Tropical Algebra

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The *tropical semiring* $(\mathbb{R}\cup\{\infty\}, \oplus, \odot)$ consists of the real numbers $\mathbb{R}$, together with an extra element ∞ called infinity. The arithmetic operations of addition and multiplication are

$$x \oplus y \;:=\; \min(x,y) \qquad \text{and} \qquad x \odot y \;:=\; x+y.$$

The *tropical sum* of two numbers is their minimum, and the *tropical product* of two numbers is their usual sum. Here are some examples of how to do arithmetic in the tropical world:

$$3 \oplus 7 \;=\; 3 \qquad \text{and} \qquad 3 \odot 7 \;=\; 10.$$

Tropical addition and tropical multiplication are both *commutative*:

$$x \oplus y \;=\; y \oplus x \qquad \text{and} \qquad x \odot y \;=\; y \odot x.$$

These two arithmetic operations are also associative, and the times operator $\odot$ takes precedence when plus $\oplus$ and times $\odot$ occur in the same expression. The *distributive law* holds:

$$x \odot (y \oplus z) \;\;=\;\; x \odot y \;\oplus\; x \odot z.$$

Here is a numerical example to show distributivity:

$$\begin{aligned} 3 \odot (7 \oplus 11) \;\;&=\;\; 3 \odot 7 \;\;=\;\; 10, \\ 3 \odot 7 \;\oplus\; 3 \odot 11 \;\;&=\;\; 10 \oplus 14 \;\;=\;\; 10. \end{aligned}$$

Both arithmetic operations have an identity element. Infinity is the *identity element* for addition and zero is the *identity element* for multiplication:

$$x \oplus \infty \;=\; x \qquad \text{and} \qquad x \odot 0 \;=\; x.$$

We also note the following identities involving the two identity elements:

$$x \odot \infty \;=\; \infty \qquad \text{and} \qquad x \oplus 0 \;=\; \begin{cases} 0 & \text{if } x \geq 0, \\ x & \text{if } x < 0. \end{cases}$$

There is no subtraction in tropical arithmetic. There is no real number x that we can call "17 minus 8" because the equation $8 \oplus x = 17$ has no solution x. Tropical division is defined to be classical subtraction, so $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ satisfies all ring axioms except for the existence of an additive inverse. Such algebraic structures are called *semirings*, whence the name tropical semiring. It is essential to remember that "0" is the multiplicative identity element. For instance, all coefficients in the *Binomial Theorem* are zero. Note the identity

$$\begin{aligned}(x \oplus y)^3 \quad &= \quad (x \oplus y) \odot (x \oplus y) \odot (x \oplus y) \\ &= \quad 0 \odot x^3 \ \oplus \ 0 \odot x^2 y \ \oplus \ 0 \odot x y^2 \ \oplus \ 0 \odot y^3.\end{aligned}$$

Of course, the zero coefficients can here be dropped. The following holds for all $x, y \in \mathbb{R}$:

$$(x \oplus y)^3 \quad = \quad x^3 \ \oplus \ x^2 y \ \oplus \ x y^2 \ \oplus \ y^3 \quad = \quad x^3 \ \oplus \ y^3.$$

The familiar algebra of vectors and matrices make sense over the tropical semiring. For instance, the tropical scalar product in $\mathbb{R}^3$ of a row vector with a column vector is the scalar

$$\begin{aligned}(u_1, u_2, u_3) \odot (v_1, v_2, v_3)^{\mathrm{T}} \quad &= \quad u_1 \odot v_1 \ \oplus \ u_2 \odot v_2 \ \oplus \ u_3 \odot v_3 \\ &= \quad \min\{u_1 + v_1,\, u_2 + v_2,\, u_3 + v_3\}.\end{aligned}$$

Here is the product of a column vector and a row vector of length three:

$$\begin{aligned}&(u_1, u_2, u_3)^T \odot (v_1, v_2, v_3) \\ &= \begin{pmatrix} u_1 \odot v_1 & u_1 \odot v_2 & u_1 \odot v_3 \\ u_2 \odot v_1 & u_2 \odot v_2 & u_2 \odot v_3 \\ u_3 \odot v_1 & u_3 \odot v_2 & u_3 \odot v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 & u_1 + v_2 & u_1 + v_3 \\ u_2 + v_1 & u_2 + v_2 & u_2 + v_3 \\ u_3 + v_1 & u_3 + v_2 & u_3 + v_3 \end{pmatrix}.\end{aligned} \tag{1}$$

Any matrix which can be expressed as such a product has *tropical rank one*.

Given a $d \times n$-matrix A, we might be interested in computing its image $\{A \odot \mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$, and in solving the linear systems $A \odot \mathbf{x} = \mathbf{b}$ for various right hand sides $\mathbf{b}$. For an introduction to tropical linear systems and their applications we recommend the books on *Max-linear Systems* by Butkovič [1] and *Essentials of Tropical Combinatorics* by Joswig [2].

