on $\mathbb{P}(K^{\binom{n}{k}})$. Namely, a minor of A is replaced by the corresponding maximal minor of M_W, and then the quadric is homogenized by the special minor that corresponds to the identity matrix in (1). The collection of all these homogeneous quadratic equations gives a full polynomial description of $G(k,n)$.

The Grassmannian $G(k,n)$ is an irreducible subvariety of $\mathbb{P}(K^{\binom{n}{k}})$ because it is the image of a polynomial map $\mathfrak{i}$, namely the image of the space $K^{k \times n}$ of all $k \times n$ matrices under taking all maximal minors. □

Note that we have proved that as a set $G(k,n)$ may be defined by quadratic equations. In fact, with slightly more effort one can show that $I(G(k,n))$ may be generated by quadratic polynomials, known as *Plücker relations* [2, Chapter 3]. Further below we will discuss the Plücker relations for $k = 2$. From a more algebraic perspective the equations vanishing on $G(k,n)$ are exactly the *polynomial relations among maximal minors*. We point out that finding polynomial equations among nonmaximal minors of a fixed size is an important open problem in commutative algebra.

Another fact that follows from the proof, is that the intersection $G(k,n) \cap K^{\binom{n}{k}-1}$ of the Grassmannian with the open affine is a nonlinearly embedded affine space $K^{k \times (n-k)}$.

Corollary 3. *The dimension of the Grassmannian $G(k, n)$ equals $k(n-k)$.*

Remark 4. The Grassmannian $G(k, n)$ parametrizes k-dimensional vector subspaces of an n-dimensional vector space, or equivalently $k-1$ dimensional projective subspaces of an $n-1$ dimensional projective space.

For readers familiar with the exterior power of a vector space, here is a more invariant way to describe the Grassmannian:

$$G(k, n) = \{[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}(\bigwedge^k V) : v_1, \ldots, v_k \in V \text{ are linearly independent}\}.$$

Indeed, first we may identify $\mathbb{P}(K^{\binom{n}{k}})$ with $\mathbb{P}(\bigwedge^k V)$ by fixing a basis of V and an induced basis of $\bigwedge^k V$. Expanding $v_1 \wedge \cdots \wedge v_k$ in a basis we indeed obtain the minors of $[v_1, \ldots, v_k]$. The group $GL(V)$ acts naturally on V, taking subspaces to subspaces. This induces an action on $\mathbb{P}(\bigwedge^k V)$, that restricts to the Grassmannian. Precisely $g \in GL(V)$ transforms $v_1 \wedge \cdots \wedge v_k$ to $g(v_1) \wedge \cdots \wedge g(v_k)$. We note that the action is *transitive*: for any $p_1, p_2 \in G(k, V)$ there exists a (non-unique) automorphism $g \in GL(V)$ such that $g(p_1) = p_2$. This holds because any set of k linearly independent vectors may be transformed by an invertible linear map to any other such set. Hence, $G(k, V)$ is an *orbit* under the action of $GL(V)$ on $\mathbb{P}(\bigwedge^k V)$. In fact, $G(k, V)$ is characterised as the unique closed orbit in this space. Projective algebraic varieties that are orbits of linear algebraic groups are called *homogeneous*, the Grassmannians being prominent examples. Homogeneous varieties are always smooth. Indeed, any algebraic variety always contains a smooth point and an action of a group must take a smooth point to a smooth point - a version of this statement is given in Exercise 2.

Example 5. We consider $G(2, 4)$. The Grassmannian is the image of a map:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \mapsto (af - be : ag - ce : ah - de : bg - cf : bh - df : ch - dg) \in \mathbb{P}^5.$$

Alternatively, fixing a basis (v_1, v_2, v_3, v_4) of V we may write:

$$(av_1 + bv_2 + cv_3 + dv_4) \wedge (ev_1 + fv_2 + gv_3 + hv_4) =$$

$$(af-be)v_1 \wedge v_2 + (ag-ce)v_1 \wedge v_3 + (ah-de)v_1 \wedge v_4 + (bg-cf)v_2 \wedge v_3 + (bh-df)v_2 \wedge v_4 + (ch-dg)v_3 \wedge v_4.$$

This Grassmannian has dimension 4, i.e. it is a hypersurface in $\mathbb{P}^5$. We write the coordinates on $\mathbb{P}^5$ as $(p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$. The indices refer to the minors of a 2×4-matrix. Following the proof of Theorem 2, we look at matrices (1). They take the form

$$\begin{bmatrix} 1 & 0 & c & d \\ 0 & 1 & g & h \end{bmatrix}.$$

The expansion of the rightmost 2×2-minor yields the inhomogeneous quadratic equation $p_{34} = ch - dg = (-p_{23})p_{14} - (-p_{24})p_{13}$. We homogenize this equation with the extra variable p_{12}. We conclude that $G(2, 4)$ is the hypersurface in $\mathbb{P}^5$ defined by the *Plücker quadric*

$$p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}. \tag{2}$$

In what follows we want to describe special subvarieties of Grassmannians, on the example of $G(2,4)$. Let us fix a complete flag in $\mathbb{P}(K^4)$, i.e. $f_0 = \mathbb{P}^0 \subset f_1 = \mathbb{P}^1 \subset f_2 = \mathbb{P}^2 \subset \mathbb{P}^3$. Our aim is to group families of projective lines, i.e. subvarieties of $G(2,4)$, according to how they intersect the flag. Clearly the flag distinguishes a point in $G(2,4)$, namely $X_0 := f_1 \in G(2,4)$. There is also a distinguished one dimensional variety X_1: lines l such that $f_0 \in l \subset f_2$. The most interesting is the case of two dimensional subvarieties. There are two types of those:

1. X_2 consisting of lines l such that $f_0 \in l$ and
2. $X_{2'}$ consisting of lines l such that $l \in f_2$.

There is also one three dimensional variety X_3, consisting of all lines that intersect the given line f_1. The varieties $X_1, X_2, X_{2'}$ and X_3 we described are called *Schubert subvarieties*. In Exercise 1 you will generalize their construction to arbitrary Grassmannians.

We now describe the geometry of above varieties explicitly in $\mathbb{P}^5$, as in Example 5. We assume $f_i = \langle v_0, \dots v_i \rangle$, for $i = 0,1,2$. The point $f_1 \in G(2,4)$ is given by vanishing of all the coordinates p_{ij} apart from p_{12}. We have $f_0 \in l \subset f_2$ if and only if l admits the following matrix representation:

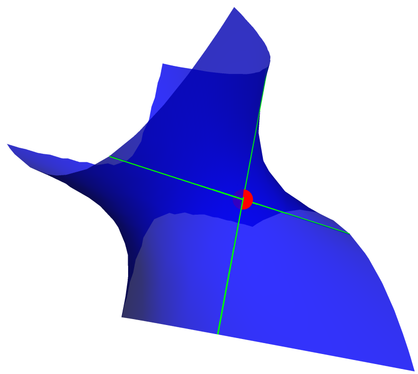
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f & g & 0 \end{bmatrix}.$$

In particular, the subvariety X_1 is given by vanishing of all p_{ij} apart from p_{12} and p_{13} and, as those two minors can be arbitrary, it is a line $\mathbb{P}^1 \subset G(2,4) \subset \mathbb{P}^5$. Similarly, X_2 is a $\mathbb{P}^2$ with coordinates p_{12}, p_{13}, p_{14} and $X_{2'}$ is a different $\mathbb{P}^2$ with coordinates p_{12}, p_{13}, p_{23}. Recall from Example 5 that $G(2,4)$ is a four dimensional quadric in $\mathbb{P}^5$.

In general, for a nondegenerate $2k$ dimensional quadric $Q \subset \mathbb{P}^{2k+1}$ and any $L = \mathbb{P}^{k-1} \subset Q$ there exist exactly two k-dimensional subspaces that contain L and are contained in Q. These are X_2 and $X_{2'}$ for $L = X_1$. For $k = 1$ this is a classical fact of projective geometry. A quadratic surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Through any point $p = L$ there pass precisely two lines contained in the quadric. In Figure 1 the blue quadric contains the red point p and the two lines are green. This three dimensional picture may be seen in fact as a cut of $\mathbb{P}^5$ with a codimension two projective space H defined by $p_{12} = p_{34} = 0$. The red point p equals $H \cap X_1$. The quadric is simply $H \cap G(2,4)$ and the two green lines correspond to $X_2 \cap H$ and $X_{2'} \cap H$.

For readers familiar with algebraic topology, we remark that Schubert varieties represent cohomology classes in the Grassmannian when $K = \mathbb{C}$. *Schubert calculus* is the study of intersection of these classes. When intersecting Schubert varieties as cohomology classes one should consider them coming from *different* flags. As we have seen in our construction $X_2 \cap X_{2'} = X_1$. However, if we consider all lines in $\mathbb{P}^3$ going through a fixed point and all lines contained in an *unrelated* $\mathbb{P}^2 \subset \mathbb{P}^3$, we see that both sets of lines do not have common elements. Hence, the intersection of the two classes is empty. On the other hand, if we consider lines going through two points, or contained in two planes, in each case we obtain exactly one line. The selfintersections of X_2 and $X_{2'}$ (as cohomology classes) give one (point).