For a first application of tropical algebra, consider the problem of finding shortest paths in a weighted directed graph. We fix a directed graph G with n nodes labeled $1, 2, \ldots, n$. Every directed edge (i, j) in G has an associated length d_{ij} which is a non-negative real number. If (i, j) is not an edge of G then we set $d_{ij} = +\infty$. We represent G by its $n \times n$ *adjacency matrix* $D_G = (d_{ij})$ with zeros on the diagonal and whose off-diagonal entries are the edge lengths d_{ij}. The matrix D_G need not be symmetric; we allow $d_{ij} \neq d_{ji}$ for some i, j. However, if G is an undirected graph, then we represent G as a directed graph with two directed edges (i, j) and (j, i) for each undirected edge $\{i, j\}$. In that special case, D_G is a symmetric matrix, and we think of $d_{ij} = d_{ji}$ as the distance between node i and node j.

Consider the $n \times n$-matrix with entries in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ that results from tropically multiplying the given adjacency matrix D_G with itself $n - 1$ times:

$$D_G^{\odot(n-1)} \quad = \quad D_G \odot D_G \odot \cdots \odot D_G. \tag{2}$$

Proposition 1. *Let G be a weighted directed graph on n nodes with adjacency matrix D_G. The entry of the matrix $D_G^{\odot(n-1)}$ in row i and column j equals the length of a shortest path from node i to node j in the graph G.*

Proof. Let $d_{ij}^{(r)}$ denote the minimum length of any path from node i to node j which uses at most r edges in G. We have $d_{ij}^{(1)} = d_{ij}$ for any two nodes i and j. Since the edge weights d_{ij} were assumed to be non-negative, a shortest path from node i to node j visits each node of G at most once. In particular, any shortest path in the directed graph G uses at most $n-1$ directed edges. Hence the length of a shortest path from i to j equals $d_{ij}^{(n-1)}$.

For $r \geq 2$ we have a recursive formula for the length of a shortest path:

$$d_{ij}^{(r)} \quad = \quad \min\bigl\{d_{ik}^{(r-1)} + d_{kj} \,:\, k = 1, 2, \ldots, n\bigr\}. \tag{3}$$

Using tropical arithmetic, this formula can be rewritten as follows:

$$\begin{aligned} d_{ij}^{(r)} \quad &= \quad d_{i1}^{(r-1)} \odot d_{1j} \;\oplus\; d_{i2}^{(r-1)} \odot d_{2j} \;\oplus\; \cdots \;\oplus\; d_{in}^{(r-1)} \odot d_{nj}. \\ &= \quad (d_{i1}^{(r-1)}, d_{i2}^{(r-1)}, \ldots, d_{in}^{(r-1)}) \odot (d_{1j}, d_{2j}, \ldots, d_{nj})^T. \end{aligned}$$

From this it follows, by induction on r, that $d_{ij}^{(r)}$ equals the entry in row i and column j of the $n \times n$ matrix $D_G^{\odot r}$. Indeed, the right hand side of the recursive formula is the tropical product of row i of $D_G^{\odot(r-1)}$ and column j of D_G, which is the (i,j) entry of $D_G^{\odot r}$. In particular, $d_{ij}^{(n-1)}$ is the entry in row i and column j of $D_G^{\odot(n-1)}$. This proves the claim. □

The above algorithm is an instance of what is known as *Dynamic Programming* in Computer Science. For us, running that algorithm means performing the matrix multiplication

$$D_G^{\odot r} \quad = \quad D_G^{\odot(r-1)} \odot D_G \qquad \text{for } r = 2, \ldots, n-1.$$

We next consider the notion of the *tropical determinant*. Fix an $n \times n$ matrix $X = (x_{ij})$. As there is no negation in tropical arithmetic, we define this determinant as the tropical sum over the tropical diagonal products obtained by taking all $n!$ permutations π of $\{1, 2, \ldots, n\}$:

$$\text{tropdet}(X) \quad := \quad \bigoplus_{\pi \in S_n} x_{1\pi(1)} \odot x_{2\pi(2)} \odot \cdots \odot x_{n\pi(n)}. \tag{4}$$

Here S_n is the *symmetric group* of permutations of $\{1, 2, \ldots, n\}$. Evaluating the tropical determinant means solving the classical *assignment problem* of combinatorial optimization. Imagine a company that has n jobs and n workers, and each job needs to be assigned to exactly one of the workers. Let x_{ij} be the cost of assigning job i to worker j. The company wishes to find the cheapest assignment $\pi \in S_n$. The optimal total cost equals

$$\min\bigl\{x_{1\pi(1)} + x_{2\pi(2)} + \cdots + x_{n\pi(n)} \;:\; \pi \in S_n\bigr\}. \tag{5}$$

This minimum is precisely the tropical determinant (4) of the matrix $X = (x_{ij})$:

Proposition 2. *The tropical determinant solves the assignment problem.*

In the assignment problem we seek the minimum over $n!$ quantities. This appears to require exponentially many operations. However, there is a polynomial-time algorithm. It was developed by Harold Kuhn in 1955 who called it the *Hungarian Assignment Method*. This algorithm maintains a price for each job and a partial assignment of workers and jobs. At each iteration, an unassigned worker is chosen and a shortest augmenting path from this person to the set of jobs is chosen. The total number of arithmetic operations is $O(n^3)$.

In classical arithmetic, the complexity of evaluating determinants and permanents differs greatly. The determinant of an $n \times n$ matrix can be computed in $O(n^3)$ steps, namely by *Gaussian elimination*, while computing the permanent of an $n \times n$ matrix is a hard problem. Leslie Valiant proved that computing permanents is *#P-complete*. In tropical arithmetic, computing the permanent is easier, thanks to the Hungarian Assignment Method. We can think of the Hungarian Method as a certain tropicalization of Gaussian Elimination.