Figure 1: Quadric $p_{23}p_{14} - p_{13}p_{24} = 0$ obtained by cutting $G(2,4)$ with an $H = \mathbb{P}^3$ in the affine chart $p_{13} + p_{24} = 1$. The green lines are equal to $X_2 \cap H$ and $X_{2'} \cap H$. The red point is $X_1 \cap H$.



The discussion above can be summarized by the following multiplicative relations that hold in the cohomology ring of $G(2,4)$. Recall that multiplication represents intersection:

$$[X_3][X_3] = [X_2] + [X_{2'}], \;\; [X_2][X_2] = [X_{2'}][X_{2'}] = [X_3][X_1] = 1, \;\; [X_2][X_{2'}] = 0.$$

With this we can now answer questions like the following: *How many lines pass through four general lines in $\mathbb{P}^3$?* The set of such lines is a finite set in $G(2,4)$. It is the the intersection of four hypersurfaces, all of the form X_3. Since intersections are represented by multiplication in the cohomology ring, the following computation shows that the answer is *two*:

$$[X_3]^4 \; = \; ([X_3][X_3])^2 \; = \; ([X_2] + [X_{2'}])^2 \; = \; [X_2]^2 + 2[X_2][X_{2'}] + [X_{2'}]^2 = 1 + 2 \cdot 0 + 1 = 2.$$

Here is another question that can be answered by Schubert calculus: *How many lines are simultaneously tangent to four general quadratic surfaces in $\mathbb{P}^3$?* Suppose a given quadric Q in $\mathbb{P}^3$ is represented as a symmetric 4×4-matrix. Then the condition for a line to be tangent to Q is expressed by the vanishing of the quadratic form $P(\wedge_2 Q)P^T$ in the Plücker coordinates $P = (p_{12}, p_{13}, \ldots, p_{34})$ of that line. Therefore, the cohomology class of lines tangent to Q is $2[X_3]$. Therefore, the number of lines tangent to four given general quadrics in $\mathbb{P}^3$ equals

$$(2[X_3])^4 \; = \; 16[X_3]^4 \; = \; 16 \cdot 2 \; = \; 32.$$

Of course, in order to actually compute the 32 lines over $\mathbb{C}$, when we are given four concrete quadrics in $\mathbb{P}^3$, we would need to carry out some serious Gröbner basis computations.

Remark 6. Grassmannians are named after Hermann Grassmann, a XIX century German polymath. However, it was Julius Plücker who first realized that lines in 3-space may be studied as a four-dimensional object [3]. The (earlier) discoveries of Grassmann were much more fundamental: he was the one to realize that algebraic setting of geometry allows to consider objects not in three-dimensional space, but in any dimension.

We now discuss the homogeneous prime ideal $I(\mathrm{Gr}(k, n))$ of the Grassmannian $\mathrm{Gr}(k, n)$. A complete answer is known, in terms of certain quadratic relations that form a Gröbner basis. These are known as *straightening relations*. For a derivation and explanation see e.g. [4, Chapter 3]. These relations are implemented in the computer algebra system `Macaulay2`, where one obtains generators for $I(\mathrm{Gr}(k, n))$ with the convenient command `Grassmannian`.

We here present the answer in the special case $k = 2$. The corresponding Grassmannian $\mathrm{Gr}(2, n)$ is the space of lines in $\mathbb{P}^{n-1}$. It is convenient to write the $\binom{n}{2}$ Plücker coordinates as the entries of a skew-symmetric $n \times n$-matrix $P = (p_{ij})$. We are interested in the principal submatrices of P having size 4×4. One such submatrix is given by taking the first four rows and first four columns. The determinant of that matrix is the square of the Plücker quadric (2). One refers to the square root of the determinant of a skew-symmetric matrix of even order as its *pfaffian*. Thus the 4×4 pfaffians of our matrix P are the $\binom{n}{4}$ quadrics

$$\underline{p_{il}p_{jk}} - p_{ik}p_{jl} + p_{ij}p_{kl} \qquad \text{for} \quad 1 \leq i < j < k < l \leq n. \tag{3}$$

Theorem 7. *The $\binom{n}{4}$ quadrics listed in (3) form the reduced Gröbner basis of the Plücker ideal $I(\mathrm{Gr}(2, n))$, for any monomial ordering on the polynomial ring in the $\binom{n}{2}$ variables p_{ij} that selects the underlined leading terms.*

Proof. The argument in the proof of Theorem 2 shows that the quadrics (3) cut out $\mathrm{Gr}(2, n)$ as a subset in $\mathbb{P}^{\binom{n}{2}-1}$. In other words, our Grassmannian is given as the set of skew-symmetric $n \times n$-matrices whose 4×4 pfaffians vanish. These are skew-symmetric matrices of rank 2.