Eigenvectors and eigenvalues of square matrices are a central topic in linear algebra. Let us now see their counterparts in tropical linear algebra. We fix an $n \times n$-matrix $A = (a_{ij})$ whose entries a_{ij} are in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. An *eigenvalue* of A is a real number λ such that

$$A \odot \mathbf{v} \;=\; \lambda \odot \mathbf{v} \qquad \text{for some } \mathbf{v} \in \mathbb{R}^n. \tag{6}$$

We say that $\mathbf{v}$ is an *eigenvector* of the tropical matrix A. The arithmetic operations in the equation (6) are tropical. For instance, for $n = 2$, the left hand side of (6) equals

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11} \odot v_1 \oplus a_{12} \odot v_2 \\ a_{21} \odot v_1 \oplus a_{22} \odot v_2 \end{pmatrix} = \begin{pmatrix} \min\{a_{11} + v_1, a_{12} + v_2\} \\ \min\{a_{21} + v_1, a_{22} + v_2\} \end{pmatrix}.$$

The right hand side of (6) is equal to

$$\lambda \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad = \quad \begin{pmatrix} \lambda \odot v_1 \\ \lambda \odot v_2 \end{pmatrix} \quad = \quad \begin{pmatrix} \lambda + v_1 \\ \lambda + v_2 \end{pmatrix}.$$

Let $G(A)$ denote the directed graph with adjacency matrix A. Its nodes are labeled by $[n] = \{1, 2, \ldots, n\}$. There is an edge from node i to node j if and only if $a_{ij} < \infty$, and the edge has length a_{ij}. The *normalized length* of a directed path $i_0, i_1, \ldots, i_k$ in $G(A)$ is the sum (in classical arithmetic) of the lengths of the edges divided by the length k of the path. Thus the normalized length is $(a_{i_0 i_1} + a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k})/k$. If $i_k = i_0$ then the path is a *directed cycle*, and this quantity is the normalized length of the cycle. Recall that a directed graph is *strongly connected* if there is a directed path from any node to any other node.

Theorem 3. *Let A be an $n \times n$-matrix such that $G(A)$ is strongly connected. Then A has precisely one eigenvalue $\lambda(A)$. It equals the minimum normalized length of a directed cycle.*

Proof. Let $\lambda = \lambda(A)$ be the minimum of the normalized lengths over all directed cycles in $G(A)$. We first prove that $\lambda(A)$ is the only possibility for an eigenvalue. Suppose that $\mathbf{z} \in \mathbb{R}^n$ is any eigenvector of A, and let γ be the corresponding eigenvalue. For any cycle $(i_1, i_2, \ldots, i_k, i_1)$ in $G(A)$ we have

$$a_{i_1 i_2} + z_{i_2} \geq \gamma + z_{i_1}, \;\; a_{i_2 i_3} + z_{i_3} \geq \gamma + z_{i_2},$$
$$a_{i_3 i_4} + z_{i_4} \geq \gamma + z_{i_3}, \; \ldots \;, \; a_{i_k i_1} + z_{i_1} \geq \gamma + z_{i_k}.$$

Adding the left-hand sides and the right-hand sides, we find that the normalized length of the cycle is greater than or equal to γ. In particular, we have $\lambda(A) \geq \gamma$. For the reverse inequality, start with any index i_1. Since $\mathbf{z}$ is an eigenvector with eigenvalue γ, there exists i_2 such that $a_{i_1 i_2} + z_{i_2} = \gamma + z_{i_1}$. Likewise, there exists i_3 such that $a_{i_2 i_3} + z_{i_3} = \gamma + z_{i_2}$. We continue in this manner until we reach an index i_l which was already in the sequence, say, $i_k = i_l$ for $k < l$. By adding the equations along this cycle, we find that

$$\begin{aligned}&(a_{i_k i_{k+1}} + z_{i_{k+1}}) + (a_{i_{k+1} i_{k+2}} + z_{i_{k+2}}) + \cdots + (a_{i_{l-1} i_l} + z_{i_l}) \\ = \quad &(\gamma + z_{i_k}) \; + \; (\gamma + z_{i_{k+1}}) \; + \; \cdots \; + \; (\gamma + z_{i_{l-1}}).\end{aligned}$$

We conclude that the normalized length of the cycle $(i_k, i_{k+1}, \ldots, i_l = i_k)$ in $G(A)$ is equal to γ. In particular, $\gamma \geq \lambda(A)$. This proves that $\gamma = \lambda(A)$.

It remains to prove the existence of an eigenvector. Let B be the matrix obtained from A by (classically) subtracting $\lambda(A)$ from every entry in A. All cycles in $G(B)$ have non-negative length, and there exists a cycle of length zero. Using tropical matrix operations we define

$$B^+ \; = \; B \oplus B^2 \oplus B^3 \oplus \cdots \oplus B^n.$$

This matrix is known as the *Kleene plus* of the matrix B. The entry B^+_{ij} in row i and column j of B^+ is the length of a shortest path from node i to node j in the weighted directed graph $G(B)$. Since this graph is strongly connected, we have $B^+_{ij} < \infty$ for all i and j.

Fix any node j that lies on a zero length cycle of $G(B)$. Let $\mathbf{x} = B^+_{\cdot j}$ denote the jth column vector of the matrix B^+. We have $x_j = B^+_{jj} = 0$, as there is a path from j to itself of length zero, and there are no negative weight cycles. This implies $B^+ \odot \mathbf{x} \leq B^+_{\cdot j} = \mathbf{x}$. Next note that $(B \odot \mathbf{x})_i = \min_l (B_{il} + x_l) = \min_l (B_{il} + B^+_{lj}) \geq B^+_{ij} = x_i$, since lengths of shortest paths obey the triangle inequality. In vector notation this states $B \odot \mathbf{x} \geq \mathbf{x}$. Since tropical linear maps preserve coordinatewise inequalities among vectors, we have $B^2 \odot \mathbf{x} \geq B \odot \mathbf{x}$, and $B^3 \odot \mathbf{x} \geq B^2 \odot \mathbf{x}$, etc. Therefore, $B^+ \odot \mathbf{x} = B \odot \mathbf{x} \oplus B^2 \odot \mathbf{x} \oplus \cdots \oplus B^n \odot \mathbf{x} = B \odot \mathbf{x}$. This yields $\mathbf{x} \leq B \odot \mathbf{x} = B^+ \odot \mathbf{x} \leq \mathbf{x}$. This means that $B \odot \mathbf{x} = \mathbf{x}$, so $\mathbf{x}$ is an eigenvector of B with eigenvalue 0. We conclude that $\mathbf{x}$ is an eigenvector with eigenvalue λ of our matrix A:

$$A \odot \mathbf{x} \; = \; (\lambda \odot B) \odot \mathbf{x} \; = \; \lambda \odot (B \odot \mathbf{x}) \; = \; \lambda \odot \mathbf{x}.$$

This completes the proof of Theorem 3. □

The eigenvalue λ of a tropical $n \times n$-matrix can be computed efficiently. Given a matrix $A = (a_{ij})$, one sets up the following *linear program* with $n+1$ decision variables $v_1, \ldots, v_n, \lambda$:

$$\text{Maximize } \gamma \text{ subject to } \quad a_{ij} + v_j \geq \gamma + v_i \text{ for all } 1 \leq i, j \leq n. \tag{7}$$

Proposition 4. *The unique eigenvalue $\lambda(A)$ of the given $n \times n$-matrix $A = (a_{ij})$ coincides with the optimal value γ^* of the linear program* (7).

Proof. See [3, Proposition 5.1.2]. □

We next determine the *eigenspace* of the matrix A, which is the set

$$\mathrm{Eig}(A) \quad = \quad \big\{ \mathbf{x} \in \mathbb{R}^n \,:\, A \odot \mathbf{x} \,=\, \lambda(A) \odot \mathbf{x} \big\}.$$

The set $\mathrm{Eig}(A)$ is closed under tropical scalar multiplication: if $\mathbf{x} \in \mathrm{Eig}(A)$ and $c \in \mathbb{R}$ then $c \odot \mathbf{x}$ is also in $\mathrm{Eig}(A)$. We can thus identify $\mathrm{Eig}(A)$ with its image in the quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^{n-1}$. Here $\mathbf{1} = (1, 1, \ldots, 1)$. This space is called the *tropical projective torus*; cf. [2, Section 1.4]. We saw that every eigenvector of the matrix A is also an eigenvector of the matrix $B = (-\lambda(A)) \odot A$ and vice versa. Hence the eigenspace $\mathrm{Eig}(A)$ is equal to

$$\mathrm{Eig}(B) \quad = \quad \big\{ \mathbf{x} \in \mathbb{R}^n \,:\, B \odot \mathbf{x} \,=\, \mathbf{x} \big\}.$$

Theorem 5. *Let B_0^+ be the submatrix of the Kleene plus B^+ given by the columns whose diagonal entry B_{jj}^+ is zero. The image of this matrix (with respect to tropical multiplication of vectors on the right) is equal to the desired eigenspace:*

$$\mathrm{Eig}(A) \;=\; \mathrm{Eig}(B) \;=\; \mathrm{Image}(B_0^+).$$

Proof. See [3, Theorem 5.1.3]. □

Example 6. We demonstrate the computation of eigenvalues and eigenvectors for $n = 3$. In our first example, the minimal normalized cycle lengths are attained by the loops:

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 4 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix} \quad \Rightarrow \quad \lambda(A) = 3 \quad \Rightarrow \quad B = B^+ = B_0^+ = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The eigenspace is the tropical linear span in $\mathbb{R}^3$ of the column vectors of B. Its image in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ is the hexagon with vertices $(0,1,1)$, $(0,0,1)$, $(1,0,1)$, $(1,0,0)$, $(1,1,0)$ and $(0,1,0)$. In our second example, the shortest normalized cycle is the loop between nodes 1 and 2:

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 4 & 3 \end{pmatrix} \quad \Rightarrow \quad \lambda(A) = 1 \quad \Rightarrow \quad B = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 1 \\ 3 & 3 & 2 \end{pmatrix} \quad \Rightarrow \quad B^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}.$$