By Hilbert's Nullstellensatz, we now know that the radical of $I(\mathrm{Gr}(2, n)$ is generated by (3). We need to argue that this ideal is radical. However, this follows from the assertion that (3) form a Gröbner basis. Indeed, the leading monomials $p_{il}p_{jk}$ are square-free, so they generate a radical monomial ideal. However, if the initial ideal $\mathrm{in}(J)$ of some ideal J is radical then also J itself is radical. So, all we need to do is to verify the Gröbner basis property for our quadrics. That Gröbner basis is then automatically a reduced Gröbner basis because none of the two trailing terms in (3) is a multiple of some other leading term.

To verify the Göbner basis property, we argue as follows. For $n = 4$, it is trivial because there is only one generator. For $n = 5, 6, 7$, this is a direct computation, e.g. using `Macaulay2`. One checks that the S-polynomial of any two quadrics in (3) reduces to zero. Suppose that $n \geq 8$ and consider two Plücker quadrics. These involve at most 8 distinct indices. If the number of distinct indices is 7 or less then we are done by the aforementioned computation, which already verified the claim upto $n = 7$. Hence we may assume that all eight indices occurring in the two Plücker quadrics are distinct. In that case, the two underlined leading monomials are relatively prime. Here, Buchberger's Second Criterion applies, and we can conclude that the S-polynomial automatically reduces to zero. In conclusion, all S-polynomials formed by pairs from (3) reduce to zero. This completes the proof. ∎

Exercises

1. Fix a complete flag in $\mathbb{P}^n$. Construct a bijection between:

   - subvarieties of $G(k,n)$, that can be defined as $l \in G(k,n)$ that intersect each element of the flag in at most the given dimension, and
   - Young diagrams contained in a $k \times (n-k)$ rectangle.

   The codimension (or dimension - depending on the construction you choose) of the subvariety in $G(k,n)$ equals the number of boxes in the corresponding Young diagram.

2. Let G be a subgroup of $GL(V)$ and let $X \subset V$ be a variety, such that the action of G on V restricts to X. Prove that if x is a smooth point of X and $g \in G$, then gx is also a smooth point. Hint: Consider the action of G on the polynomial ring.

3. Consider $G(2,4) \times G(2,4) \subset \mathbb{P}^5 \times \mathbb{P}^5$. Describe the locus of pairs of lines $(l_1, l_2) \in G(2,4) \times G(2,4)$ such that l_1 intersects l_2 in $\mathbb{P}^3$. Hint: Present both lines as 2×4 matrices. Note that two lines in $\mathbb{P}^3$ intersect if and only if they do not span the whole ambient space. Apply Laplace expansion of the determinant.

4. For a variety $X \subset \mathbb{P}^n$, one considers a subset of $G(k+1, n+1)$ of $\mathbb{P}^k \subset X$. This is known as the Fano variety of k dimensional subspaces of X. Fix a nondegenerate quadric $Q \subset \mathbb{P}^3$. Describe the Fano variety of lines in it. Hint: One may solve this exercise either theoretically or using algebra software. Also Figure 1 gives a hint about the answer.

5. How many real lines in 3-space can be simultaneously tangent to four given spheres?

6. The two lines incident to four given real lines in $\mathbb{P}^3$ can be either real or complex. In the latter case they form a complex conjugate pair. Write down a polynomial in the $24 = 4 \cdot 6$ Plücker coordinates of four given lines whose sign distinguishes the two cases.

7. How many lines in $\mathbb{P}^3$ are simultaneously incident to two given lines and tangent to two given quadratic surfaces?

8. Consider the set of all lines in $\mathbb{P}^3$ that are tangent to the cubic Fermat surface $\{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\}$. This set is an irreducible hypersurface in the Grassmannian $\mathrm{Gr}(2,4)$. Compute a polynomial in $p_{12}, p_{13}, \ldots, p_{34}$ that defines this hypersurface.

9. Write down a minimal generating set for the ideal of the Grassmannian $\mathrm{Gr}(3,6)$.

10. Prove that the determinant of a skew-symmetric $n \times n$-matrix is zero if n is odd, and it is the square of a polynomial when n is even.

11. Examine the monomial ideal that is generated by the underlined initial monomials in (3). Express this ideal as the intersection of prime ideals. How many primes occur?

12. Fix six general planes $\mathbb{P}^2$ in $\mathbb{P}^4$. How many lines in $\mathbb{P}^4$ intersect all six planes?

13. Let $n = 2k$ and suppose that the $n \times n$-matrix $A = (a_{ij})$ in (1) is symmetric, i.e. its entries satisfy the equations $a_{ij} = a_{ji}$. Write these equations in terms of the $\binom{2n}{n}$ Plücker coordinates. The resulting subvariety of $\mathrm{Gr}(n, 2n)$ is the *Lagrangian Grassmannian.*

References

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