The eigenspace of A is the tropical linear space spanned by the first column of B_+:

$$\mathrm{Eig}(A) \;=\; \mathrm{Eig}(B) \;=\; \big\{ c \odot (0,0,3)^T \,:\, c \in \mathbb{R} \big\} \;=\; \big\{ (c, c, c+3)^T \,:\, c \in \mathbb{R} \big\}$$

So, here $\mathrm{Eig}(A)$ is just a single point in the tropical projective 2-torus $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. ◇

We computed the eigenspace of a square matrix as the image of another matrix. This motivates the study of images of tropical linear maps $\mathbb{R}^m \to \mathbb{R}^n$. Such images are <u>not</u> tropical linear spaces. They are known as *tropical polytopes*. Indeed, one defines *tropical convexity* in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ by taking tropical linear combinations. Tropical convexity is a rich and beautiful theory with many applications. For textbook introductions see [2, Chapter 5] and [3, §5.2].

We give a brief illustration in the case $m = n = 3$. The image of a 3×3-matrix X is the set of all tropical linear combinations of three vectors in $\mathbb{R}^3$. We represent this set by its image in the plane $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. That image is a *tropical triangle*, because it is the tropical convex hull of three points in the plane. It is possible that this triangle degenerates because three points are tropically collinear in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. This happens when the minimum in the tropical determinant (4) is attained twice. In that case, the matrix X is called *tropically singular*.

Example 7. Let $T = \text{image}(A)$ be the tropical triangle defined by either of the matrices

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad A' = \begin{pmatrix} -1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Each point in the quotient $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ can be represented uniquely by a vector $(u, v, 0)^T$ with last coordinate zero. The tropical triangle T consists of the segment between $(-1,-1,0)^T$ and $(0,0,0)^T$, the segment between $(0,0,3)^T$ and $(0,1,0)^T$, the segment between $(2,1,0)^T$ and $(1,1,0)^T$, and the classical triangle with vertices $(0,0,0)^T$, $(0,1,0)^T$ and $(1,1,0)^T$. $\diamondsuit$.

There are five distinct combinatorial types of tropical triangles in the plane. Similarly, there are 35 types of tropical quadrilaterals in the plane. They are shown in [3, Figure 5.2.4].

Up to this point, this section has explored the tropical counterparts of concepts from linear algebra. In what follows we move on to nonlinear algebra, and we discuss the tropical counterparts of algebraic varieties. This will also show how the tropical semiring arises naturally from the familiar arithmetic operations over a field K.

We fix an algebraically closed field with a valuation, for instance the field of *Puiseux series* in a variable t with complex coefficients. This field is denoted $K = \mathbb{C}\{\{t\}\}$. It contains the field $\mathbb{C}(t)$ of rational functions and its algebraic closure $\overline{\mathbb{C}(t)}$. Indeed, every algebraic function can be expanded into a Puiseux series with integer exponents. The *valuation* of a scalar c in K is the exponent $\text{val}(a) \in \mathbb{Q}$ of the smallest term $c_a t^a$ that appears with nonzero coefficient in the expansion of c. Here two examples of scalars in K and their valuations:

$$c = \frac{1}{t^2 + 2t^3 + t^5} = t^{-2} - 2t^{-1} + 4 - 9t + 20t^2 - 44t^3 + 97t^4 - 214t^5 + 472t^6 - \cdots \quad \text{has } \text{val}(c) = -2.$$

$$c' = t^{2/7}\sqrt{1 - t^{2/3}} = t^{2/7} - \frac{1}{2}t^{20/21} - \frac{1}{8}t^{34/21} - \frac{1}{16}t^{16/7} - \frac{5}{128}t^{62/21} - \cdots \quad \text{has } \text{val}(c') = \frac{2}{7}.$$

Every polynomial of degree d in $K[x]$ has d distinct roots, counting multiplicities.

Example 8 (Puiseux series)**.** Every cubic polynomial in $K[x]$ has three roots, easily found using computer algebra. For instance, the three roots of $f(x) = tx^3 - x^2 + 3tx - 2t^5$ are

$$\begin{gathered} t^{-1} - 3t - 9t^3 - 54t^5 + 2t^6 - 405t^7 + 18t^8 - 3402t^9 + 180t^{10} - 30618t^{11} + 1890t^{12} + \cdots \\ 3t + 9t^3 - \tfrac{2}{3}t^4 + 54t^5 - 2t^6 + \tfrac{10931}{27}t^7 - 18t^8 + 3402t^9 - \tfrac{43756}{243}t^{10} + 30618t^{11} + \cdots \\ \tfrac{2}{3}t^4 + \tfrac{4}{27}t^7 + \tfrac{16}{243}t^{10} - \tfrac{8}{81}t^{12} + \tfrac{80}{2187}t^{13} - \tfrac{80}{729}t^{15} + \tfrac{448}{19683}t^{16} - \tfrac{224}{2187}t^{18} + \cdots \end{gathered} \tag{8}$$

The valuations of the three roots are -1, 1 and 4. These valuations characterize the asymptotic behavior of the roots when the parameter t is a real number very close to zero.

Consider any polynomial in n variables with coefficients in the Puiseux series field K:

$$f \;=\; c_1\mathbf{x}^{\mathbf{a}_1} \,+\, c_2\mathbf{x}^{\mathbf{a}_2} \,+\, \cdots \,+\, c_s\mathbf{x}^{\mathbf{a}_s}. \tag{9}$$

The tropicalization of f is the following expression in tropical arithmetic:

$$\mathrm{trop}(f) \;=\; \mathrm{val}(c_1)\odot\mathbf{x}^{\odot\mathbf{a}_1} \,\oplus\, \mathrm{val}(c_2)\odot\mathbf{x}^{\odot\mathbf{a}_2} \,\oplus\, \cdots \oplus\, \mathrm{val}(c_s)\odot\mathbf{x}^{\odot\mathbf{a}_s}.$$

To evaluate this *tropical polynomial* at a point $\mathbf{u} = (u_1,\ldots,u_n)$, we take the minimum of

$$\mathrm{val}(c_i)\odot\mathbf{u}^{\odot\mathbf{a}_i} \;=\; \mathrm{val}(c_i)\odot u_1^{\odot a_{i1}}\odot\cdots\odot u_n^{\odot a_{in}} \;=\; c_i + a_{i1}u_1 + \cdots + a_{in}u_n \quad \text{over } \; i\in\{1,\ldots,s\}.$$

If this minimum is attained at least twice then we say that $\mathbf{u}$ is a *tropical zero* of $\mathrm{trop}(f)$.

Proposition 9. *If* $\mathbf{z} = (z_1,\ldots,z_n)\in K^n$ *is a zero of a polynomial* f *in* $K[\mathbf{x}]$ *then its coordinatewise valuation* $\mathrm{val}(\mathbf{z}) = \big(\mathrm{val}(z_1),\ldots,\mathrm{val}(z_n)\big)\in\mathbb{Q}^n$ *is a tropical zero of* $\mathrm{trop}(f)$.

Proof. Note that the valuation of the Puiseux series $c_i\mathbf{z}_i^{\mathbf{a}}$ equals $\mathrm{val}(c_i)\odot\mathbf{u}^{\odot\mathbf{a}_i}$. The sum of these r Puiseux series is zero in K, so the terms of lowest valuation must cancel. This implies that the minimum valuation is attained by two or more of the expressions $\mathrm{val}(c_i)\odot\mathbf{u}^{\odot\mathbf{a}_i}$. By definition, this means that the vector $\mathbf{u}\in\mathbf{Q}^n$ is a tropical zero of $\mathrm{trop}(f)$. $\square$

A celebrated result due to Kapranov states that the converse holds as well. Namely, if $f\in K[\mathbf{x}]$ and $\mathbf{u}\in\mathbb{Q}^n$ is a tropical zero of $\mathrm{trop}(f)$ then there exists a point $\mathbf{z}\in K^n$ such that $f(\mathbf{z}) = 0$ and $\mathrm{val}(\mathbf{z}) = \mathbf{u}$. We refer to [3, Theorem 3.1.3] the proof and further details.

Example 10 $(n = 1)$**.** If f is the cubic polynomial in Example 8 then its tropicalization is

$$\mathrm{trop}(f) \;=\; 1\odot x^{\odot 3} \,\oplus\, 0\odot x^{\odot 2} \,\oplus\, 1\odot x \,\oplus\, 5.$$

The tropical zeros are the rational numbers x such that the minimum of $1+3x$, $0+2x$, $1+x$ and 5 is attained twice. There are three solutions: $x = -1$, $x = 1$ and $x = 4$. Each of these is the valuation of an element in K that is a zero of f. These solutions are listed in (8). $\diamondsuit$

The extra element $+\infty$ arises naturally from the arithmetic in a field K with valuation because $\mathrm{val}(0) = \infty$. Sometimes it is preferable to restrict tropical algebra to $\mathbb{R}$, or to $\mathbb{Q}$, thus excluding $+\infty$. This is accomplished by disallowing zero coordinates among the solutions of a polynomial equation. To be precise, we set $K^* = K\backslash\{0\}$ and we introduce the *algebraic torus* $(K^*)^n$. The ring of polynomial functions on $(K^*)^n$ is the *Laurent polynomial ring*

$$K[\mathbf{x}^{\pm}] \;:=\; K[x_1^{\pm 1}, x_2^{\pm 1},\ldots,x_n^{\pm 1}].$$

Its elements are polynomials as in (9) but we now allow negative integers among the coordinates of the exponent vectors $\mathbf{a}_i$. For every $\mathbf{u}\in\mathbb{R}^n$, the initial form $\mathrm{in}_{\mathbf{u}}(f)$ is the subsum of terms $\overline{c_i}\mathbf{x}^{\mathbf{a}_i}$ in (9) for which $\mathrm{val}(c_i)\odot\mathbf{u}^{\odot\mathbf{a}_i}$ is minimal. Here $\overline{c_i}$ is the term of lowest order in the Puiseux series c_i. For instance, if c is the scalar in the middle line in (8) then $\overline{c} = 3t$.

Lemma 11. *For a Laurent polynomial $f \in K[\mathbf{x}^{\pm}]$ and $\mathbf{u} \in \mathbb{R}^n$, the following are equivalent:*

$\mathrm{in}_{\mathbf{u}}(f)$ *is a unit in* $K[\mathbf{x}^{\pm}]$ $\Leftrightarrow$ $\mathrm{in}_{\mathbf{u}}(f)$ *is a not monomial* $\Leftrightarrow$ $\mathbf{u}$ *is a tropical zero of* $\mathrm{trop}(f)$.

Fix any ideal I in $K[\mathbf{x}^{\pm}]$ and let $\mathcal{V}(I)$ be its variety in the algebraic torus $(K^*)^n$. We define the *tropical variety* associated with the ideal I to be the following subset of $\mathbb{R}^n$:

$$\mathrm{trop}(\mathcal{V}(I)) \;=\; \bigl\{\, \mathbf{u} \in \mathbb{R}^n \;:\; \mathbf{u} \text{ is a tropical zero of } \mathrm{trop}(f) \text{ for all } f \in I \,\bigr\}.$$

We also refer to this set as the *tropicalization* of the variety $\mathcal{V}(I)$.

The study of tropical varieties is the subject of tropical algebraic geometry. Two important results are the *Fundamental Theorem* ([3, Theorem 3.2.3]) and the *Structure Theorem* ([3, Theorem 3.3.5]). The former extends the theorem of Kapranov mentioned above. It states that the set of rational points in $\mathrm{trop}(\mathcal{V}(I))$ is the image of the classical variety $\mathcal{V}(I) \subset (K^*)^n$ under the coordinatewise valuation map. The latter states that $\mathrm{trop}(\mathcal{V}(I))$ is a balanced polyhedral complex, whose dimension agrees that the dimension of $\mathcal{V}(I)$. Numerous concrete examples of such polyhedral complexes are found in the books [2] and [3].

Example 12. Fix $n = 9$ and let $\mathbf{x} = (x_{ij})$ be a 3×3-matrix whose entries are unknowns.

Let I be the ideal generated by the nine 2×2-minors of $\mathbf{x}$. Then $\mathcal{V}(I)$ is the 5-dimensional variety of 3×3-matrices of rank 1 in $(K^*)^{3\times 3}$. The tropical variety $\mathrm{trop}(\mathcal{V}(I))$ is the set of 3×3-matrices in (1), that is, the matrices of tropical rank one. This is the linear subspace of dimension 5 in $\mathbb{R}^{3\times 3}$ defined by the tropical 2×2-determinants $u_{ij} \odot u_{kl} \oplus u_{ik} \odot u_{kj}$. Of course, this minimum is attained twice if and only if $u_{ij} + u_{kl} - u_{ik} - u_{kj} = 0$. Every matrix $\mathbf{u} = (u_{ij})$ that satisfies these linear equations, and has its entries in $\mathbb{Q}$, arises as the valuation $u = \mathrm{val}(\mathbf{z})$ of a rank one matrix $\mathbf{z} = (z_{ij})$ with entries in K^*. We can just take $\mathbf{z} = (t^{u_{ij}})$.

The situation becomes more interesting when we pass from rank 1 to rank 2. Let J be the principal ideal generated by the determinant of $\mathbf{x}$. The $\mathcal{V}(J)$ is a hypersurface of degree three in $(K^*)^{3\times 3}$. The tropical hypersurface $\mathrm{trop}(\mathcal{V}(J))$ is defined by the tropical determinant

$$\mathrm{tropdet}(\mathbf{u}) \;=\; u_{11} \odot u_{22} \odot u_{33} \;\oplus\; u_{11} \odot u_{23} \odot u_{32} \;\oplus\; \cdots \;\oplus\; u_{13} \odot u_{22} \odot u_{31}. \tag{10}$$

Thus $\mathrm{trop}(\mathcal{V}(J))$ is set of all 3×3-matrices $\mathbf{u} = (u_{ij})$ such that this minimum is attained twice. For such a matrix, there is more than one optimal assignment of the three workers to the three jobs in (4). The set $\mathrm{trop}(\mathcal{V}(J))$ is a polyhedral fan of dimension 8. It is a cone with apex $\mathrm{trop}(\mathcal{V}(I)) \simeq \mathbb{R}^5$ over the 2-dimensional polyhedral complex shown in Figure 1.

The six triangles represent matrices $\mathbf{u}$ where the minimum in (10) is attained by two permutations in S_3 that have the same sign. The nine squares on the right in Figure 1 are glued to form a torus. These represent matrices $\mathbf{u}'$ where the minimum in (10) is attained by two permutations in S_3 that have opposite signs. Concrete examples for the two cases are

$$\mathbf{u} \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}' \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

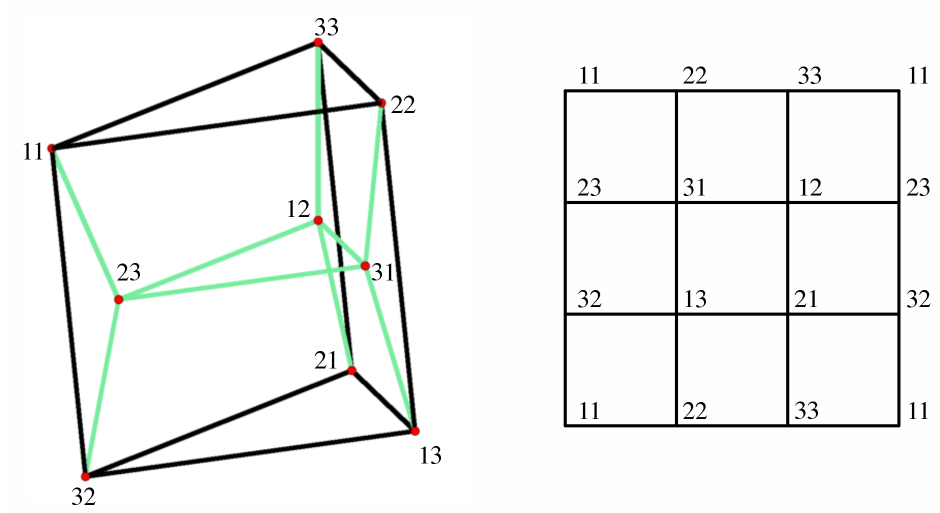

Figure 1: Combinatorics of the hypersurface defined by the tropical 3×3-determinant.

Here are classical matrices of rank 1 that map to $\mathbf{u}$ and $\mathbf{u}'$ under tropicalization:

$$\mathbf{z} = \begin{pmatrix} t+1 & -1+t & 2t \\ t & 1 & 1+t \\ 1 & t & 1+t \end{pmatrix} \qquad \text{and} \qquad \mathbf{z}' = \begin{pmatrix} 1 & 2 & t \\ 2 & 4 & 5 \\ 3t & 6t & 7 \end{pmatrix}$$

The supports of the matrices $\mathbf{u} = \mathrm{trop}(\mathbf{z})$ match the labels of the corresponding 2-cells in Figure 1. The matrix $\mathbf{u}$ has support $13, 21, 32$, which labels the bottom triangle on the left. The matrix $\mathbf{u}$ has support $13, 23, 31, 32$, which labels the middle left square on the right. $\diamondsuit$

We close with a remark on lifting Proposition 1 from tropical algebra to algebra over the field K. Given a directed graph G with rational edge weights d_{ij}, we now define a new adjacency matrix A_G. The entry of A_G in row i and column j equals t^{ij} if (i,j) is an edge of G, and 0 otherwise. By construction, the valuation of the matrix A_G is the earlier adjacency matrix D_G. Moreover, the matrix in (2) is the valuation of the classical matrix power of A_G:

$$D_G^{\odot(n-1)} = (\mathrm{val}(A_G))^{\odot(n-1)} = \mathrm{val}\big(A_G^{n-1}\big). \tag{11}$$

Indeed, the (i,j) entry of A_G^{n-1} is the generating function for all paths. To be precise, it is the Puiseux polynomial $\sum_\ell c_\ell t^\ell$, where c_ℓ is the number of paths from i to j having length ℓ.

Exercises

1. Let u, v, w be real numbers and let x, y, z be variables. What are the coefficients in the expansion of the expression $(u \odot x \,\oplus\, v \odot y \,\oplus\, w \odot z)^{\odot n}$ in tropical arithmetic?

2. Prove that the tropical multiplication of square matrices is an associative operation.

3. Draw the graph of the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto 1 \oplus 2 \odot x \oplus 3 \odot x^{\odot 2} \oplus 6 \odot x^{\odot 3} \oplus 10 \odot x^{\odot 4}$. What are the tropical zeros of this tropical polynomial?

4. How would you define the tropical characteristic polynomial of a square matrix? Compute your characteristic polynomial for the 3×3-matrices in Example 6.

5. Draw the graph of the function $\mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto 1 \oplus 2 \odot x \oplus 3 \odot y \oplus 6 \odot xy \oplus 10 \odot xy^{\odot 2}$. What are the tropical zeros of this tropical polynomial?

6. Let G be the directed graph on n nodes with edge weights $d_{ij} = i \cdot j$ for $i, j \in \{1, 2, \ldots, n\}$. Compute the tropical powers $D_G^{\odot i}$ of the matrix D_G for $i = 1, 2, \ldots, n-1$. What are their tropical ranks? Interpret the entries of these matrices in terms of paths.

7. Take G from above with $n = 5$. Compute the powers A_G^i of the matrix A_G for $i < n$. What are their ranks? Interpret the entries in terms of paths. Verify equation (11).

8. Take G from above with $n = 3$. Find the eigenvalues and eigenspaces of A_G. Find the tropical eigenvalue and the tropical eigenspace of D_G. Do you see a relationship?

9. Take G from above with $n = 10$. Compute the determinant of A_G and the tropical determinant of D_G. Do you see a relationship? Can you generalize to arbitrary n?

10. Take G from above. The matrix D_G defines a tropical linear map from $\mathbb{R}^n$ to itself. Determine the image of this map for $n = 2, 3, 4$. Draw pictures in $\mathbb{R}^n/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^{n-1}$.

11. Consider the quartic polynomial $f(x) = t + t^2x + t^3x^2 + t^6x^3 + t^{10}x^4$ in $K[x]$. Identity its four roots. Write the first 10 terms of these Puiseux series. What are their valuations?

12. Let J be the ideal generated by the determinant of a symmetric 3×3-matrix. This lives in a Laurent polynomial ring with six variables. Determine the tropical hypersurface $\operatorname{trop}(\mathcal{V}(J))$. Write a discussion analogous to Example 12. Draw the analog to Figure 1.

13. Analyze the complexity of the algorithm described in Proposition 1. How would you improve the computation of $D_G^{\odot(n-1)}$? What happens if some weights of the edges of G are negative? What happens if the graph contains cycles of negative total weight? How would you detect if such a cycle exists?

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